## 2.2. LAPLACE'S EQUATION

Among the most important of all partial differential equations are undoubtedly *Laplace's equation*

(1) $$\Delta u = 0$$

and *Poisson's equation*

(2) $$-\Delta u = f.^{*}$$

In both (1) and (2), $x \in U$ and the unknown is $u : \bar{U} \to \mathbb{R}$, $u = u(x)$, where $U \subset \mathbb{R}^n$ is a given open set. In (2) the function $f : U \to \mathbb{R}$ is also given. Remember from §A.3 that the *Laplacian* of $u$ is $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$.

**DEFINITION.** *A $C^2$ function $u$ satisfying* (1) *is called a* harmonic *function.*

**Physical interpretation.** Laplace's equation comes up in a wide variety of physical contexts. In a typical interpretation $u$ denotes the density of some quantity (e.g. a chemical concentration) in equilibrium. Then if $V$ is any smooth subregion within $U$, the net flux of $u$ through $\partial V$ is zero:

$$\int_{\partial V} \mathbf{F} \cdot \boldsymbol{\nu} \, dS = 0,$$

$\mathbf{F}$ denoting the flux density and $\boldsymbol{\nu}$ the unit outer normal field. In view of the Gauss–Green Theorem (§C.2), we have

$$\int_V \operatorname{div} \mathbf{F} \, dx = \int_{\partial V} \mathbf{F} \cdot \boldsymbol{\nu} \, dS = 0,$$

and so

(3) $$\operatorname{div} \mathbf{F} = 0 \quad \text{in } U,$$

since $V$ was arbitrary. In many instances it is physically reasonable to assume the flux $\mathbf{F}$ is proportional to the gradient $Du$ but points in the opposite direction (since the flow is from regions of higher to lower concentration). Thus

(4) $$\mathbf{F} = -aDu \quad (a > 0).$$

*I prefer to write (2) with the minus sign, to be consistent with the notation for general second-order elliptic operators in Chapter 6.

Substituting into (3), we obtain Laplace's equation

$$\operatorname{div}(Du) = \Delta u = 0.$$

If $u$ denotes the

$$\begin{cases} \text{chemical concentration} \\ \text{temperature} \\ \text{electrostatic potential,} \end{cases}$$

equation (4) is

$$\begin{cases} \text{Fick's law of diffusion} \\ \text{Fourier's law of heat conduction} \\ \text{Ohm's law of electrical conduction.} \end{cases}$$

See Feynman–Leighton–Sands [**F-L-S,** Chapter 12] for a discussion of the ubiquity of Laplace's equation in mathematical physics. Laplace's equation arises as well in the study of analytic functions and the probabilistic investigation of Brownian motion.

### 2.2.1. Fundamental solution.

**a. Derivation of fundamental solution.** One good strategy for investigating any partial differential equation is first to identify some explicit solutions and then, provided the PDE is linear, to assemble more complicated solutions out of the specific ones previously noted. Furthermore, in looking for explicit solutions, it is often wise to restrict attention to classes of functions with certain symmetry properties. Since Laplace's equation is invariant under rotations (Problem 2), it consequently seems advisable to search first for *radial* solutions, that is, functions of $r = |x|$.

Let us therefore attempt to find a solution $u$ of Laplace's equation (1) in $U = \mathbb{R}^n$, having the form

$$u(x) = v(r),$$

where $r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and $v$ is to be selected (if possible) so that $\Delta u = 0$ holds. First note for $i = 1, \ldots, n$ that

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}\left(x_1^2 + \cdots + x_n^2\right)^{-1/2} 2x_i = \frac{x_i}{r} \quad (x \neq 0).$$

We thus have

$$u_{x_i} = v'(r)\frac{x_i}{r}, \; u_{x_i x_i} = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

for $i = 1, \ldots, n$, and so

$$\Delta u = v''(r) + \frac{n-1}{r} v'(r).$$

Hence $\Delta u = 0$ if and only if

$$v'' + \frac{n-1}{r} v' = 0. \tag{5}$$

If $v' \neq 0$, we deduce

$$\log(|v'|)' = \frac{v''}{v'} = \frac{1-n}{r},$$

and hence $v'(r) = \frac{a}{r^{n-1}}$ for some constant $a$. Consequently if $r > 0$, we have

$$v(r) = \begin{cases} b \log r + c & (n = 2) \\ \frac{b}{r^{n-2}} + c & (n \geq 3), \end{cases}$$

where $b$ and $c$ are constants.

These considerations motivate the following

**DEFINITION.** *The function*

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \log |x| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & (n \geq 3), \end{cases} \tag{6}$$

*defined for $x \in \mathbb{R}^n$, $x \neq 0$, is the* fundamental solution of Laplace's equation.

The reason for the particular choices of the constants in (6) will be apparent in a moment. (Recall from §A.2 that $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$.)

We will sometimes slightly abuse notation and write $\Phi(x) = \Phi(|x|)$ to emphasize that the fundamental solution is radial. Observe also that we have the estimates

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, \quad |D^2\Phi(x)| \leq \frac{C}{|x|^n} \quad (x \neq 0) \tag{7}$$

for some constant $C > 0$.

**b. Poisson's equation.** By construction the function $x \mapsto \Phi(x)$ is harmonic for $x \neq 0$. If we shift the origin to a new point $y$, the PDE (1) is unchanged; and so $x \mapsto \Phi(x - y)$ is also harmonic as a function of $x$, $x \neq y$. Let us now take $f : \mathbb{R}^n \to \mathbb{R}$ and note that the mapping $x \mapsto \Phi(x - y)f(y)$ $(x \neq y)$ is harmonic for each point $y \in \mathbb{R}^n$, and thus so is the sum of finitely many such expressions built for different points $y$.

This reasoning might suggest that the convolution

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

(8)
$$= \begin{cases} -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log(|x-y|) f(y)\, dy & (n=2) \\ \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}}\, dy & (n \geq 3) \end{cases}$$

will solve Laplace's equation (1). *However, this is wrong.* Indeed, as intimated by estimate (7), $D^2\Phi(x-y)$ is *not* summable near the singularity at $y = x$, and so naive differentiation through the integral sign is unjustified (and incorrect). We must proceed more carefully in calculating $\Delta u$.

Let us for simplicity now assume $f \in C_c^2(\mathbb{R}^n)$; that is, $f$ is twice continuously differentiable, with compact support.

**THEOREM 1** (Solving Poisson's equation). *Define* $u$ *by* (8). *Then*

(i) $u \in C^2(\mathbb{R}^n)$

*and*

(ii) $-\Delta u = f$ *in* $\mathbb{R}^n$.

We consequently see that (8) provides us with a formula for a solution of Poisson's equation (2) in $\mathbb{R}^n$.

**Proof.** 1. We have

(9)
$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\, dy;$$

hence

$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \left[\frac{f(x+he_i-y) - f(x-y)}{h}\right] dy,$$

where $h \neq 0$ and $e_i = (0, \ldots, 1, \ldots, 0)$, the 1 in the $i^{\text{th}}$-slot. But

$$\frac{f(x+he_i-y) - f(x-y)}{h} \to f_{x_i}(x-y)$$

uniformly on $\mathbb{R}^n$ as $h \to 0$, and thus

$$u_{x_i}(x) = \int_{\mathbb{R}^n} \Phi(y) f_{x_i}(x-y)\, dy \quad (i = 1, \ldots, n).$$

Similarly

(10)
$$u_{x_i x_j}(x) = \int_{\mathbb{R}^n} \Phi(y) f_{x_i x_j}(x-y)\, dy \quad (i, j = 1, \ldots, n).$$

As the expression on the right-hand side of (10) is continuous in the variable $x$, we see $u \in C^2(\mathbb{R}^n)$.

2. Since $\Phi$ blows up at 0, we will need for subsequent calculations to isolate this singularity inside a small ball. So fix $\varepsilon > 0$. Then

$$\begin{aligned} \Delta u(x) &= \int_{B(0,\varepsilon)} \Phi(y)\Delta_x f(x-y)\,dy + \int_{\mathbb{R}^n - B(0,\varepsilon)} \Phi(y)\Delta_x f(x-y)\,dy \\ &=: I_\varepsilon + J_\varepsilon. \end{aligned} \tag{11}$$

Now

$$|I_\varepsilon| \le C\|D^2 f\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\varepsilon)} |\Phi(y)|\,dy \le \begin{cases} C\varepsilon^2|\log\varepsilon| & (n=2) \\ C\varepsilon^2 & (n\ge 3). \end{cases} \tag{12}$$

An integration by parts (see §C.2) yields

$$\begin{aligned} J_\varepsilon &= \int_{\mathbb{R}^n - B(0,\varepsilon)} \Phi(y)\Delta_y f(x-y)\,dy \\ &= -\int_{\mathbb{R}^n - B(0,\varepsilon)} D\Phi(y)\cdot D_y f(x-y)\,dy \\ &\quad + \int_{\partial B(0,\varepsilon)} \Phi(y)\frac{\partial f}{\partial \nu}(x-y)\,dS(y) \\ &=: K_\varepsilon + L_\varepsilon, \end{aligned} \tag{13}$$

$\boldsymbol{\nu}$ denoting the *inward* pointing unit normal along $\partial B(0,\varepsilon)$. We readily check

$$|L_\varepsilon| \le \|Df\|_{L^\infty(\mathbb{R}^n)} \int_{\partial B(0,\varepsilon)} |\Phi(y)|\,dS(y) \le \begin{cases} C\varepsilon|\log\varepsilon| & (n=2) \\ C\varepsilon & (n\ge 3). \end{cases} \tag{14}$$

3. We continue by integrating by parts once again in the term $K_\varepsilon$, to discover

$$\begin{aligned} K_\varepsilon &= \int_{\mathbb{R}^n - B(0,\varepsilon)} \Delta\Phi(y) f(x-y)\,dy - \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y)\,dS(y) \\ &= -\int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y)\,dS(y), \end{aligned}$$

since $\Phi$ is harmonic away from the origin. Now $D\Phi(y) = \frac{-1}{n\alpha(n)}\frac{y}{|y|^n}$ $(y \ne 0)$ and $\boldsymbol{\nu} = \frac{-y}{|y|} = -\frac{y}{\varepsilon}$ on $\partial B(0,\varepsilon)$. Consequently $\frac{\partial\Phi}{\partial\nu}(y) = \boldsymbol{\nu}\cdot D\Phi(y) = \frac{1}{n\alpha(n)\varepsilon^{n-1}}$ on $\partial B(0,\varepsilon)$. Since $n\alpha(n)\varepsilon^{n-1}$ is the surface area of the sphere $\partial B(0,\varepsilon)$, we have

$$\begin{aligned} K_\varepsilon &= -\frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} f(x-y)\,dS(y) \\ &= -⨍_{\partial B(x,\varepsilon)} f(y)\,dS(y) \to -f(x) \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{15}$$

(Remember from §A.3 that a slash through an integral denotes an average.)

4. Combining now (11)–(15) and letting $\varepsilon \to 0$, we find $-\Delta u(x) = f(x)$, as asserted. □

Theorem 1 is in fact valid under far less stringent smoothness requirements for $f$: see Gilbarg–Trudinger [**G-T**].

**Interpretation of fundamental solution.** We sometimes write

$$-\Delta\Phi = \delta_0 \quad \text{in } \mathbb{R}^n,$$

$\delta_0$ denoting the Dirac measure on $\mathbb{R}^n$ giving unit mass to the point 0. Adopting this notation, we may formally compute

$$\begin{aligned} -\Delta u(x) &= \int_{\mathbb{R}^n} -\Delta_x \Phi(x-y) f(y)\, dy \\ &= \int_{\mathbb{R}^n} \delta_x f(y)\, dy = f(x) \quad (x \in \mathbb{R}^n), \end{aligned}$$

in accordance with Theorem 1. This corrects the faulty calculation (9).

### 2.2.2. Mean-value formulas.

Consider now an open set $U \subset \mathbb{R}^n$ and suppose $u$ is a harmonic function within $U$. We next derive the important *mean-value formulas*, which declare that $u(x)$ equals both the average of $u$ over the sphere $\partial B(x,r)$ and the average of $u$ over the entire ball $B(x,r)$, provided $B(x,r) \subset U$. These implicit formulas involving $u$ generate a remarkable number of consequences, as we will momentarily see.

**THEOREM 2** (Mean-value formulas for Laplace's equation). *If $u \in C^2(U)$ is harmonic, then*

$$u(x) = ⨍_{\partial B(x,r)} u\, dS = ⨍_{B(x,r)} u\, dy \tag{16}$$

*for each ball $B(x,r) \subset U$.*

**Proof.** 1. Set

$$\phi(r) := ⨍_{\partial B(x,r)} u(y)\, dS(y) = ⨍_{\partial B(0,1)} u(x+rz)\, dS(z).$$

Then

$$\phi'(r) = ⨍_{\partial B(0,1)} Du(x+rz)\cdot z\, dS(z),$$

and consequently, using Green's formulas from §C.2, we compute

$$\begin{aligned}\phi'(r) &= ⨍_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r}\, dS(y)\\ &= ⨍_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\, dS(y)\\ &= \frac{r}{n} ⨍_{B(x,r)} \Delta u(y)\, dy = 0.\end{aligned}$$

Hence $\phi$ is constant, and so

$$\phi(r) = \lim_{t\to 0} \phi(t) = \lim_{t\to 0} ⨍_{\partial B(x,t)} u(y)\, dS(y) = u(x).$$

2. Observe next that our employing polar coordinates, as in §C.3, gives

$$\begin{aligned}\int_{B(x,r)} u\, dy &= \int_0^r \left( \int_{\partial B(x,s)} u\, dS \right) ds\\ &= u(x) \int_0^r n\alpha(n) s^{n-1} ds = \alpha(n) r^n u(x).\end{aligned}$$

□

**THEOREM 3** (Converse to mean-value property). *If $u \in C^2(U)$ satisfies*

$$u(x) = ⨍_{\partial B(x,r)} u\, dS$$

*for each ball $B(x,r) \subset U$, then $u$ is harmonic.*

**Proof.** If $\Delta u \not\equiv 0$, there exists some ball $B(x,r) \subset U$ such that, say, $\Delta u > 0$ within $B(x,r)$. But then for $\phi$ as above,

$$0 = \phi'(r) = \frac{r}{n} ⨍_{B(x,r)} \Delta u(y)\, dy > 0,$$

a contradiction. □

### 2.2.3. Properties of harmonic functions.

We now present a sequence of interesting deductions about harmonic functions, all based upon the mean-value formulas. Assume for the following that $U \subset \mathbb{R}^n$ is open and bounded.