TOPICS IN ADVANCED ANALYSIS - PDES

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DISCLAIMER: this is an α-version of the notes

1. Possible extensions

Subjects which can be further studied:

(1) Linear transport equation for vector fields in BV or singular integrals of an L^1 functions.
(2) Traffic models on network
(3) Local existence of smooth solutions to Euler
(4) Riemann problem for 1-d systems
(5) Incompressible Euler in the framework of optimal transport
(6) Incompressible Euler for vorticity in L^p or mesures

2. Preliminaries

We will always work in an open subset of $\mathbb{R}^n$, and not distinguish between the gradient and the differential. The scalar product is denoted by $\cdot$.

The notation for derivatives is

$$\partial_{x_i} u = \partial_i u,$$
$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n, \quad \partial^\alpha u = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_n}^{\alpha_n} u.$$

Often if the space if $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ we will write

$$\partial_t u = u_t, \quad (\partial_{x_1} u, \dots, \partial_{x_n} u) = \nabla u = Du.$$

Similarly for $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ the divergence operators acts only on $\mathbb{R}^d$.

We will denote the Lagrangian derivative as

$$\frac{d}{dt} u = \frac{d}{dt} u(t, X(t, y)) = u_t + \dot{X} \cdot \nabla u.$$

The convolution $*$ will be denoted with u^ϵ:

$$u^\epsilon(x) = \phi^\epsilon * u = \int u(y) \frac{\phi(x/\epsilon)}{\epsilon^n} dx.$$

The space of probabilities on X is $\mathcal{P}(X)$, the space of measures $\mathcal{M}(X)$.

Part 1
Basic notions

Definition 2.1. A *k-th order PDE (Partial Differential Equation)* is a relation of the form

$$F(D^k u, D^{k-1}u, \dots, u, x) = 0,$$

where

(1) $u : \mathbb{R}^n \to \mathbb{R}$ is the unknown,
(2) $D^k u = (D^\alpha u)_\alpha$, α multi-index with $|\alpha| = k$,
(3) $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \mathbb{R} \times U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open.

The PDE is said to be

linear: F is linear in $u, Du, \dots$: its form is

$$\sum_{|\alpha|\leq k} a_\alpha(x) D^\alpha u = f(x);$$

homogeneous: linear with $f = 0$: in this case $u = 0$ is always a solution, and the set of solutions is a linear space;

semilinear: linear in the maximal derivatives:

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, \dots, u, x) = 0;$$

Quasilinear: linear in the maximal derivatives when the other are fixed:

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \dots, u, x) D^\alpha u + a_0(D^{k-1}u, \dots, u, x) = 0;$$

fully nonlinear: all the others.

Definition 2.2. A *system of PDEs* of order k

$$F(D^k u, \dots, u, x) = 0$$

is as above with only difference that $u : \mathbb{R}^n \to \mathbb{R}^m$ and

$$F : \mathbb{R}^{mn^k} \times R^{mn^{k-1}} \times \dots, \mathbb{R}^m \times R^n \to \mathbb{R}.$$

In general the PDE is supplemented with data u_0 on (part of) the boundary of U in order to have single out one solution u.

The same classification as before for *linear, homogeneous, semilinear, quasilinear, fully nonlinear system* of PDEs.

The PDEs we will consider in this course are:

transport: if $b(t, x) \in \mathbb{R}^n$ is a vector field, then the transport is *advective* if

$$u_t + b(t, x) \cdot \nabla u = 0,$$

conservative if

$$u_t + \mathrm{div}(b(t, x)u) = 0,$$

with a *source* if

$$u_t + b(t, x) \cdot \nabla u = f(t, x, u) \quad \text{or} \quad u_t + \mathrm{div}(b(t, x)u) = f(t, x, u);$$

Laplace: for $u : \mathbb{R}^n \supset U \to \mathbb{R}$

$$-\Delta u = -\sum_{i=1}^n \partial_i^2 u = 0;$$

Poisson:

$$-\Delta u = -\sum_{i=1}^n \partial_i^2 u = f(x);$$

Heat:

$$u_t - \Delta u = f(t, x);$$

Wave:

$$\Box u = u_{tt} - \Delta u = f;$$

First order PDEs:

$$F(Du, u, x) = 0;$$

Hamilton-Jacobi:

$$u_t + H(x, Du) = 0;$$

Conservation laws: for $u : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^m$, $F = (F_i)_{i=1}^n$, $F_i : \mathbb{R}^m \to \mathbb{R}^m$,

$$u_t + \mathrm{div}(F(u)) = u_t + \sum_{i=1}^n \partial_{x_i} F_i(u) = 0;$$

Incompressible Euler: for $u : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$,

$$u_t + (u \cdot \nabla)u + \nabla P = 0, \quad \mathrm{div}\, u = 0,$$

and $P : [0,\infty)\tilde{\mathbb{R}}^n \mapsto \mathbb{R}^n$ is the pressure.

3. Exercises

(1) Catalog the above PDEs in linear, semilinear, quasilinear, ect.
(2) Show that every linear system of PDE is transformed in a linear algebra problem by Fourier transform $\hat{u}(\xi)$.
(3) Characteristic speed: apply the above exercise to the PDE

$$\sum_{i=1}^n A_i \partial_i u = 0, \quad u : \mathbb{R}^n \to \mathbb{R}^m,$$

and show that the frequencies $\xi \in \mathbb{R}^d$ for $\hat{u}(\xi) \neq 0$ corresponds to all directions $d \in \mathbb{R}^n$ and vectors $v \in \mathbb{R}^m$ such that there is a solution of the form $u(x) = vw(d \cdot x)$, $w : \mathbb{R} \to \mathbb{R}^m$.

Part 2
Linear Transport Equation

Follow [Ambrosio, Lecture notes on transport equations].

The linear transport PDEs are of the form

$$u_t + b \cdot \nabla u = g(t,x)u \text{ (advective)}, \quad \rho_t + \operatorname{div}(b\rho) = g(t,x)\rho, \text{ (conservative or continuity)} \tag{3.1}$$

with

$$u, \rho \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^n), \quad b \in L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n), \ \operatorname{div} b \in L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^n), \quad g \in L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^n).$$

The PDEs are considered distributionally, with the advective transport written as

$$u_t + \operatorname{div}(bu) = g(t,x)u + u \operatorname{div} b.$$

The PDE is supplemented with an initial data $u(t=0) = u_0$, $\rho(t=0) = \rho_0$, so that the distributional formulation is for $\phi \in \mathcal{D}(\mathbb{R}^{d+1})$

$$\text{advective} \quad \int_0^\infty \int u\big(\phi_t + b \cdot \nabla\phi + \operatorname{div} b\phi + g\phi\big)dxdt + \int \phi(t=0)u_0 dx = 0,$$

$$\text{continuity} \quad \int_0^\infty \int \rho\big(\phi_t + b \cdot \nabla\phi + g\phi\big)dxdt + \int \phi(t=0)\rho_0 dx = 0.$$

Remark 3.1. These are not the most general assumptions: it is sufficient to have

$$\rho \in \mathcal{M}(\mathbb{R}^{d+1}), \quad b, g \in L^1_{\mathrm{loc}}(\rho)$$

for the continuity equation.

The advective PDEs is solved by assuming that there is solution to

$$\rho_t + \operatorname{div}(b\rho) = 0, \quad \rho \in [C^{-1}, C],$$

and using duality to define u solution if

$$(u\rho)_t + \operatorname{div}(u\rho b) = g(t,x)u\rho. \tag{3.2}$$

The solution u may depend on the function ρ chosen.

In between assumptions can be $b \in L^p_{\mathrm{loc}}$, $\rho \in L^{p'}_{\mathrm{loc}}$, $\frac{1}{p} + \frac{1}{p'} = 1$.

Remark 3.2. The initial data can be considered as a source term,

$$\rho_t + \operatorname{div}(b\rho) = \rho_0 \delta_{t=0} \times \mathscr{L}^d.$$

This idea leads to the notion of *divergence-measure vector fields*

$$\operatorname{div}_{t,x}(\rho(1,b)) = \rho_t + \operatorname{div}(b\rho) \in \mathcal{M},$$

sometimes refereed as 1-*dimensional normal current*.

4. The smooth case

Assume

$$b \in C^1(\mathbb{R}^+ \times \mathbb{R}^n), \quad \frac{b}{1+|x|} \in L^\infty.$$

Let $X(t,y)$ be the unique flow of b

$$\dot X(t,y) = b(t, X(t,y)), \quad X(0,y) = y, \quad J(t,y) = \det \nabla X(t,y), \ \dot J = \operatorname{div} b(t, X(t,y))J.$$

The trajectories $t \mapsto X(t,y)$ are called *characteristics*.

Lemma 4.1. *The distributional solutions to* (3.1) *satisfies*

$$\frac{d}{dt} u(t, X(t,y)) = g(t, X(t,y))u(t, X(t,y)), \quad \frac{d}{dt}(\rho J) = g(t, X(t,y))\rho J, \tag{4.1}$$

with initial data $u_0(y)$, $\rho_0(y)$.

Proof. Being $X(t,y)$ smooth, we can use test functions of the form

$$\phi(t, X(t,y)) = \psi(t)\varphi(y),$$

so that the weak formulation of the continuity equation reads as

$$\begin{aligned} 0 &= \int_0^\infty \int \rho\big(\phi_t + b \cdot \nabla \phi + g\phi\big) dx dt + \int \phi(t=0)\rho_0 dx \\ &= \int_0^\infty \int \rho\big(\psi'\varphi + g\psi\varpi\big) dx dt + \int \psi(0)\varphi\rho_0 dx \\ (t,x) = (t, X(t,y)) \qquad &= \int \bigg[\int_0^\infty \rho J \psi' dt + \rho_0 J \psi\bigg] \varphi dy. \end{aligned}$$

Being φ arbitrary, then for $\mathscr{L}^d$-a.e. y

$$\int_0^\infty \rho J \psi' dt + \rho_0 J \psi = 0,$$

which is the distributional formulation of the second ODE in the statement.

Concerning u, we can test with

$$\phi(t, X(t,y)) = \frac{\psi(t)\varphi(y)}{J(t,y)}$$

and notice that

$$\begin{aligned} \partial_t \phi + b \cdot \nabla \phi + \operatorname{div} b\phi &= \frac{d}{dt}\phi + \operatorname{div} b\phi \\ &= \frac{\psi'\varphi}{J} - \frac{\psi\varphi}{J^2}\dot{J} + \operatorname{div} b\phi \\ &= \frac{\psi'\varphi}{J} - \frac{\psi\varphi}{J}\operatorname{div} b + \operatorname{div} b\phi = \frac{\psi'\varphi}{J}, \end{aligned}$$

so that the same computation above applies. □

Corollary 4.2. *Under the above smoothness and growth assumptions on b and if $g \in L^\infty$, there exists a unique distributional solution.*

Proof. The distributional solutions to (3.1) are

$$u(t, X(t,y)) = e^{\int_0^t g(s,X(s,y))ds} u_0(y), \quad \rho(t, X(t,y)) = \frac{e^{\int_0^t g(s,X(s,y))ds}\rho_0(y)}{J(t,y)}. \tag{4.2}$$

Note that $g(t, X(t,y)$ is meaningful because $\ln J$ is locally bounded. □

5. Existence of solutions

Consider

$$b = b_1 + b_2, \quad b \in L^1_{\mathrm{loc}}, \frac{b_2}{1+|x|} \in L^\infty,$$

and let b^ϵ be its regularization by convolution: then b^ϵ satisfies the assumption of the previous section: let $u^\epsilon(t)$ be the solution of Lemma 4.1.

Proposition 5.1. *The solution $u^\epsilon(t)$ are uniformly bounded in L^∞ and up to subsequences weakly converges to a weak solution to the transport equation.*

Proof. If the family of solutions u^ϵ is uniformly bounded, then there is a subsequence weakly converging in L^∞: since $b^\epsilon \to b$ in L^1_{loc}, then every weak limit u is a weak solution (just pass to the limit of the weak formulation).

Being $g \in L^\infty$, one gets

$$\|u^\epsilon(t)\|_\infty \leq \|u_0\|^\infty e^{\|g\|_\infty t}.$$

Being $\operatorname{div} b \in L^\infty$, similarly for the Jacobian

$$e^{-\|diveb\|_\infty t} \leq J(t,y) \leq e^{\|diveb\|_\infty t}.$$

The formulas of Lemma 4.1 yield the uniform bounds. □

6. A counterexample to uniqueness

Consider the space $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ (in order to avoid boundaries), parametrized as $[0,1]^2$, and define

$$b_0(t,x) = \begin{cases} (3\mathbf{I}_{[0,1/2]}(y), 0) & t \in [0,1/6), \\ (0, 3\mathbf{I}_{[1/4,3/4]}(x)) & t \in [1/6,1/3), \\ (3/2\mathbf{I}_{[0,1/2]}(y), 0) & t \in [1/3,1/2). \end{cases}$$

Hence the solution to

$$u_t + b \cdot \nabla u = 0, \quad u_0 = \mathrm{sign}(\sin(2\pi x)),$$

satisfies

$$u(t = 1/2) = \mathrm{sign}\big(\sin(4\pi x)\big) = u_0(2x).$$

Extend b by periodicity and set

$$b(t,x) = b_0(t, 2^n x), \quad \text{for } t \in [1 - 2^{-n}, 1 - 2^{-n-1}],$$

so that the solution u is

$$u(t = 1 - 2^{-n}) = u_0(2^n x).$$

Lemma 6.1. *It holds*

$$u(t) \underset{t\to 1}{\rightharpoonup} 0.$$

Proof. It is sufficient to observe that

$$\int_{2^{-n}[k,k+1]^2} u(t,x) = 0$$

for all $t \geq 1 - 2^{-n}$. □

Let now

$$\tilde{u}(t,x) = u(1-t,x), \quad \tilde{b}(t,x) = -b(1-t,x).$$

Proposition 6.2. *The function $\tilde{u}$ is a nonzero weak solution to*

$$\tilde{u}_t + \tilde{b} \cdot \nabla \tilde{u} = 0, \quad \tilde{u}(t=0) = 0.$$

The proof is immediate.

In particular the vector field $\tilde{b}$ has not uniqueness of solutions.

7. Renormalization

Assume as before that

$$b = b_1 + (1 + |x|)b_2, \quad b_1 \in L^1, b_2 \in L^\infty, \mathrm{div}\, b \in L^\infty.$$

Definition 7.1. A distributional solution to

$$u_t + b \cdot \nabla u = g(t,x)u, \quad u(t=0) = u_0,$$

is *renormalizable* if for all $\beta \in C^1(\mathbb{R})$ bounded it holds

$$\beta(u)_t + b \cdot \beta(u) = g(t,x)\beta'(u)u, \quad \beta(u)(t=0) = \beta(u_0).$$

The last condition means that the initial data form $\beta(u)$ is the function $\beta(u_0)$.

It is immediate to see that every smooth solution is renormalized, and also if b is smooth then every distributional solution in $L^1_l oc$ is renormalized.

Definition 7.2. If every distributional solution $u \in L^\infty$ is renormalized, then b has the *renormalization property*.

Proposition 7.3. *If b has the renormalization property, then there exists only one solution in L^∞.*

Proof. By taking the difference of two solutions, it is enough to prove that the only solution with $u_0 = 0$ is $u = 0$. Considering the function $\beta(u) = u^2/(1+u^2)$, we have to prove that the only positive solution is 0.

Let

$$b = b_1 + b_2, \quad b_1 \in L^1, \ \frac{|b_2|}{1+|x|} \le C, \ \operatorname{div} b \in L^\infty,$$

and consider the family of test functions

$$\phi_R(t,x) = \phi_0\bigg(\frac{e^{2Ct}x}{R}\bigg), \quad \phi_0(y) = \phi_0(|y|) = \begin{cases} 1 & |y| \le 1, \\ 0 & |y| \ge 2, \end{cases}$$

$$\partial_t \phi_R = 2Cx \cdot \nabla \phi_R.$$

From the weak formulation we get

$$\begin{aligned} \frac{d}{dt}\int \phi_R v &= \int v\big(\partial_t \phi_R + b \cdot \nabla \phi_R\big) \\ &= \int v b_1 \cdot \nabla \phi_R + \int v(b_2 + 2Cx)\cdot \nabla \phi_R \\ &\le \int v b_1 \cdot \nabla \phi_R - C\int v|x||\nabla \phi_R|, \end{aligned}$$

because

$$|b_2| \le C(1+|x|), \quad \nabla \phi_R \parallel (-x).$$

Hence we discover that for any ϵ

$$\begin{aligned} \int v(t)\phi_R(t) &= \int_0^t \int v(s) b_1(s)\cdot \nabla \phi_R(s) dx ds - C\int_0^s \int v(s)|x||\nabla \phi_R(s)| dx ds \\ &\le \bigg(\int_0^t \int_{|x|\in Re^{-t}(1,2)} v(s)\big(|b_1(s)| - C|x|\big)|\nabla \phi_R(s)| dx\bigg) ds < \epsilon, \end{aligned}$$

for $R \gg 1$, being $b_1 \in L^1$. □

Remark 7.4. The solution $\tilde{u}$ of Proposition 6.2 satisfies for all $\beta \in C^1$

$$\beta(\tilde{u})_t + b \cdot \nabla \beta(\tilde{u}) = 0,$$

but the initial data will be

$$\frac{\beta(1) + \beta(-1)}{2} \ne \beta(0).$$

Hence it is not renormalized.

8. Weakly differentiable vector fields

Here we prove that vector fields such that $\nabla b \in L^p$ have the renormalization property.

Let ϕ be a convolution kernel: taking $\phi^\epsilon *$ (PDE) we obtain

$$u^\epsilon_t + b * \nabla u^\epsilon = b * \nabla u^\epsilon - (b * \nabla u)^\epsilon, \quad u^\epsilon(t=0) = u^\epsilon_0.$$

the last term is called the *commutator*, because it is the commutator

$$[b \cdot \nabla \cdot, (\cdot)^\epsilon] u.$$

Multiplying by $\beta'(u^\epsilon)$, and using the chain rule for smooth functions

$$\beta(u^\epsilon)_t + b \cdot \nabla \beta(u^\epsilon) = \beta'(u^\epsilon)\big(b * \nabla u^\epsilon - (b * \nabla u)^\epsilon\big). \tag{8.1}$$

Observing that

- $u^\epsilon \to u$ in L^1_{loc},
- $\beta(u^\epsilon) \to \beta(u)$ in L^1_{loc},

it is thus sufficient to prove that distributionally

$$\beta'(u^\epsilon)\big(b * \nabla u^\epsilon - (b * \nabla u)^\epsilon\big) \to 0$$

in order to pass to the limit to the PDE.

Theorem 8.1 (DiPerna-Lions)**.** *If $\nabla b \in L^p_{\mathrm{loc}}$, $u \in L^\infty$, then*

$$\beta'(u^\epsilon)\big(b * \nabla u^\epsilon - (b * \nabla u)^\epsilon\big) \underset{L^1_{\mathrm{loc}}}{\to} 0.$$

Proof. The formula we have to estimate is

$$\begin{aligned}
&\beta(u^\epsilon(x))\bigg(\int u(y)b(x)\cdot\nabla\phi^\epsilon(x-y)dy - \int u(y)b(y)\cdot\nabla\phi^\epsilon(x-y)dy + \int u(y)\operatorname{div} b(y)\phi^\epsilon(x-y)dy\bigg) \\
&= \beta(u^\epsilon(x))\bigg(\int u(y)\big(b(x)-b(y)\big)\cdot\nabla\phi^\epsilon(x-y)dy + \int u(y)\operatorname{div} b(y)\phi^\epsilon(x-y)dy\bigg) \\
&= \beta(u^\epsilon(x))\bigg(\int u(x-\epsilon z)\bigg(\frac{b(x)-b(x-\epsilon z)}{\epsilon}\bigg)\cdot\nabla\phi(z)dz + \int u(y)\operatorname{div} b(y)\phi^\epsilon(x-y)dy\bigg).
\end{aligned}$$

For every direction z it holds

$$\bigg\|\frac{b(x)-b(x-\epsilon z)}{\epsilon}\bigg\|_{L^p(B_R(0))} \leq \|\nabla b\cdot z\|_{L^p(B_{R+\epsilon|z|}(0))},$$

$$\frac{b(x)-b(x-\epsilon z)}{\epsilon} \underset{L^p}{\to} \nabla b(x)z,$$

so that from the boundedness of u^ϵ and its pointwise a.e. convergence to u we have

$$\begin{aligned}
&\beta(u^\epsilon(x))\int u(x-\epsilon z)\bigg(\frac{b(x)-b(x-\epsilon z)}{\epsilon}\bigg)\cdot\nabla\phi(z)dz \\
&\underset{L^p}{\to} \beta'(u(x))u(x)\sum_{ij}\partial_j b_i(x)\int z_j\partial_i\phi(z)dz \\
&= \beta'(u(x))u(x)\sum_{ij}\partial_j b_i(x)(-\delta_{ij}) = -\beta'(u(x))u(x)\operatorname{div} b(x).
\end{aligned}$$

Adding the term

$$\beta(u^\epsilon(x))\int u(y)\operatorname{div} b(y)\phi^\epsilon(x-y)dy \underset{L^p_{\mathrm{loc}}}{\to} \beta'(u(x))u(x)\operatorname{div} b(x)$$

wo obtain that the desired limit 0. □

Corollary 8.2. *Every vector field $b \in L^1_{\mathrm{loc}}$ such that $\nabla b \in L^p$, $p \in [1,\infty]$, has the renormalization property.*

Corollary 8.3. *If $b = b_1 + b_2$, $b_1 \in L^1$ and $b_2/(1+|x|) \in L^\infty$, $\nabla b \in L^p_{\mathrm{loc}}$ and $\operatorname{div} b \in L^\infty$, then for every initial data $u_0 \in L^\infty$ there is a unique solution in L^∞.*

Remark 8.4. The proof of renormalization works also if $u \in L^{p'}$, $1/p = 1/p' = 1$, but there are counterexamples for other ranges of exponents, e.g. $b \in W^{1,p}$, $u \in L^{q'}$, with $1/q + 1/p = 1 + 1/(n+1)$ (dual of the embedding exponent).

In the case $b \in L^1_{\mathrm{loc}}\mathrm{BV}_x$, $\operatorname{div} b \in L^\infty$, the proof shows that the r.h.s. of (8.1) converges to 0 only weakly: the proof is much more delicate.

9. Relation between the ODE and the PDE

This section extends the relation

$$u(t,x) \text{ solution to advective transport} \quad\Leftrightarrow\quad u(t,X(t,y)) = u_0(y),\ \dot X = b(t,X),$$

to the weak setting. Note that the ODE is meaningless for a single trajectory (b is defined in the equivalence class up to negligible sets), while the flow $X : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ as a function in L^1_{loc} is well defined.

For simplicity in this section we assume that $b \in L^\infty$ and the initial data u_0 has compact support, even if the general result holds for normal 1-currents.

We need the following notions:

(1) the space
$$\Gamma = \Big\{ \gamma \in \mathrm{Lip}([0,1],\mathbb{R}^n), \gamma(0) \in \mathrm{supp}\, u_0 \Big\}$$
is compact in $C^0([0,1],\mathbb{R}^n)$;
(2) the space of measures $\eta \in \mathcal{M}(\Gamma)$ is weakly compact, being the dual of the continuous functions on Γ;
(3) the evaluation map $e(t,\gamma) = \gamma(t)$ is continuous on Γ, and to every $\eta \in \mathcal{M}(\Gamma)$ is associate $\xi(t) \in \mathcal{M}(\mathbb{R}^n)$ by the formula $\xi = e(t)_\sharp \eta$, i.e.
$$\int \phi(x)\xi(t,dx) = \int \phi(\gamma(t))\eta(d\gamma), \quad \phi \in C(\mathbb{R}^n).$$
The above inequality holds also for Borel integrable functions by bounded and monotone convergence.

Proposition 9.1. *Assume that η is concentrated on the set of solutions to the ODE $\dot X = b(t,X)$, i.e. $b(t,X(t))$ is in $L^1([0,1])$ and for all $t \in [0,1]$*
$$X(t) = X(0) + \int_0^t b(s,X(s))ds.$$
Then $\xi(t)$ is a measure valued solution to
$$\xi_t + \mathrm{div}(b\xi) = 0. \tag{9.1}$$

Proof. Use the weak formulation and the definition of $\xi(t) = e(t)_\sharp \eta$ to obtain
$$\begin{aligned}
0 &= \int \bigg[\int_0^\infty \frac{d}{dt}\phi(t,\gamma(t))dt + \phi(\gamma(0)) \bigg] \eta(d\gamma) \\
&= \int \bigg[\int_0^\infty \phi_t(t,\gamma(t)) + b(t,\gamma(t)) \cdot \nabla\phi(t,\gamma(t)) \bigg] \eta(d\gamma) \\
&= \int_0^\infty \int (\phi_t + b\cdot\nabla\phi)(t,x)\xi(t,dx)dt + \int \phi(0,x)\xi(0,dx).
\end{aligned}$$
□

Remark 9.2. Being the PDE (9.1) invariant if we change zb on $\mathscr{L}^1 \otimes \xi(t)$-negligible sets, it follows that the set of trajectories affected by the choice of the representative b in the equivalence class is η-negligible.

The converse of the above proposition is the next

Theorem 9.3 (Smirnov-Ambrosio)**.** *Assume that b is a bounded Borel function and $t \mapsto \xi(t) \in \mathcal{P}(\mathbb{R}^n)$ is a Borel map such that distributionally*
$$\xi(t)_t + \mathrm{div}(b(t)\xi(t)) = 0.$$
Then there exists $\eta \in \mathcal{P}(\Gamma)$ such that $\xi(t) = e(t)_\sharp \eta$.

Proof. Approximate the transport PDE as
$$\xi^\epsilon_t + \mathrm{div}(b^\epsilon \xi^\epsilon) = 0, \quad b^\epsilon = \frac{(b\xi)^\epsilon}{\xi^\epsilon}.$$
In order to avoid division by 0, we assume here that the convolution kernel is unbounded, for example $G(\epsilon)$ fundamental solution to the Heat equation at time ϵ.

It is fairly easy to see that
$$b^\epsilon \in L^\infty \cap C^\infty,$$
so that by Lemma 4.1 we have the explicit formula for solutions as
$$\int \phi(X(t,y))\xi^\epsilon(t,X(t,y))dy = \int \phi(y)\xi^\epsilon(0,y)dy.$$
Equivalently, defining the measure $\eta^\epsilon \in \mathcal{P}(\Gamma)$ by
$$\int \psi(\gamma)\eta^\epsilon(d\gamma) = \int \psi(X^\epsilon(\cdot,y))\xi^\epsilon_0(y)dy, \quad \psi \in C(\Gamma),$$

we have that $\xi^\epsilon = e(t)_\sharp \eta^\epsilon$.

Being Γ compact, up to subsequences $\eta^\epsilon \rightharpoonup \eta$, and it is immediate to see that

$$\xi(t) = e(t)_\sharp \eta,$$

and

$$\begin{aligned} \int \int \phi b \xi(t, dy) dt &= \lim_\epsilon \int \int \phi (b\xi)^\epsilon dy dt \\ &= \lim_\epsilon \int \int \phi b^\epsilon \xi^\epsilon dy dt \\ &= \lim_\epsilon \int \int \phi(\gamma(t)) \dot\gamma dt \eta^\epsilon(d\gamma) \\ &= \int \int \phi \dot\gamma dt \eta(d\gamma), \end{aligned}$$

because

$$\gamma \mapsto \int \phi(\gamma(t)) \dot\gamma(t) dt$$

is continuous on Γ ($\dot\gamma$ converges weakly in L^∞).

It remains to check that η is concentrated on solutions to the ODE $\dot X = b(t, X)$.

Consider $\tilde b$ smooth bounded vector field and compute

$$\begin{aligned} &\int \left| \gamma(t) - \gamma(0) - \int_0^t \tilde b(s, \gamma(s)) ds \right| \eta^\epsilon \\ &= \int \left| X^\epsilon(t, y) - y - \int_0^t \tilde b(s, X^\epsilon(s, y) ds \right| \xi_0^\epsilon(y) dy \\ &\leq \int \int_0^1 |b^\epsilon - \tilde b|(t, X^\epsilon(t, y)) \xi_0^\epsilon(y) dy dt \\ &= \int_0^1 \int |b^\epsilon(t, x) - \tilde b(t, x)| \xi^\epsilon(t, y) dy dt \\ &= \int_0^1 \int \big| (\xi b)^\epsilon - \tilde b \xi^\epsilon \big| dy dt \\ &\leq \int_0^1 \int \big| (\xi b)^\epsilon - (\tilde b \xi)^\epsilon \big| dy dt + \int_0^1 \int |(\tilde b \xi)^\epsilon - \tilde b \xi^\epsilon| dy dt. \end{aligned} \tag{9.2}$$

Since

$$\gamma \mapsto \gamma(t) - \gamma(0) - \int_0^t \tilde b(s, \gamma(s)) ds$$

is continuous on Γ,

$$\begin{aligned} \int_0^1 \int \big| (\xi b)^\epsilon - (\tilde b \xi)^\epsilon \big| dy dt &\leq \int \int_0^1 \int G(\epsilon, y - x) |b(x) - \tilde b(x)| \xi(dx) dt dy \\ &\leq \int_0^1 \int |b - \tilde b| \xi(t, dy) dt, \end{aligned}$$

and being $\tilde{\mathbf{b}}$ smooth

$$\lim_\epsilon \int_0^1 \int |(\tilde b \xi)^\epsilon - \tilde b \xi^\epsilon| dy dt \leq \lim_\epsilon \int_0^1 \int \int G(\epsilon, y - x) |\tilde b(y) - \tilde b(x)| \xi(dx) dy dt = 0,$$

we can pass to the limit to the inequality (9.2) obtaining

$$\int \left| \gamma(t) - \gamma(0) - \int_0^t \tilde b(s, \gamma(s)) ds \right| \eta^\epsilon \leq \int_0^1 \int |b - \tilde b| \xi(t, dy) dt.$$

Letting the r.h.s. tend to 0 and observing that as a consequence

$$\tilde b(t, \gamma(t)) \to \tilde b(t, \gamma(t)) \quad \text{in } L^1(\mathscr{L}^1 \times \eta),$$

one obtains

$$\int \left| \gamma(t) - \gamma(0) - \int_0^t b(s,\gamma(s))ds \right| \eta^\epsilon = 0.$$

Hence, η-a.e. γ satisfies

$$\gamma(t) = \gamma(0) + \int_0^t b(s,\gamma(s))ds$$

for all rational times t, and being γ continuous it holds for all $t \in [0,1]$. □

10. Uniqueness of the regular Lagrangian flow

Here we assume that $b \in L^\infty$, $\operatorname{div} b \in L^\infty$, but the same result holds in $b \in W^{1,p}$, $\operatorname{div} b \in L^\infty$ and even weaker settings. It is still an open question which is the weakest setting where well posedness can be proved.

Definition 10.1. A Borel function $X : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a *Regular Lagrangian Flow (RLF)* for the vector field $b(t,x)$ if

1. for $\mathscr{L}^n$-a.e. y the function $t \mapsto X(t,y)$ is a solution to the ODE $\dot{X}(t,y) = b(t, X(t,y))$;
2. for every Borel set A, $t \in [0,T]$ it holds

$$\frac{\mathscr{L}^n(A)}{C} \leq \mathscr{L}^n(X(t,A) \leq C\mathscr{L}^n(A).$$

In the following we will need the next theorem.

Theorem 10.2 (Disintegration)**.** *Let X, Y be separable metric spaces, $\mu \in \mathcal{P}(X)$ and $f : X \to Y$ a Borel function. Then there is a map*

$$Y \ni y \mapsto \mu_y \in \mathcal{P}(X)$$

such that:

(1) define the image probability $m = f_\sharp \mu$ by

$$\forall B \subset Y \ \textit{Borel}\ \big(m(B) = \mu(f^{-1}(B))\big);$$

(2) for all $A \subset X$ Borel it holds

$$y \mapsto \mu_y(A) \quad \textit{is } m\textit{-measurable};$$

(3) it holds

$$\mu(A) = \int \mu_y(A) m(dy);$$

(4) μ_y is unique up to an m-negligible set.

The statement of the previous theorem is written

$$\mu = \int \mu_y m(dy).$$

Let η be the Smirnov representation of a solution $\rho \in [1/C, C]$ to the continuity equation (note that here η is just σ-compact, i.e. it is concentrated on a countable family of compact sets). Using the Borel map $\Gamma \ni \gamma \mapsto e(0,\gamma) = \gamma(0) \in \mathbb{R}^n$, we can write by the previous theorem

$$\eta = \int \eta_y \rho_0(y) \mathscr{L}^n(dy),$$

where we used $e(0)_\sharp \eta = \rho(t=0)\mathscr{L}^n$.

Proposition 10.3. *If there is a $\rho_0\mathscr{L}^d$-positive set $A \subset \mathbb{R}^n$ of initial data such that η_y is not a Dirac delta δ_{γ_y}, then the PDE has no uniqueness.*

Proof. The assumption implies that there are two disjoint sets Γ_1, Γ_2 and a time $\bar{t}$ such that

$$0 < e(0)_\sharp \eta \llcorner_{\Gamma_1} = e(0)_\sharp \eta \llcorner_{\Gamma_2} \leq \mathcal{L}^d,$$

$$\forall \gamma \in \Gamma_1 \ \big(\gamma(\bar{t}) \in A_1\big), \quad \forall \gamma \in \Gamma_2 \ \big(\gamma(\bar{t}) \in A_2\big),$$

$$A_1 \cap A_2 = \emptyset.$$

Hence the solutions
$$\rho_1(t) = e(t)_\sharp \eta \llcorner_{\Gamma_1}, \quad \rho_2(t) = e(t)_\sharp \eta \llcorner_{\Gamma_2},$$
satisfy
$$\rho_1(0) = \rho_2(0) = e(0)_\sharp \eta \llcorner_{\Gamma_1} = e(0)_\sharp \eta \llcorner_{\Gamma_2}, \quad \rho_1(\bar{t}) \times \rho_2(\bar{t}) = 0,$$
hence they are different. □

Theorem 10.4. *There exists a unique RLF.*

Proof. Since we know that the PDE as existence and uniqueness, the above proposition implies that $\eta_y = \delta_{\gamma_y}$. By considering a partition of $\mathbb{R}^n$ into boxes of measure 1, we obtain a flow $X(t, y) = \gamma_y(t)$ defined for $\mathscr{L}^n$-a.e. y.

Being every solution in L^∞ with the estimate
$$\|\rho(s)\|_\infty \leq e^{\|\operatorname{div} b\|_\infty |t-s|} \|\rho(t\|_\infty \quad \forall s, t,$$
it follows from the formula
$$\int_A \rho(s, x) dx = \int_{X(s,A)} \rho(s, x) dx$$
that
$$e^{-\|\operatorname{div} b\|_\infty |t-s|} \mathscr{L}^n(X(s, A)) \leq \mathscr{L}^n(X(t, A)) \leq e^{\|\operatorname{div} b\|_\infty |t-s|} \mathscr{L}^n(X(s, A)).$$
□

11. Exercises

(1) Prove the duality formula (3.2) is true for smooth solution.

(2) Consider the PDE
$$a(x, y)u_x + b(x, y)u + y + c(x, y)u = d(x, y).$$
   (a) Classify the PDE.
   (b) Assume $a \neq 0$ in some open bounded domain Ω, and a, b, c, d smooth: find the characteristic curves and write the formula for the solution.
   (c) Assuming Ω smooth, specify on which part of the boundary the initial data can be assigned.
   (d) Assume that $(a, b) \not 0$ on $\bar{\Omega}$: deduce whether assigning the boundary data on $\partial\Omega$ yields uniqueness.

(3) Solve in $\Omega = \{y > 0\}$
$$xu_x - yu_y = u - y, \quad u(y^2, y) = y.$$

(4) Solve
$$u_x + yu_y - u_z = u - y, \quad u(x, y, 1) = x + y.$$

(5) Prove that $t \mapsto \xi(t) = e(t)_\sharp \eta$ is weakly continuous for all $\eta \in \mathcal{M}(\Gamma)$.

Part 3
Laplace Equation

Follow Ch. 2.2 of [Evans, PDE]

Part 4
Heat Equation

Follow Ch. 2.3 of [Evans, PDE]

Part 5
Wave Equation

Follow Ch. 2.4 of [Evans, PDE]

Part 6
First Order PDEs

Follow Ch. 3.2 of [Evans, PDE]

Part 7
Hamilton-Jacobi Equation

Follow Ch. 3.3 and Ch. 10 of [Evans, PDE]

Part 8

Systems of conservation laws

Let $u : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^m$ be the vector of m conserved quantities, $F : \mathbb{R}^m \to \mathbb{R}^{m\times n}$, $F = (F_1, \dots, F_n)$ with $F_i : \mathbb{R}^m \to \mathbb{R}^m$, the matrix of fluxes: the $m \times m$ system of conservation laws (with initial data u_0) in n dimension is

$$u_t + \operatorname{div} F(u) = u_t + \sum_{i=1}^n \partial_{x_i} F_i(u) = 0, \quad u(t=0) = u_0. \tag{11.1}$$

The solution is in general not regular, hence it must be considered in the sense of distributions.

The PDE is invariant for the rescaling $(t,x) \mapsto \lambda(t,x)$: the initial data and solution invariant for this scaling form the Riemann problem

$$u_0(r\xi) = v(\xi),\ \xi \in \mathbb{S}^{n-1}, \quad u(t,x) = u(x/t).$$

A particular case of invariant solution is the *jump or shock*:

$$u(t,x) = \begin{cases} u^- & \sigma t < n\cdot x, \\ u^+ & \sigma t > n \cdot x, \end{cases} \quad \sigma \in \mathbb{R}, n \in \mathbb{S}^{n-1}.$$

In order to be a weak solution it must satisfy

$$-\sigma(u^+ - u^-) + \sum_i n_i(F_i(u^+) - F_i(u^-)) = 0,$$

by just applying the divergence formula to the distributional formulation of (11.1).

Proposition 11.1. *Assume that $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$ is piecewise Lipschitz, with discontinuities across Lipschitz surfaces. Then u is a weak solution to (11.1) iff*

(1) in the Lipschitz regions

$$u_t + \sum_i DF_i D_x u = 0, \quad \mathscr{L}^{n+1}\text{-a.e.};$$

(2) if $(-\sigma, n)$ is the normal to the discontinuity surfaces S in $\mathscr{H}^n$-a.e. (t,x), normalized so that $|n| = 1$, and u^-, u^+ are the traces when crossing S in the direction $(-\sigma, n)$, then the RH condition holds

$$-\sigma(u^+ - u^-) + \sum_i n_i(F_i(u^+) - F(_i(u^-)) = 0, \quad \mathscr{H}^n\text{-a.e. } (t,x) \in S.$$

Proof. It follows directly from the weak formulation. □

Example 11.2. Consider the simplest nonlinear equation, Burgers equation

$$u_t + (u^2/2)_x = 0, \quad u, x \in \mathbb{R}.$$

If the initial data is

$$u_0(x) = -\tanh(x),$$

then the solution by characteristics (see Part 6) is

$$x(y) = y - t\tanh y, \quad u(t, y - t\tanh y) = \tanh y.$$

Clearly after $t = 1$ the map $y \mapsto x(y)$ is not invertible anymore.

It is thus natural study *discontinuous solutions*, in order to prolong the existence interval.

Example 11.3. Consider again Burgers equation $u_t + (u^2/2)_x = 0$, $x, u \in \mathbb{R}$, with initial data

$$u_0(x) = \operatorname{sign}(x).$$

By Rankine-Hoguniot conditions, one weak solution is

$$u(t,x) = u_0(x),$$

but another solution is obtained by characteristics

$$u(t,x) = \begin{cases} -1 & x \leq -t, \\ x/t & -t < x < t, \\ 1 & x \geq t. \end{cases}$$

Hence weak solutions are not unique.

If there are function $L(u) \in \mathbb{R}, M(u) \in \mathbb{R}^n$ such that

$$DM(u) = DL(u)DF(u), \quad \partial_i M_i(u) = \sum_j \partial_j L(u) \partial_i F_j(u),$$

then for smooth solutions u an additional conservation law holds:

$$\partial_t L + \operatorname{div} M = 0.$$

In general, the condition for existence of the *companion conservation law* above is overdetermined when the dimension of the system is larger than $m = 2$. For physical systems, there is always a convex function $\eta : \mathbb{R}^m \to \mathbb{R}$, the *entropy*, and an entropy flux $q : \mathbb{R}^m \to \mathbb{R}^n$ such that $Dq_i = D\eta DF_i$.

Lemma 11.4. *Assume that the system admits a convex entropy η: then if the weak solution $u \in L^\infty$ is the L^1_{loc}-limit of $u^\epsilon \in L^\infty$ where*

$$u^\epsilon_t + \operatorname{div} F(u^\epsilon) = \epsilon \Delta u^\epsilon, \quad u^\epsilon(t = 0) = u_0,$$

then the dissipation of entropy holds:

$$\eta_t + \operatorname{div} q \leq 0.$$

Proof. Indeed multiplying the PDE of u^ϵ for $\mathcal{D}\eta(u^\epsilon)$ one obtains

$$\eta(u^\epsilon)_t + \operatorname{div}(q(u^\epsilon)) = \epsilon D\eta(u^\epsilon)\Delta u^\epsilon = \epsilon \Delta \eta(u^\epsilon) - \epsilon |D\eta(u^\epsilon) \nabla u^\epsilon|^2 \leq \epsilon \Delta \eta(u^\epsilon).$$

Writing the distributional formulation and letting $\epsilon \to 0$ one obtains the conclusion. □

In particular, a natural request is the following.

Definition 11.5. A weak solution $u \in L^\infty$ is called *entropy solution* if for all convex entropies/entropy fluxes η, q it holds

$$\eta_t + \operatorname{div} q \leq 0$$

in distributions.

It is easy to see that the solution $u(t,x) = u_0$ of Example 11.3 is not entropic.

In general, the entropy condition is not enough to select a unique weak solution.

Being the system invariant for the scaling $(t,x) \mapsto \epsilon(t,x)$, the following definition is natural.

Definition 11.6. The *Riemann problem* is the self similar solution to (11.1) of the form

$$u(t,x) = u(x/t), \quad u_0(x) = u(x/|x|).$$

When $n > 1$, the Riemann problem is as complicated as the the general solution. For $n = 1$ there are instead explicit formulas.

12. Entropy coordinates and symmetric systems

Consider a system of conservation laws with a uniformly convex entropy η: the condition for the entropy means that $\nabla u \nabla F_i = \nabla Q_i$. Differentiating

$$\nabla^2 Q_i = \nabla^2 \eta \nabla F_i + \nabla \eta \nabla^2 F_i.$$

Since the first and last are symmetric (being second derivatives), then

Lemma 12.1. *The matrix*

$$\sum_j \partial^2_{jk} \eta \partial_i F_{ij}$$

is symmetric for every $i = 1, \dots, n$.

Define the new variable

$$v = \nabla\eta(u), \quad u = \nabla\eta^*(v),$$

where η^* is the Legendre transform of η: recall that $\nabla^2\eta^*(v) = (\nabla^2\eta(\nabla\eta^*(v)))^{-1}$. Then

$$\nabla^2\eta(\nabla\eta^*(v))^{-1}\nabla v_t = \nabla^2\eta(\nabla\eta^*(v))^{-1}\nabla^2\eta(u)u_t = u_t = -\sum_i DF_i\nabla^2\eta^*(v)\partial_i v.$$

Defining the symmetric matrices

$$A_0(v) = \nabla^2\eta(\nabla\eta^*(v))^{-1}, \quad A_i(v) = DF_i(\nabla\eta*(v))\nabla^2\eta^*(v),$$

we conclude

Lemma 12.2. *If there is uniformly convex entropy η, the system of conservation laws can be written in the symmetric form*

$$A_0(v)v_t + \sum_i A_i(v)\partial_i v = 0,$$

with A_0, A_i symmetric and $A_0 \in [C^{-1}, C]\mathrm{id}$.

13. Local existence of solutions

13.1. **The linear case.** Assume first that the system is linear:

$$u_t + \sum_i A_i\partial_i u = 0. \tag{13.1}$$

Taking the Fourier transform

$$\hat{u}(t, xi) = \frac{1}{(2\pi)^{n/2}}\int u(t,x)e^{-i\xi\cdot x}dx,$$

one obtains

$$\frac{d}{dt}\hat{u}(t,\xi) = i\bigg(\sum_i \xi_i A_i\bigg)u(t,\xi).$$

The above ODE is homogeneous in ξ: indeed setting

$$\hat{A}(\xi) = \sum_i \xi_i A_i,$$

then

$$\hat{u}(t,\xi) = e^{i|\xi|\hat{A}(\xi/|\xi|)t}u_0(\xi).$$

It is clear then that if $\hat{A}(\xi)$ is not diagonalizable with real eigenvalues, then there are harmonics growing with arbitrarily large time;

Example 13.1. Assume indeed that there is $\bar{\xi} \in \mathbb{S}^{n-1}$ such that $\hat{A}(\bar{\xi})$ has one eigenvalue not real, namely $\lambda = a + ib$ with eigenvector v. Then an initial datum of the form

$$u_0(x) = v\phi(\bar{\xi}\cdot x), \quad \hat{u}_0(\xi) = v\hat{\phi}(\bar{\xi}\cdot\xi),$$

satisfies

$$\hat{u} = e^{i(\xi\cdot\hat{\xi})\hat{A}(\bar{\xi})t}v\hat{\phi} = e^{(ia-b)t\bar{\xi}\cdot\xi}v\hat{\phi}.$$

The solution is of order $e^{bt\bar{\xi}\cdot\xi}$, i.e. as $\bar{\xi}\cdot\xi \to \infty$ there are harmonics with arbitrary large growth for every fixed $t > 0$.

Definition 13.2. The linear system (13.1) is *hyperbolic* if for every $\xi \in \mathbb{S}^{n-1}$ the matrix $\hat{A}(\xi) = \sum_i \xi_i A_i$ is diagonalizable with real eigenvalues. If the matrix $e^{iA(\xi)}$ is bounded on $\mathbb{R}^n$, then it is *uniformly hyperbolic*.

The system of conservation laws (11.1) is called *hyperbolic/uniformly hyperbolic* if for every $\bar{u} \in \mathbb{R}^m$ the linearized system

$$v_t + \sum_i DF_i(\bar{u})\partial_i v = 0$$

is hyperbolic/uniformly hyperbolic.

In the one-dimensional case $n = 1$, the system is *strictly hyperbolic* if the eigenvalues of $DF(\bar{u})$ are separated.

Note that in 1-space dimension, hyperbolic = uniformly hyperbolic.

Lemma 13.3. *If the linear system is symmetric, then the system is uniformly hyperbolic and there is a unique solution for every initial data: in particular, symmetric systems are uniformly hyperbolic.*

Proof. The existence and uniqueness for uniformly hyperbolic systems is a direct consequence of the definition and the Fourier representation of solutions.

The assumptions gives that the system is written as

$$A_0 \frac{d}{dt}\hat{u} = \bigg(\sum_i \xi_i A_i \bigg) \hat{u}.$$

Being A_0 definite positive, by a linear change of coordinate we can assume it to be id, and being $\sum_i \xi_i A_i$ symmetric it follows that $|e^{i \sum_i \xi_i A_i}| = 1$. □

13.2. **The general case.** Consider a system of conservation laws with a uniformly convex entropy η: since also

$$\eta(u) - \eta(0) - D\eta(0) \cdot u$$

is an entropy with flux $q(u) - q(0) - D\eta(0) \cdot F$, we can assume w.l.o.g. that $\eta(u) \sim u^2$, $\|\eta(u)\|_1 \sim \|u\|_2^2$. The entropy dissipation gives immediately that the L^2-norm is bounded:

$$\eta(u(t)) \leq \eta(u(0)).$$

This is not sufficient for global existence, but it can give a bootstrap argument for estimating the growth of the H^s-norms of u.

Theorem 13.4. *If $u_0 \in H^s(\mathbb{R}^n, \mathbb{R}^m)$, $s > \frac{n}{2} + 1$, then there is locally a unique entropy solution.*

Proof. The uniqueness of smooth solutions is in the next proposition. Here we study the existence of regular solutions. The initial estimate is the bound of the L^2-norm.

Differentiating α-times, $|\alpha| = k$,

$$A_0(v)v_t + \sum_i A_i(v)\partial_i v = 0,$$

one obtains

$$A_0(v)(\partial^\alpha v)_t + \sum_i A_i \partial_i (\partial^\alpha v) = S_\alpha(t,x),$$

where the term $S_\alpha(t,x)$ satisfies depends only on the derivative of v up to order k. Multiplying by $\partial^\alpha v$ and integrating

$$\begin{aligned} \frac{d}{dt} \int \frac{\partial^\alpha v A_0 \partial^\alpha v}{2} dx &= \int \partial^\alpha v A_0 (\partial^\alpha v)_t + \int \frac{\partial^\alpha v (DA_0 v_t) \partial^\alpha v}{2} dx \\ &= - \sum_i \int \partial^\alpha v A_i (\partial^\alpha v)_{x_i} + \int \partial^\alpha v S_\alpha(t,x) dx + \int \frac{\partial^\alpha v (DA_0 v_t) \partial^\alpha v}{2} dx \\ &= \int \partial^\alpha v S_\alpha(t,x) dx + \int \frac{\partial^\alpha v (DA_0 v_t + \sum_i DA_i \partial_i v) \partial^\alpha v}{2} dx. \end{aligned}$$

A tedious computation based on Gagliardo-Niremberg inequality gives

$$\bigg| \int \partial^\alpha v S_\alpha(t,x) dx \bigg| \leq \mathcal{O}(\|v\|_{C^1}) \|v\|_{H^k}^2,$$

and similarly

$$\bigg| \int \frac{\partial^\alpha v (DA_0 v_t + \sum_i DA_i \partial_i v) \partial^\alpha v}{2} dx \bigg| \leq C \|Dv\|_\infty \|v\|_{H^k}^2.$$

Hence we get

$$\frac{d}{dt} \|v\|_{H^k}^2 \leq C \|v\|_{C^1} \|v\|_{H^k}^2.$$

If now $s > \frac{n}{2} + 1$, Sobolev embedding gives

$$\frac{d}{dt} \|v\|_{H^k}^2 \leq C \|v\|_{C^1} \|v\|_{H^k}^2 \leq C (\|v\|_{H^k}^2)^2.$$

Gronwall estimates gives that the norm is bounded for a small time interval. □

The next proposition is the *weak-strong uniqueness principle*: if there is a classical/regular solution, then it is unique in the class of weak solutions ("weak" depends on the context).

Proposition 13.5. *If there is a classical solution u, then every weak entropy solution u' with $u'_0 = u_0$ coincides with u.*

Proof. Define the relative entropy

$$\eta(u'|u) = \eta(u') - \eta(u) - D\eta(u) \cdot (u' - u), \quad \eta(u|u') \sim (u' - u)^2.$$

Then

$$\begin{aligned}
\frac{d}{dt} \int \eta(u'|u)dx &= \int \partial_t \eta(u') - \partial_t \eta(u) - (\partial_t D\eta(u)) \cdot (u' - u) - D\eta(u) \cdot (u'_t - u_t)dx \\
&\leq - \int D^2\eta : u_t \times (u' - u)dx + \int D\eta(u) \operatorname{div}(F(u') - F(u))dx \\
&= - \int D^2\eta : u_t \times (u' - u)dx - \int D^2\eta(u) : Du \times (F(u') - F(u))dx \\
&= - \int D^2\eta : Du(F(u') - F(u) - DF(u)(u' - u))dx \leq \mathcal{O}(1)\|u' - u\|_2 = \mathcal{O}(1)\eta(u'|u),
\end{aligned}$$

where the constant in from depends on the norm $\|u\|_{C^1}$ and the functions F, η. We have used the symmetry of $D^2\eta DF$ in the last line.

A Gronwall estimate gives $\eta(u'(t)|u(t)) \leq e^{\mathcal{O}(t)}\eta(u'_0, u_0) = 0$. □

14. The scalar equation

For the scalar conservation law

$$u_t + \operatorname{div} F(u) = 0, \quad F : \mathbb{R} \to \mathbb{R},$$

the theory is complete.

The approach is to prove

(1) existence of solutions by vanishing viscosity,
(2) uniquness of entropy solutions.

Proposition 14.1. *Let $u_0 \in L^\infty$: then the solution to the parabolic PDE*

$$u_t + \operatorname{div} F(u) = \Delta u, \quad u(t = 0) = u_0,$$

satisfies

(1) $\|u(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty}$,
(2) $\|u(t) - u'(t)\|_1 \leq \|u_0 - u'_0\|_1$,
(3) $\text{Tot.Var.}(u(t)) \leq \text{Tot.Var.}(u_0)$,
(4) $\|u(t) - u(s)\|_1 \leq \mathcal{O}(\text{Tot.Var.}(u_0))(\sqrt{|t - s|} + |t - s|)$.

Proof. First of all, by standard regularization of parabolic equations, the solution is smooth until the L^∞-norm is bounded. Netx, the bound in L^∞ holds because of maximum principle for parabolic equations: this gives the first point. Hence the solution exists for all t and it is smooth.

By computing for η convex

$$\partial_t \eta(u - u') + \eta'(u - u') \operatorname{div}(F(u) - F(u')) = \Delta\eta(u - u') - |\eta'|^2|D(u - u')|^2 \leq \Delta\eta(u - u'),$$

which distributionally is

$$\int \eta(u(T, x) - u'(T, x))dx \leq \int_0^T \int \eta''(u - u')(F(u) - F(u'))(t, x)dtdx + \int |u_0 - u'_0|dx.$$

Letting $\eta(\cdot) \to |\cdot|$ in C^0 we conclude that

$$\int |u - u'|(T, x)dx \leq \int |u_0 - u'_0|dx.$$

This proves the second.

Applying the above estimate to $u(t,x+h)-u(t,x)$ we obtain

$$\int |u(T,x+h)-u'(T,x)|dx \leq \int |u_0(x+h)-u_0'(x)|dx \leq |h|\text{Tot.Var.}(u_0),$$

and this implies that $\text{Tot.Var.}(u(T)) \leq \text{Tot.Var.}(u_0)$.

By Duhamel formula, if $G(t,x)=\frac{e^{-|x|^2/t}}{\sqrt{2\pi t}}$ is the heat kernel,

$$u(t)=G(t-s)*u(s)-\int_0^{t-s} G(t-s-\tau)*\operatorname{div} F(u(s+\tau))d\tau.$$

Hence for $0<t-s\leq 1$

$$\begin{aligned}
\|u(t)-u(s)\|_1 &\leq \|(G(t-s)-\mathrm{id})*u(s)\|_1+\int_0^{t-s}\|DF\|_\infty \text{Tot.Var.}(u(s))ds \\
&\leq \int \bigg| \int_\infty^x G(t-s,y)dy-\mathbf{1}_{\mathbb{R}^+}(x)\bigg| dx\text{Tot.Var.}(u(s))+\mathcal{O}(1)\text{Tot.Var.}(u_0)(t-s) \\
&\leq \mathcal{O}(1)\sqrt{t-s}\text{Tot.Var.}(u_0).
\end{aligned}$$

We have used the uniform bound on the $TV(u(t))$. □

Corollary 14.2. *There exists an entropy solution for every initial data $u_0\in L^\infty$.*

Proof. If u^1 solves

$$u_t^1+\operatorname{div} F(u^1)=\Delta u^1,\quad u^1(0)=u_0^1,$$

then $u^\epsilon(t,x)=u^1(\epsilon t,\epsilon x)$ solves

$$u_t^\epsilon+\operatorname{div} F(u^\epsilon)=\epsilon\Delta u^\epsilon,\quad u^\epsilon(0)=u_0^1(\epsilon x).$$

Hence the rescaled estimates of Proposition 14.1 become

$$\text{Tot.Var.}(u^\epsilon(t))\leq \text{Tot.Var.}(u_0^\epsilon),\quad \|u^\epsilon(t)-(u')^\epsilon(t)\|_1\leq \|u^\epsilon(0)-(u')^\epsilon(0)\|_1,$$

$$\|u^\epsilon(t)-u^\epsilon(s)\|_1\leq \mathcal{O}(\text{Tot.Var.}(u_0^\epsilon))(\sqrt{\epsilon|t-s|}+|t-s|).$$

Hence, $u_0,t\mapsto u^\epsilon(t)$ is family of uniformly continuous ($\sqrt{\epsilon\cdot}+\cdot$-Hölder in time and 1-Lipschitz in u_0) functions from $\{\text{Tot.Var.}(u_0)\leq C\}\cap L^\infty \times[0,T]$ with values in the compact subset $\text{Tot.Var.}(u)\cap L^\infty$ of L^1_{loc}: by Ascoli-Arzela' up to subsequences one obtains a limit function $u_0,t\mapsto u(t)$, now Lipschitz in times and 1-Lipschitz in u_0.

Being 1-Lipschitz in u_0, it can be extended uniformly to the whole $L^1\cap L^\infty$. Being the limit in L^1, the function u is certainly entropic. □

It remains to prove the uniqueness, since a-priori there can be several different Lipschitz semigroup $u_0,t\mapsto u(t)$.

Theorem 14.3. *There exists a unique entropic solution.*

Proof. Consider two entropy solutions u,u': by doubling the variables we have that

$$\eta(t,x,s,y)=|u(t,x)-u'(s,y)|$$

is a convex entropy for u and u':

$$\partial_t|u(t,x)-u'(s,y)|+\operatorname{div}_x(\operatorname{sign}(u-u')(F(u)-F(u'))\leq 0,\quad \partial_s|u(t,x)-u'(s,y)|+\operatorname{div}_y(\operatorname{sign}(u-u')(F(u)-F(u'))\leq 0.$$

Testing with

$$\phi\bigg(\frac{t+s}{2},\frac{x+y}{2}\bigg)\psi\bigg(\frac{t-s}{2},\frac{x-y}{2}\bigg)>0,$$

the weak formulation is

$$\frac{1}{2}\int\int|u-u'|(\phi_\tau\psi+\phi\psi_\tau)+\operatorname{sign}(u-u')(F(u)-F(u')(D_z\phi\psi+\phi D_z\psi)dxdt+\int|u_0-u'(s,y)|\phi\psi dx\geq 0,$$

$$\frac{1}{2}\int\int|u-u'|(\phi_\tau\psi-\phi\psi_\tau)+\operatorname{sign}(u-u')(F(u)-F(u')(D_z\phi\psi-\phi D_z\psi)dyds+\int|u(t,x)-u_0'|\phi\psi dy\geq 0.$$

Integrating and adding both one recovers

$$
\begin{aligned}
0 \leq & \int\int |u-u'|\phi_\sigma\psi + \mathrm{sign}(u-u')(F(u)-F(u')D_w\phi\psi dxdtdsdy \\
& + \int\int |u_0 - u'(s,y)|\phi\psi dxdsdy + \int |u(t,x)-u_0'|\phi\psi dtdxdy \\
= & \, 2^{2n+2}\int \psi(\tau,z)\int |u(\sigma+\tau,w+z)-u'(\sigma-\tau,w-z)|\phi_\sigma(\sigma,w) + \mathrm{sign}(u-u')(F(u)-F(u')D_w\phi d\sigma dw d\tau dz \\
& + 2^{2n+1}\int |u_0(w+z)-u'(-\tau,w-z)|\phi\psi dxdy + 2^{2n+1}\int |u(\tau,w+z)-u_0'(w-z)|\phi\psi dxdy.
\end{aligned}
$$

Taking $\psi = \psi(\cdot/\epsilon)/\epsilon^{n+1}$ we obtain

$$
\begin{aligned}
& \int \psi(\tau,z)\int |u(\sigma+\epsilon\tau,w+\epsilon z)-u'(\sigma-\epsilon\tau,w-\epsilon z)|\phi_\sigma(\sigma,w) + \mathrm{sign}(u-u')(F(u)-F(u')D_w\phi d\sigma dw d\tau dz \\
& \quad + \frac{1}{2}\int |u_0(w+\epsilon z)-u'(-\epsilon\tau,w-\epsilon z)|\phi\psi dxdy + \frac{1}{2}\int |u(\epsilon\tau,w+\epsilon z)-u_0'(w-z)|\phi\psi dxdy \geq 0.
\end{aligned}
$$

The continuity in L^1 of the map $t, u_0 \mapsto u(t)$ implies that

$$u(t+\epsilon\tau, x+\epsilon z) \to u(t,x) \quad \text{in } L^1_{\mathrm{loc}},$$

and then letting $\epsilon \to 0$ we obtain

$$\int |u(\sigma,w)-u'(\sigma,w)|\phi_\sigma(\sigma,w) + \mathrm{sign}(u-u')(F(u)-F(u')D_w\phi d\sigma dw + \int |u_0(w)-u_0'(w)|\phi(0,w)dxdy \geq 0.$$

This is the weak formulation of the PDE

$$\partial_t|u-u'| + \mathrm{div}(\mathrm{sign}(u-u')(F(u)-F(u')) \leq 0, \quad |u-u'|(t=0) = |u_0-u_0'|.$$

Consider now the test function

$$\varphi(y) = \begin{cases} 1 & |y| \leq 1, \\ \text{smooth} & 1 < |y| < 2, \\ 0 & |y| \geq 2. \end{cases}$$

We have by the weak formulation

$$
\begin{aligned}
& \int |u(T,x)-u'(T,x)|\psi(x/R)dx \\
& = \int_0^T \int \varphi'((|x|+C(T-t))/R)\big(\mathrm{sign}(u-u')(F(u)-F(u'))\cdot(-x/(R|x|)) - C|u-u'|\big)dxdt.
\end{aligned}
$$

Hence is $C \geq \mathrm{Lip}\, F$ then

$$\int |u(T,x)-u'(T,x)|\psi(x/R)dx = 0.$$

Letting $R \to \infty$ we conclude. □

15. The scalar 1-dimensional case

In the 1-d case scalar, there are explicit formulas for the Riemann problem, as well as constructive methods for solutions.

Lemma 15.1. *Every convex function is an entropy for*

$$u_t + f(u)_x = 0, \quad u, f(u), x \in \mathbb{R}.$$

Proof. Just observe that

$$q(u) = \int^u \eta'(v)f'(v)dv.$$

□

Since up to a linear part one can write

$$\eta(u) = \int \eta''(v)[u-v]^+ dv,$$

it is enough to study the Kruzkhov entropies $\eta_k = [u-k]^+$, with flux $q_k = (f(u)-f(k))\mathbf{1}_{u\geq k}$.

The Rankine-Hugoniot conditions become

$$-\sigma(u^+ - u^-) + f(u^+) - f(u^-) = 0, \quad u^\pm = \lim_{x\to\bar{x}\pm} u(x),$$

i.e.

$$\sigma = \frac{f(u^+)-f(u^-)}{u^+-u^-} = \fint_{u^-}^{u^+} f'(v)dv.$$

This means that the speed of the jump is the average speed between u^-, u^+.

Proposition 15.2. *The jump $[u^-, u^+]$ is entropic iff the segment connecting $(u^-, f(u^-)), (u^+, f(u^+))$ is*

(1) below $f\llcorner[u^-, u^+]$ if $u^- < u^+$,
(2) above $f\llcorner[u^+, u^-]$ if $u^- > u^+$.

Proof. We prove just the first: the second follows by changing $f \mapsto -f(-u)$.

For every k, compute the dissipation for $\eta_k(u) = [u-k]^+$: the only cases are when $u^- \leq k \leq u^+$,

$$\partial_t \eta_k + \partial_x q_k = -\big[\sigma(u^+-k) + f(u^+) - f(k)\big](1+\sigma^2)^{-1/2}\mathcal{H}^1_{x=\sigma t} \leq 0.$$

This implies that

$$f(k) \geq f(u^+) + \sigma(k-u^+),$$

which is the desired condition. □

Consider now the Riemann problem

$$u_t + f(u)_x = 0, \quad u_0 = \begin{cases} u^- & x<0, \\ u^+ & x>0. \end{cases}$$

Assume first that $u^- < u^+$, consider the function

$$g(u) = (\mathrm{conv}_{[u^-,u^+]} f)(u),$$

and define

$$u(\xi) = \begin{cases} u^- & \xi < g'(u^-), \\ (g')^{-1}(\xi) & g'(u^-) \leq \xi \leq g'(u^+), \\ u^+ & \xi > g'(u^+). \end{cases} \tag{15.1}$$

For the case $u^- > u^+$, the symmetric formula is

$$g(u) = (\mathrm{conc}_{[u^+,u^-]} f)(u), \quad u(\xi) = \begin{cases} u^- & \xi < g'(u^-), \\ (g')^{-1}(\xi) & g'(u^-) \leq \xi \leq g'(u^+), \\ u^+ & \xi > g'(u^+). \end{cases}$$

Note that in the first case the solution is increasing, while in the second decreasing.

Proposition 15.3. *The function $u(t,x) = u(\xi = x/t)$ of (15.1) is the unique entropic solution to the Riemann problem $[u^-, u^+]$.*

Proof. The function u is piecewise Lipschitz with jumps across the (possibly countably many) discontinuities where g^{-1} has a jump: each of these jumps $[u_k^-, u_k^+]$, $k = 1, 2, \dots$, has speed

$$\sigma_k = g'(v) = \frac{f(u^+)-f(u^-)}{u^+-u^-}, \quad v \in [u_k^-, u_k^+],$$

and, being g the convex envelope of f, it is below f, i.e. it is entropic.

For the parts where $(g')^{-1}$ is differentiable, we have

$$u_t = -((g')^{-1})'(\xi)\frac{\xi}{t}, \quad f(u)_x = f'((g')^{-1})((g')^{-1})'(\xi)\frac{1}{t} = -\xi((g')^{-1})'(\xi)\frac{1}{t},$$

where we have used that in the points where $(g')^{-1}$ is not discontinuous then

$$g' = f' \quad \text{and} \quad g'((g')^{-1}(\xi)) = \xi.$$

□

15.1. **Wavefront tracking algorithm.** Assume first that $f(u)$ is piecewise affine, with corners at the points $2^{-\ell}\mathbb{Z}$:

$$\frac{df}{du} = a_k \quad u \in [k, k+1]2^{-\ell}.$$

Then the solution formula (15.1) gives finitely many discontinuities s_j, traveling with speed σ_j.

The scheme thus works as follows.

(1) Consider an initial data made of a piecewise constant function u_0 taking values in $2^{-\ell}\mathbb{Z}$: let x_i be the positions of the discontinuities, and let

$$\text{Tot.Var.}(u_0) = \sum_i |u_0(x_i+) - u_0(x_i-)|.$$

(2) Solve every Riemann problem in x_i as in Proposition 15.3, and let the finitely many discontinuities travel with their speed until they meet with discontinuities coming from other Riemann problems: we can assume by moving slightly the initial points that these collisions are only binary (even if the scheme works in any situation). The number of discontinuities is at most

$$2^\ell \text{Tot.Var.}(u_0).$$

(3) When two discontinuities $[u^-, u^m]$, $[u^m, u^+]$ collide at $t > 0$, then two cases may happen:
   (a) either the jumps are both decreasing or increasing, in which case the solution of the new Riemann problem $[u^-, u^+]$ is a single discontinuity: this is fairly easy to verify from the requirement that both jumps are entropic. Hence the number of discontinuities decreases by 1;
   (b) the jumps has opposite sign: then, assuming for simplicity $u^- < u^+ < u^m$, the total variation decreases of

$$\text{Tot.Var.}(u(t+)) - \text{Tot.Var.}(u^-) = u^+ - u^- - \big[(u^m - u^-) + (u^m - u^+)\big] = 2(u^m - u^+),$$

   i.e. it decreases of at least $2^{-\ell}$.

(4) Then the number of collisions where the number of jumps increases is finite

$$\leq 2^\ell \text{Tot.Var.}(u_0),$$

and for the other collisions the number of waves decreases.

It is easy to see that from the last point the number of collisions is finite, in particular the scheme can proceed for all $t > 0$. We obtain from the fact that every discontinuity is entropic

Proposition 15.4. *The wavefront solution constructed above is the unique entropy solution for the piecewise affine flux.*

If $f^\ell \to f$ in $W^{1,\infty}$, and u^ℓ is the solution with the same initial data with bounded total variation, it is possible to show that the solutions are compact and converge to the entropy solution (which is unique).

Part 9

Incompressible Euler

The PDE is

$$u_t + (u\cdot\nabla)u + \nabla P = 0, \quad \operatorname{div} u = 0. \tag{15.2}$$

It describes the motion of a fluid which is incompressible ($\operatorname{div} u = 0$), u begin the velocity of the fluid.

16. Least action principle

We consider for simplicity the space $(t,x) \in \mathbb{R}^+ \times \mathbb{T}^n$ and the manifold

$$\mathcal{M} = \Big\{ T : \mathbb{T} \to \mathbb{T},\ \text{measure preserving, smooth} \Big\}.$$

If $[0,1] \ni t \mapsto T_t \in \mathcal{M}$ is a curve in $\mathcal{M}$, then its length is

$$L(\{T_t\}) = \bigg(\int_0^1 \bigg\| \frac{dT_t}{dt} \bigg\|_2^2 dt \bigg)^{1/2}.$$

The distance among maps is computed by the minimal length of curves connecting two maps, i.e. the geodesic distance.

Define the velocity field

$$v(t, T_(x)) = \frac{dT_t}{dt}(t,x),$$

and compute for a test function ϕ

$$\begin{aligned} 0 &= \frac{d}{dt}\int \phi(T_t(x))dx = \int \nabla\phi(T_t(x)) v(t, T_t(x))dx \\ &= \int \nabla\phi(x) v(t,x)dx = -\int \phi \operatorname{div} v dx. \end{aligned}$$

where in the first equality we used the measure preserving property of T_t. Hence v is measure preserving.

Compute now the minimal action between two configurations $\mathrm{id}, \bar{T}$:

$$\min_{\{T_t\}: T_0 = \mathrm{id}, T_1 = \bar{T}} \int\int \bigg| \frac{dT}{dt}(t,x) \bigg|^2 dx dt.$$

Consider the perturbations

$$T_t^\epsilon = S_t^\epsilon \circ T_t, \quad S^\epsilon = \mathrm{id} + \epsilon b(t,x) + \mathcal{O}(\epsilon^2) \text{ measure preserving, which implies } \operatorname{div} b_t = 0,$$

and compute

$$\begin{aligned} 0 &= \frac{d}{d\epsilon}\int_0^1 \int \bigg| \frac{dT^\epsilon}{dt}(t,x) \bigg|^2 dx dt \\ &= \frac{d}{d\epsilon}\int_0^1 \int \bigg| \frac{dT}{dt}(t,x) + \epsilon \frac{db}{dt}(t,x) + \mathcal{O}(\epsilon^2) \bigg|^2 dx dt \\ &= 2\int_0^1 \int \frac{dT_t}{dt} \cdot \frac{db}{dt} dx dt + \mathcal{O}(\epsilon) \\ &= -2\int_0^1 \int \frac{d^2 T_t}{d^2 t} \cdot b dx dt + \mathcal{O}(\epsilon). \end{aligned}$$

Hence we deduce that from $b(t,x)$ being arbitrary (with $\operatorname{div} b = 0$)

$$\frac{d^2 T_t}{d^2 t} = \frac{d}{dt} v(t, T_t(x)) = \partial_t v(T_t(x)) + (v(t, T_t(x)) \cdot \nabla) v(t, T_t(x)) \perp b \quad \forall b (\operatorname{div} b = 0).$$

This means that $d^2 T_t / d^2 t$ is a gradient,

$$v_t + (v\cdot\nabla)v + \nabla P = 0.$$

For nonsmooth maps T_t, it is possible to develop a similar theory in the framework of optimal transport.

17. Vorticity equation and 2-d Euler

Here we assume $n = 2$, similar computations can be done for $n > 2$, but the analysis is more complicated and the equations contains additional terms.

Define the vorticity for $u \in \mathbb{R}^2$, as

$$\omega = \mathrm{curl} u = \partial_1 u_2 - \partial_2 u_1, \quad u = (u_1, u_2).$$

Applying curl· to Incompressible Euler we obtain

$$\omega_t + u \cdot \nabla \omega = 0. \tag{17.1}$$

If we assume that $u \in L^2$, the vector field u cab ne recovered from ω by the formula

$$u = \nabla^\perp \Delta^{-1} \omega = \frac{1}{2\pi} \int \frac{(x-y)^\perp}{|x-y|^2} \omega(y) dy. \tag{17.2}$$

Remark 17.1. It is possible to prove that if $\omega \in L^p$, then $v \in W^{1,p}$, so that one can solve (17.1) and prove existence of a solution in L^p. However, the uniqueness is proved only for $\omega \in L^\infty$.

17.1. **Well posedness for $\omega \in L^\infty$.** We prove that in this case the vector field u is $L \log L$-Lipschitz.

Proposition 17.2. *If ω in bounded with compact support, then the vector field u given by (17.2) satisfies*

$$\|u\|_\infty \leq C, \quad |u(x') - u(x)| \leq C|x' - x| \bigg(1 + \log \bigg(1 + \frac{1}{|x' - x|} \bigg) \bigg),$$

for some constant depending on $\|\omega\|_\infty$, $\mathrm{diam\,supp}\,\omega$.

Proof. First,

$$|u(x)| \leq \|\omega\|_\infty \int_{x - \mathrm{supp}\,\omega} \frac{1}{2\pi |z|} dz \leq \|\omega\|_\infty \, \mathrm{diam\,supp}\,\omega$$

gives the first estimate.

Write

$$K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2},$$

and compute

$$\begin{aligned} |u(x+h) - u(x)| &= \bigg| \int \big(K(x+h-y) - K(x-y) \big) \omega(y) dy \bigg| \\ &\leq \|\omega\|_\infty \int_{|y-x| \leq 2h} \big| K(x+h-y) - K(x-y) \big| dy \\ &\quad + \|\omega\|_\infty \int_{|y-x| > 2h \cap \mathrm{supp}\,\omega} \big| K(x+h-y) - K(x-y) \big| dy \\ &\leq 2\|\omega\|_\infty \int_{|z| \leq 2h} \frac{1}{2\pi |z|} dz + \|\omega\|_\infty \int_{|y-x| > 2h \cap \mathrm{supp}\,\omega} \big| \nabla K(z(y)) h \big| dy \\ &\leq 2\|\omega\|_\infty h + C\|\omega\|_\infty h \int_h^{h + \mathrm{diam\,supp}\,\omega} \frac{1}{|z|^2} |z| d|z| \\ &\leq C\|\Omega\|_\infty h \bigg(1 + \log \bigg(1 + \frac{\mathrm{diam\,supp}\,\omega}{h} \bigg) \bigg). \end{aligned}$$

□

Corollary 17.3. *If $\omega(t)$ is assigned, then the vector field $u(t)$ generates a unique flow.*

Note that this flow is defined pointwise, not a.e. as the Regular Lagrangian Flow.

Proof. Indeed, it is well known that if

$$\frac{dx}{dt} = b(t,x), \quad |b(t,x+h) - b(t,x)| \leq \zeta(|h|), \int_0^\epsilon \frac{dz}{\omega(z)} = \infty \text{ for all } \epsilon > 0,$$

then the ODE has uniqueness. Existence follows from Peano's Theorem. □

However, since u depends on ω, one cannot deduce uniqueness: indeed it is open if there are multiple solutions for $\omega \in L^p$, $p < \infty$ (numerically it seems so).

For the case $\omega \in L^\infty$, instead, the proof of well posedness follows the following points.

17.1.1. *Existence of a solution.*

(1) Define iteratively
$$\partial \omega^n + u^{n-1} \cdot \nabla \omega^n, \quad u^{n-1} = K * \omega^{n-1}.$$
(2) The definition of u^n gives by Proposition 17.2 that
$$\|u^{n-1}(t)\|_\infty \leq C \operatorname{diam} \operatorname{supp} \omega^{n-1}(t),$$
so that it holds
$$\operatorname{diam} \operatorname{supp} \omega^n \leq \operatorname{diam} \operatorname{supp} \omega_0 + C \int_0^t \operatorname{diam} \operatorname{supp} \omega^{n-1}(s) ds.$$
Hence uniformly in n
$$\operatorname{diam} \operatorname{supp} \omega_n(t) \leq \operatorname{diam} \operatorname{supp} \omega_0 e^{Ct}.$$
In particular we have a uniform bound on the support of ω^n.

(3) The first PDE is well defined, indeed if X^{n-1} is the (unique) measure preserving flow generated by u^{n-1} then
$$\omega_n(t, X^{n-1}(t,y)) = \omega_0(y).$$
In particular the bound $\|\omega_0\|_\infty$ is preserved.

(4) From the $L \log L$-Lipschitz estimate of u^{n-1},
$$\frac{d}{dt}|X(t,y_1) - X(t,y_2)| \leq C|X(t,y_1) - X(t,y_2)| \bigg(1 + \log\bigg(1 + \frac{1}{|X(t,y_1) - X(t,y_2)|}\bigg)\bigg),$$
which implies
$$-\ln\bigg(1 + \ln\bigg(1 + \frac{1}{|X(t,y_1) - X(t,y_2)|}\bigg)\bigg) + \ln\bigg(1 + \ln\bigg(1 + \frac{1}{|y_1 - y_2|}\bigg)\bigg) \leq Ct.$$
Hence the flows are uniformly compact in C^0.

(5) As $n \to \infty$, up to subsequences
$$\omega^n(t, x = \omega_0(X^{-1}(t,x)) \to_{L^\infty_t L^1_x} \omega_0(X^{-1}(t,x)) = \omega(t,x),$$
and then
$$u^n = K * \omega^n \to_{C^0_{t,x}} u = K * \omega.$$
(6) We can thus pass to the limit of the weak formulation obtain that ω, u is a solution of the vorticity equation.

We summarize.

Proposition 17.4. *There is a weak L^∞-solution to the vorticity equation for $\omega_0 \in L^\infty$.*

17.1.2. *Uniqueness.* We consider two solutions ω_1, ω_2 and define $w = u_1 - u_2$: from the Euler equation
$$w_t + (u_1 \cdot \nabla) w + (w \cdot \nabla) u_2 + \nabla(P_1 - P_2) = 0.$$
Computing the energy and using that $\operatorname{div} w = 0$ we get
$$\begin{aligned}\frac{d}{dt} \int \frac{|w|^2}{2} dx &= -\int (u_1 \cdot \nabla) \frac{w^2}{2} dx - \sum_{ij} \int w_i w_j \partial_j u_{2,i} dx \\ &= \sum_{ij} \int w_i w_j \partial_j u_{2,i} dx \leq \|\nabla u_2\|_p \|w\|^2_{2p/(p-1)} \\ &\leq \|\nabla u_2\|_p \|w\|_\infty^{2/p} \bigg(\int \frac{|w|^2}{2} dx\bigg)^{1-1/p}.\end{aligned}$$
We next use the following Calderon-Zygmund estimate:
$$\|\nabla u\|_p \leq Cp \|\omega\|_\infty.$$
This estimate is based on the theory of singular integrals.

Using this estimate one obtains

$$\frac{d}{dt}\int \frac{|w|^2}{2}dx \leq Cp\bigg(\int \frac{|w|^2}{2}dx\bigg)^{1-1/p},$$

which gives

$$\int \frac{|w|^2}{2}dx \leq (Ct)^p.$$

If $t < 1/C$, letting $p \to \infty$ one obtains uniqueness in the interval of time $[0, 1/2C]$.

Repeating the argument for all intervals $[k, k+1]/2C$ we conclude

Proposition 17.5. *There is a unique solution u with vorticity in L^∞ and bounded support.*

Remark 17.6. It is possible to weaken the assumptions on the bounded support of ω, e.g. $\omega \in L^\infty \cap L^1$.

18. Conservation of energy and Onsager conjecture

For $d > 2$ the solution develops singularities and in general it is not unique. Even more striking, the energy $\|u(t)\|_2^2/2$ of the solution is not preserved.

The following conjecture is due to Onsager.

Conjecture 18.1. *If $u(t) \in L^1_t C^\alpha_x \cap L^3_{t,x} \cap C([0,T], L^2_x)$, $\alpha > 1/3$, then the energy is preserved, while if $\alpha < 1/3$ there are solutions dissipating energy.*

We show here the conservation of energy if $u \in L^\infty_t C^\alpha_x$, $\alpha > 3$.

Starting with

$$u_t + \mathrm{div}(u \times u + P\mathrm{id}) = 0, \quad \mathrm{div}\, u = 0,$$

we take the convolution with a smoothing kernel $\phi^\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$ to obtain

$$u^\epsilon + \mathrm{div}(u^\epsilon \times u^\epsilon + P^\epsilon \mathrm{id}) = \mathrm{div}(u^\epsilon \times u^\epsilon - (u \times u)^\epsilon).$$

Multiplying for u^ϵ and integrating

$$\begin{aligned}\frac{d}{dt}\int \frac{|u|^2}{2}dx &= -\int u^\epsilon \,\mathrm{div}(u^\epsilon \times u^\epsilon + P^\epsilon \mathrm{id}) + \int u^\epsilon \,\mathrm{div}(u^\epsilon \times u^\epsilon - (u \times u)^\epsilon) \\ &= -\int \big(u^\epsilon \times u^\epsilon - (u \times u)^\epsilon\big) : \nabla u^\epsilon dx.\end{aligned}$$

Observing that

$$\begin{aligned}\int u(x-y) \times u(x-y)\phi^\epsilon(y)dy = u^\epsilon \times u^\epsilon - (u - u^\epsilon) \times (u - u^\epsilon) \\ + \int \phi^\epsilon(y)(u(x-y) - u(x)) \times (u(x-y) - u(x))dy,\end{aligned}$$

the last term can be written as

$$\begin{aligned}\frac{d}{dt}\int \frac{|u|^2}{2}dx &= -\int \big(u^\epsilon \times u^\epsilon - (u \times u)^\epsilon\big) : \nabla u^\epsilon dx \\ &= \int \bigg((u - u^\epsilon) \times (u - u^\epsilon) - \int \phi^\epsilon(y)(u(x-y) - u(x)) \times (u(x-y) - u(x))dy\bigg) : \nabla u^\epsilon(x)dx.\end{aligned}$$

Since $u \in C^\alpha$, we have

$$\|u(\cdot + y) - u(\cdot)\|_3 \leq C|y|^\alpha, \quad \|u - u^\epsilon\|_3 \leq C\epsilon^\alpha, \quad \|\nabla u^\epsilon\|_3 \leq C\epsilon^{\alpha-1}.$$

Hence

$$\bigg|\int (u - u^\epsilon) \times (u - u^\epsilon) : \nabla u^\epsilon(x)dx\bigg| \leq \|u - u^\epsilon\|_3^2 \|\nabla u^\epsilon\|_3 \leq C\epsilon^{3\alpha-1},$$

$$\int \phi^\epsilon(y)(u(x-y) - u(x)) \times (u(x-y) - u(x)) : \nabla u^\epsilon(x)dydx \leq \|u(\cdot + y) - u(\cdot)\|_3^2 \|\nabla u^\epsilon\|_3 \leq C\epsilon^{3\alpha-1}.$$

Letting $\epsilon \to 0$ we recover that

$$\frac{d}{dt}\int \frac{|u|^2}{2}dx = 0.$$

Proposition 18.2. *If* $u \in L_t^\infty(C_x^\alpha \cap L_x^3) \cap L_{t,x}^2$, *then the energy is conserved.*

Remark 18.3. The sharpest space for energy conservation is a Besov space with $1/3$-fractional derivative.

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