the sum taken over all multiindices. We assert this power series converges, provided

$$|x - x_0| < \frac{r}{2^{n+2}n^3 e}. \tag{23}$$

To verify this, let us compute for each N the remainder term:

$$\begin{aligned} R_N(x) &:= u(x) - \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)(x-x_0)^\alpha}{\alpha!} \\ &= \sum_{|\alpha|=N} \frac{D^\alpha u(x_0 + t(x-x_0))(x-x_0)^\alpha}{\alpha!} \end{aligned}$$

for some $0 \leq t \leq 1$, t depending on x. We establish this formula by writing out the first N terms and the error in the Taylor expansion about 0 for the function of one variable $g(t) := u(x_0 + t(x - x_0))$, at $t = 1$. Employing (22), (23), we can estimate

$$\begin{aligned} |R_N(x)| &\leq CM \sum_{|\alpha|=N} \left(\frac{2^{n+1}n^2 e}{r}\right)^N \left(\frac{r}{2^{n+2}n^3 e}\right)^N \\ &\leq CMn^N \frac{1}{(2n)^N} = \frac{CM}{2^N} \to 0 \quad \text{as } N \to \infty. \end{aligned}$$

□

See §4.6.2 for more on analytic functions and partial differential equations.

f. Harnack's inequality. Recall from §A.2 that we write $V \subset\subset U$ to mean $V \subset \bar{V} \subset U$ and $\bar{V}$ is compact.

THEOREM 11 (Harnack's inequality). *For each connected open set $V \subset\subset U$, there exists a positive constant C, depending only on V, such that*

$$\sup_V u \leq C \inf_V u$$

for all nonnegative harmonic functions u in U.

Thus in particular

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y)$$

for all points $x, y \in V$. These inequalities assert that *the values of a nonnegative harmonic function within V are all comparable*: u cannot be very small (or very large) at any point of V unless u is very small (or very large) everywhere in V. The intuitive idea is that since V is a positive distance away from ∂U, there is "room for the averaging effects of Laplace's equation to occur".

Proof. Let $r := \frac{1}{4}\operatorname{dist}(V, \partial U)$. Choose $x, y \in V$, $|x - y| \leq r$. Then

$$\begin{aligned} u(x) &= \fint_{B(x,2r)} u\,dz \geq \frac{1}{\alpha(n)2^n r^n} \int_{B(y,r)} u\,dz \\ &= \frac{1}{2^n} \fint_{B(y,r)} u\,dz = \frac{1}{2^n} u(y). \end{aligned}$$

Thus $2^n u(y) \geq u(x) \geq \frac{1}{2^n} u(y)$ if $x, y \in V$, $|x - y| \leq r$.

Since V is connected and $\bar{V}$ is compact, we can cover $\bar{V}$ by a chain of finitely many balls $\{B_i\}_{i=1}^N$, each of which has radius $\frac{r}{2}$ and $B_i \cap B_{i-1} \neq \emptyset$ for $i = 2, \ldots, N$. Then

$$u(x) \geq \frac{1}{2^{n(N+1)}} u(y)$$

for all $x, y \in V$. □

2.2.4. Green's function.

Assume now $U \subset \mathbb{R}^n$ is open, bounded, and ∂U is C^1. We propose next to obtain a general representation formula for the solution of Poisson's equation

$$-\Delta u = f \quad \text{in } U,$$

subject to the prescribed boundary condition

$$u = g \quad \text{on } \partial U.$$

a. Derivation of Green's function. Suppose $u \in C^2(\bar{U})$ is an arbitrary function. Fix $x \in U$, choose $\varepsilon > 0$ so small that $B(x, \varepsilon) \subset U$, and apply Green's formula from §C.2 on the region $V_\varepsilon := U - B(x, \varepsilon)$ to $u(y)$ and $\Phi(y - x)$. We thereby compute

$$\begin{aligned} &\int_{V_\varepsilon} u(y)\Delta\Phi(y - x) - \Phi(y - x)\Delta u(y)\,dy \\ &= \int_{\partial V_\varepsilon} u(y)\frac{\partial \Phi}{\partial \nu}(y - x) - \Phi(y - x)\frac{\partial u}{\partial \nu}(y)\,dS(y), \end{aligned} \tag{24}$$

$\boldsymbol{\nu}$ denoting the outer unit normal vector on ∂V_ε. Recall next $\Delta\Phi(x - y) = 0$ for $x \neq y$. We observe also

$$\left|\int_{\partial B(x,\varepsilon)} \Phi(y - x)\frac{\partial u}{\partial \nu}(y)\,dS(y)\right| \leq C\varepsilon^{n-1} \max_{\partial B(0,\varepsilon)} |\Phi| = o(1)$$

as $\varepsilon \to 0$. Furthermore the calculations in the proof of Theorem 1 show

$$\int_{\partial B(x,\varepsilon)} u(y)\frac{\partial \Phi}{\partial \nu}(y - x)\,dS(y) = \fint_{\partial B(x,\varepsilon)} u(y)\,dS(y) \to u(x)$$

as $\varepsilon \to 0$. Hence our sending $\varepsilon \to 0$ in (24) yields the formula

$$
\begin{aligned}
u(x) = &\int_{\partial U} \Phi(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y) \\
&- \int_U \Phi(y-x)\Delta u(y)\,dy.
\end{aligned} \tag{25}
$$

This identity is valid for any point $x \in U$ and any function $u \in C^2(\bar{U})$.

Now formula (25) would permit us to solve for $u(x)$ *if* we knew the values of Δu within U and the values of $u, \partial u/\partial \nu$ along ∂U. However, for our application to Poisson's equation with prescribed boundary values for u, the normal derivative $\partial u/\partial \nu$ along ∂U is unknown to us. We must therefore somehow modify (25) to remove this term.

The idea is now to introduce for fixed x a *corrector* function $\phi^x = \phi^x(y)$, solving the boundary-value problem

$$
\begin{cases} \Delta \phi^x = 0 & \text{in } U \\ \phi^x = \Phi(y-x) & \text{on } \partial U. \end{cases} \tag{26}
$$

Let us apply Green's formula once more, to compute

$$
\begin{aligned}
-\int_U \phi^x(y)\Delta u(y)\,dy &= \int_{\partial U} u(y)\frac{\partial \phi^x}{\partial \nu}(y) - \phi^x(y)\frac{\partial u}{\partial \nu}(y)\,dS(y) \\
&= \int_{\partial U} u(y)\frac{\partial \phi^x}{\partial \nu}(y) - \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y).
\end{aligned} \tag{27}
$$

We introduce next this

DEFINITION. Green's function *for the region U is*

$$
G(x,y) := \Phi(y-x) - \phi^x(y) \quad (x, y \in U,\ x \neq y).
$$

Adopting this terminology and adding (27) to (25), we find

$$
u(x) = -\int_{\partial U} u(y)\frac{\partial G}{\partial \nu}(x,y)\,dS(y) - \int_U G(x,y)\Delta u(y)\,dy \quad (x \in U), \tag{28}
$$

where

$$
\frac{\partial G}{\partial \nu}(x,y) = D_y G(x,y) \cdot \nu(y)
$$

is the outer normal derivative of G with respect to the variable y. Observe that the term $\partial u/\partial \nu$ does not appear in equation (28): we introduced the corrector ϕ^x precisely to achieve this.

Suppose now $u \in C^2(\bar{U})$ solves the boundary-value problem

$$
\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U, \end{cases} \tag{29}
$$

for given continuous functions f, g. Plugging into (28), we obtain

THEOREM 12 (Representation formula using Green's function). *If $u \in C^2(\bar{U})$ solves problem* (29), *then*

$$u(x) = -\int_{\partial U} g(y)\frac{\partial G}{\partial \nu}(x,y)\,dS(y) + \int_U f(y)G(x,y)\,dy \quad (x \in U). \tag{30}$$

Here we have a formula for the solution of the boundary-value problem (29), provided we can construct Green's function G for the given domain U. This is in general a difficult matter and can be done only when U has simple geometry. Subsequent subsections identify some special cases for which an explicit calculation of G is possible.

Interpreting Green's function. Fix $x \in U$. Then regarding G as a function of y, we may symbolically write

$$\begin{cases} -\Delta G = \delta_x & \text{in } U \\ \quad G = 0 & \text{on } \partial U, \end{cases}$$

δ_x denoting the Dirac measure giving unit mass to the point x.

Before moving on to specific examples, let us record the general assertion that G is symmetric in the variables x and y:

THEOREM 13 (Symmetry of Green's function). *For all $x, y \in U$, $x \neq y$, we have*

$$G(y,x) = G(x,y).$$

Proof. Fix $x, y \in U$, $x \neq y$. Write

$$v(z) := G(x,z),\ w(z) := G(y,z) \quad (z \in U).$$

Then $\Delta v(z) = 0$ $(z \neq x)$, $\Delta w(z) = 0$ $(z \neq y)$ and $w = v = 0$ on ∂U. Thus our applying Green's identity on $V := U - [B(x,\varepsilon) \cup B(y,\varepsilon)]$ for sufficiently small $\varepsilon > 0$ yields

$$\int_{\partial B(x,\varepsilon)} \frac{\partial v}{\partial \nu}w - \frac{\partial w}{\partial \nu}v\,dS(z) = \int_{\partial B(y,\varepsilon)} \frac{\partial w}{\partial \nu}v - \frac{\partial v}{\partial \nu}w\,dS(z), \tag{31}$$

$\boldsymbol{\nu}$ denoting the inward pointing unit vector field on $\partial B(x,\varepsilon) \cup \partial B(y,\varepsilon)$. Now w is smooth near x, whence

$$\left|\int_{\partial B(x,\varepsilon)} \frac{\partial w}{\partial \nu}v\,dS\right| \leq C\varepsilon^{n-1} \sup_{\partial B(x,\varepsilon)} |v| = o(1) \quad \text{as } \varepsilon \to 0.$$

On the other hand, $v(z) = \Phi(z-x) - \phi^x(z)$, where ϕ^x is smooth in U. Thus

$$\lim_{\varepsilon\to 0}\int_{\partial B(x,\varepsilon)} \frac{\partial v}{\partial \nu} w\, dS = \lim_{\varepsilon\to 0}\int_{\partial B(x,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(x-z)w(z)\, dS = w(x),$$

by calculations as in the proof of Theorem 1. Thus the left-hand side of (31) converges to $w(x)$ as $\varepsilon \to 0$. Likewise the right-hand side converges to $v(y)$. Consequently

$$G(y,x) = w(x) = v(y) = G(x,y). \qquad \square$$

b. Green's function for a half-space. In this and the next subsection we will build Green's functions for two regions with simple geometry, namely the half-space $\mathbb{R}^n_+$ and the unit ball $B(0,1)$. Everything depends upon our explicitly solving the corrector problem (26) in these regions, and this in turn depends upon some clever geometric reflection tricks.

First let us consider the half-space

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}.$$

Although this region is unbounded, and so the calculations in the previous section do not directly apply, we will attempt nevertheless to build Green's function using the ideas developed earlier. Later of course we must check directly that the corresponding representation formula is valid.

DEFINITION. *If $x = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n_+$, its* reflection *in the plane $\partial\mathbb{R}^n_+$ is the point*

$$\tilde{x} = (x_1, \dots, x_{n-1}, -x_n).$$

We will solve problem (26) for the half-space by setting

$$\phi^x(y) := \Phi(y - \tilde{x}) = \Phi(y_1 - x_1, \dots, y_{n-1} - x_{n-1}, y_n + x_n) \quad (x, y \in \mathbb{R}^n_+).$$

The idea is that the corrector ϕ^x is built from Φ by "reflecting the singularity" from $x \in \mathbb{R}^n_+$ to $\tilde{x} \notin \mathbb{R}^n_+$. We note

$$\phi^x(y) = \Phi(y - x) \quad \text{if } y \in \partial\mathbb{R}^n_+,$$

and thus

$$\begin{cases} \Delta\phi^x = 0 & \text{in } \mathbb{R}^n_+ \\ \phi^x = \Phi(y-x) & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

as required.

DEFINITION. Green's function for the half-space $\mathbb{R}^n_+$ *is*

$$G(x,y) := \Phi(y-x) - \Phi(y-\tilde{x}) \quad (x,y \in \mathbb{R}^n_+,\ x \neq y).$$

Then

$$\begin{aligned} G_{y_n}(x,y) &= \Phi_{y_n}(y-x) - \Phi_{y_n}(y-\tilde{x}) \\ &= \frac{-1}{n\alpha(n)}\left[\frac{y_n - x_n}{|y-x|^n} - \frac{y_n + x_n}{|y-\tilde{x}|^n}\right]. \end{aligned}$$

Consequently if $y \in \partial\mathbb{R}^n_+$,

$$\frac{\partial G}{\partial \nu}(x,y) = -G_{y_n}(x,y) = -\frac{2x_n}{n\alpha(n)}\frac{1}{|x-y|^n}.$$

Suppose now u solves the boundary-value problem

$$\text{(32)} \qquad \begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

Then from (30) we expect

$$\text{(33)} \qquad u(x) = \frac{2x_n}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\,dy \quad (x \in \mathbb{R}^n_+)$$

to be a representation formula for our solution. The function

$$K(x,y) := \frac{2x_n}{n\alpha(n)}\frac{1}{|x-y|^n} \quad (x \in \mathbb{R}^n_+, y \in \partial\mathbb{R}^n_+)$$

is *Poisson's kernel* for $\mathbb{R}^n_+$, and (33) is *Poisson's formula.*

We must now check directly that formula (33) does indeed provide us with a solution of the boundary-value problem (32).

THEOREM 14 (Poisson's formula for half-space). *Assume* $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$, *and define* u *by* (33). *Then*

(i) $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$,

(ii) $\Delta u = 0$ *in* $\mathbb{R}^n_+$,

and

(iii) $\displaystyle\lim_{\substack{x\to x^0 \\ x\in\mathbb{R}^n_+}} u(x) = g(x^0)$ *for each point* $x^0 \in \partial\mathbb{R}^n_+$.