and consequently, using Green's formulas from §C.2, we compute

$$\begin{aligned}\phi'(r) &= \fint_{\partial B(x,r)} Du(y)\cdot\frac{y-x}{r}\,dS(y)\\ &= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\,dS(y)\\ &= \frac{r}{n}\fint_{B(x,r)} \Delta u(y)\,dy = 0.\end{aligned}$$

Hence ϕ is constant, and so

$$\phi(r) = \lim_{t\to 0}\phi(t) = \lim_{t\to 0}\fint_{\partial B(x,t)} u(y)\,dS(y) = u(x).$$

2. Observe next that our employing polar coordinates, as in §C.3, gives

$$\begin{aligned}\int_{B(x,r)} u\,dy &= \int_0^r \left(\int_{\partial B(x,s)} u\,dS\right) ds\\ &= u(x)\int_0^r n\alpha(n)s^{n-1}ds = \alpha(n)r^n u(x).\end{aligned}$$

□

THEOREM 3 (Converse to mean-value property). *If $u \in C^2(U)$ satisfies*

$$u(x) = \fint_{\partial B(x,r)} u\,dS$$

for each ball $B(x,r) \subset U$, then u is harmonic.

Proof. If $\Delta u \not\equiv 0$, there exists some ball $B(x,r) \subset U$ such that, say, $\Delta u > 0$ within $B(x,r)$. But then for ϕ as above,

$$0 = \phi'(r) = \frac{r}{n}\fint_{B(x,r)} \Delta u(y)\,dy > 0,$$

a contradiction. □

2.2.3. Properties of harmonic functions.

We now present a sequence of interesting deductions about harmonic functions, all based upon the mean-value formulas. Assume for the following that $U \subset \mathbb{R}^n$ is open and bounded.

a. Strong maximum principle, uniqueness. We begin with the assertion that a harmonic function must attain its maximum on the boundary and cannot attain its maximum in the interior of a connected region unless it is constant.

THEOREM 4 (Strong maximum principle). *Suppose $u \in C^2(U) \cap C(\bar{U})$ is harmonic within U.*

(i) *Then*

$$\max_{\bar{U}} u = \max_{\partial U} u.$$

(ii) *Furthermore, if U is connected and there exists a point $x_0 \in U$ such that*

$$u(x_0) = \max_{\bar{U}} u,$$

then

$$u \text{ is constant within } U.$$

Assertion (i) is the *maximum principle* for Laplace's equation and (ii) is the *strong maximum principle.* Replacing u by $-u$, we recover also similar assertions with "min" replacing "max".

Proof. Suppose there exists a point $x_0 \in U$ with $u(x_0) = M := \max_{\bar{U}} u$. Then for $0 < r < \operatorname{dist}(x_0, \partial U)$, the mean-value property asserts

$$M = u(x_0) = \fint_{B(x_0,r)} u\,dy \leq M.$$

As equality holds only if $u \equiv M$ within $B(x_0, r)$, we see $u(y) = M$ for all $y \in B(x_0, r)$. Hence the set $\{x \in U \mid u(x) = M\}$ is both open and relatively closed in U and thus equals U if U is connected. This proves assertion (ii), from which (i) follows. □

Positivity. The strong maximum principle asserts in particular that if U is connected and $u \in C^2(U) \cap C(\bar{U})$ satisfies

$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U, \end{cases}$$

where $g \geq 0$, then *u is positive everywhere in U if g is positive somewhere on ∂U.*

An important application of the maximum principle is establishing the uniqueness of solutions to certain boundary-value problems for Poisson's equation.

THEOREM 5 (Uniqueness). *Let $g \in C(\partial U)$, $f \in C(U)$. Then there exists at most one solution $u \in C^2(U) \cap C(\bar{U})$ of the boundary-value problem*

$$
\begin{cases} -\Delta u = f & \text{in } U \\ \quad u = g & \text{on } \partial U. \end{cases} \tag{17}
$$

Proof. If u and $\tilde{u}$ both satisfy (17), apply Theorem 4 to the harmonic functions $w := \pm(u - \tilde{u})$. □

b. Regularity. Next we prove that if $u \in C^2$ is harmonic, then necessarily $u \in C^\infty$. Thus *harmonic functions are automatically infinitely differentiable.* This sort of assertion is called a *regularity* theorem. The interesting point is that the algebraic structure of Laplace's equation $\Delta u = \sum_{i=1}^n u_{x_i x_i} = 0$ leads to the analytic deduction that all the partial derivatives of u exist, even those which do not appear in the PDE.

THEOREM 6 (Smoothness). *If $u \in C(U)$ satisfies the mean-value property* (16) *for each ball $B(x,r) \subset U$, then*

$$u \in C^\infty(U).$$

Note carefully that u may not be smooth, or even continuous, up to ∂U.

Proof. Let η be a standard mollifier, as described in §C.4, and recall that η is a radial function. Set $u^\varepsilon := \eta_\varepsilon * u$ in $U_\varepsilon = \{x \in U \mid \operatorname{dist}(x, \partial U) > \varepsilon\}$. As shown in §C.4, $u^\varepsilon \in C^\infty(U_\varepsilon)$.

We will prove u is smooth by demonstrating that in fact $u \equiv u^\varepsilon$ on U_ε. Indeed if $x \in U_\varepsilon$, then

$$
\begin{aligned}
u^\varepsilon(x) &= \int_U \eta_\varepsilon(x-y) u(y)\, dy \\
&= \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \eta\left(\frac{|x-y|}{\varepsilon}\right) u(y)\, dy \\
&= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) \left(\int_{\partial B(x,r)} u\, dS \right) dr \\
&= \frac{1}{\varepsilon^n} u(x) \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) n\alpha(n) r^{n-1} dr \qquad \text{by (16)} \\
&= u(x) \int_{B(0,\varepsilon)} \eta_\varepsilon\, dy = u(x).
\end{aligned}
$$

Thus $u^\varepsilon \equiv u$ in U_ε, and so $u \in C^\infty(U_\varepsilon)$ for each $\varepsilon > 0$. □

c. Local estimates for harmonic functions. Now we employ the mean-value formulas to derive careful estimates on the various partial derivatives of a harmonic function. The precise structure of these estimates will be needed below, when we prove analyticity.

THEOREM 7 (Estimates on derivatives). *Assume u is harmonic in U. Then*

$$|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))} \tag{18}$$

for each ball $B(x_0, r) \subset U$ and each multiindex α of order $|\alpha| = k$.

Here

$$C_0 = \frac{1}{\alpha(n)}, \quad C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)} \quad (k = 1, \ldots). \tag{19}$$

Proof. 1. We establish (18), (19) by induction on k, the case $k = 0$ being immediate from the mean-value formula (16). For $k = 1$, we note upon differentiating Laplace's equation that u_{x_i} $(i = 1, \ldots, n)$ is harmonic. Consequently

$$\begin{aligned} |u_{x_i}(x_0)| &= \left| ⨍_{B(x_0,r/2)} u_{x_i}\, dx \right| \\ &= \left| \frac{2^n}{\alpha(n) r^n} \int_{\partial B(x_0,r/2)} u \nu_i \, dS \right| \\ &\leq \frac{2n}{r} \|u\|_{L^\infty(\partial B(x_0, \frac{r}{2}))}. \end{aligned} \tag{20}$$

Now if $x \in \partial B(x_0, r/2)$, then $B(x, r/2) \subset B(x_0, r) \subset U$, and so

$$|u(x)| \leq \frac{1}{\alpha(n)} \left(\frac{2}{r}\right)^n \|u\|_{L^1(B(x_0,r))}$$

by (18), (19) for $k = 0$. Combining the inequalities above, we deduce

$$|D^\alpha u(x_0)| \leq \frac{2^{n+1}n}{\alpha(n)} \frac{1}{r^{n+1}} \|u\|_{L^1(B(x_0,r))}$$

if $|\alpha| = 1$. This verifies (18), (19) for $k = 1$.

2. Assume now $k \geq 2$ and (18), (19) are valid for all balls in U and each multiindex of order less than or equal to $k - 1$. Fix $B(x_0, r) \subset U$ and let α

be a multiindex with $|\alpha| = k$. Then $D^\alpha u = (D^\beta u)_{x_i}$ for some $i \in \{1, \dots, n\}$, $|\beta| = k - 1$. By calculations similar to those in (20), we establish that

$$|D^\alpha u(x_0)| \leq \frac{nk}{r} \|D^\beta u\|_{L^\infty(\partial B(x_0, \frac{r}{k}))}.$$

If $x \in \partial B(x_0, \frac{r}{k})$, then $B(x, \frac{k-1}{k} r) \subset B(x_0, r) \subset U$. Thus (18), (19) for $k - 1$ imply

$$|D^\beta u(x)| \leq \frac{(2^{n+1} n(k-1))^{k-1}}{\alpha(n) \left(\frac{k-1}{k} r\right)^{n+k-1}} \|u\|_{L^1(B(x_0, r))}.$$

Combining the two previous estimates yields the bound

$$|D^\alpha u(x_0)| \leq \frac{(2^{n+1} nk)^k}{\alpha(n) r^{n+k}} \|u\|_{L^1(B(x_0, r))}. \tag{21}$$

This confirms (18), (19) for $|\alpha| = k$. □

d. Liouville's Theorem. We assert now that there are no nontrivial bounded harmonic functions on all of $\mathbb{R}^n$.

THEOREM 8 (Liouville's Theorem). *Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded. Then u is constant.*

Proof. Fix $x_0 \in \mathbb{R}^n$, $r > 0$, and apply Theorem 7 on $B(x_0, r)$:

$$\begin{aligned} |Du(x_0)| &\leq \frac{\sqrt{n} C_1}{r^{n+1}} \|u\|_{L^1(B(x_0, r))} \\ &\leq \frac{\sqrt{n} C_1 \alpha(n)}{r} \|u\|_{L^\infty(\mathbb{R}^n)} \to 0, \end{aligned}$$

as $r \to \infty$. Thus $Du \equiv 0$, and so u is constant. □

THEOREM 9 (Representation formula). *Let $f \in C_c^2(\mathbb{R}^n)$, $n \geq 3$. Then any bounded solution of*

$$-\Delta u = f \quad \text{in } \mathbb{R}^n$$

has the form

$$u(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y) \, dy + C \quad (x \in \mathbb{R}^n)$$

for some constant C.

Proof. Since $\Phi(x) \to 0$ as $|x| \to \infty$ for $n \geq 3$, $\tilde{u}(x) := \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy$ is a bounded solution of $-\Delta u = f$ in $\mathbb{R}^n$. If u is another solution, $w := u - \tilde{u}$ is constant, according to Liouville's Theorem. □

Remark. If $n = 2$, $\Phi(x) = -\frac{1}{2\pi}\log|x|$ is unbounded as $|x| \to \infty$ and so may be $\int_{\mathbb{R}^2} \Phi(x-y)f(y)\,dy$.

e. Analyticity. We refine Theorem 6:

THEOREM 10 (Analyticity). *Assume u is harmonic in U. Then u is analytic in U.*

Proof. 1. Fix any point $x_0 \in U$. We must show u can be represented by a convergent power series in some neighborhood of x_0.

Let $r := \frac{1}{4}\operatorname{dist}(x_0, \partial U)$. Then $M := \frac{1}{\alpha(n)r^n}\|u\|_{L^1(B(x_0,2r))} < \infty$.

2. Since $B(x,r) \subset B(x_0,2r) \subset U$ for each $x \in B(x_0,r)$, Theorem 7 provides the bound

$$\|D^\alpha u\|_{L^\infty(B(x_0,r))} \leq M\left(\frac{2^{n+1}n}{r}\right)^{|\alpha|}|\alpha|^{|\alpha|}.$$

Now $\frac{k^k}{k!} < e^k$ for all positive integers k, and hence

$$|\alpha|^{|\alpha|} \leq e^{|\alpha|}|\alpha|!$$

for all multiindices α. Furthermore, the Multinomial Theorem (§1.5) implies

$$n^k = (1+\cdots+1)^k = \sum_{|\alpha|=k}\frac{|\alpha|!}{\alpha!},$$

whence

$$|\alpha|! \leq n^{|\alpha|}\alpha!.$$

Combining the previous inequalities yields the estimate

$$\|D^\alpha u\|_{L^\infty(B(x_0,r))} \leq CM\left(\frac{2^{n+1}n^2e}{r}\right)^{|\alpha|}\alpha!. \tag{22}$$

3. The Taylor series for u at x_0 is

$$\sum_\alpha \frac{D^\alpha u(x_0)}{\alpha!}(x-x_0)^\alpha,$$