Proof. 1. For each fixed x, the mapping $y \mapsto G(x,y)$ is harmonic, except for $y = x$. As $G(x,y) = G(y,x)$, $x \mapsto G(x,y)$ is harmonic, except for $x = y$. Thus $x \mapsto -\frac{\partial G}{\partial y_n}(x,y) = K(x,y)$ is harmonic for $x \in \mathbb{R}^n_+$, $y \in \partial\mathbb{R}^n_+$.

2. A direct calculation, the details of which we omit, verifies

$$1 = \int_{\partial\mathbb{R}^n_+} K(x,y)\,dy \tag{34}$$

for each $x \in \mathbb{R}^n_+$. As g is bounded, u defined by (33) is likewise bounded. Since $x \mapsto K(x,y)$ is smooth for $x \neq y$, we easily verify as well $u \in C^\infty(\mathbb{R}^n_+)$, with

$$\Delta u(x) = \int_{\partial\mathbb{R}^n_+} \Delta_x K(x,y) g(y)\,dy = 0 \quad (x \in \mathbb{R}^n_+).$$

3. Now fix $x^0 \in \partial\mathbb{R}^n_+$, $\varepsilon > 0$. Choose $\delta > 0$ so small that

$$|g(y) - g(x^0)| < \varepsilon \quad \text{if } |y - x^0| < \delta,\ y \in \partial\mathbb{R}^n_+. \tag{35}$$

Then if $|x - x^0| < \frac{\delta}{2}$, $x \in \mathbb{R}^n_+$,

$$\begin{aligned} |u(x) - g(x^0)| &= \left| \int_{\partial\mathbb{R}^n_+} K(x,y)[g(y) - g(x^0)]\,dy \right| \\ &\leq \int_{\partial\mathbb{R}^n_+ \cap B(x^0,\delta)} K(x,y)|g(y) - g(x^0)|\,dy \\ &\quad + \int_{\partial\mathbb{R}^n_+ - B(x^0,\delta)} K(x,y)|g(y) - g(x^0)|\,dy \\ &=: I + J. \end{aligned} \tag{36}$$

Now (34), (35) imply

$$I \leq \varepsilon \int_{\partial\mathbb{R}^n_+} K(x,y)\,dy = \varepsilon.$$

Furthermore if $|x - x^0| \leq \frac{\delta}{2}$ and $|y - x^0| \geq \delta$, we have

$$|y - x^0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x^0|;$$

and so $|y - x| \geq \frac{1}{2}|y - x^0|$. Thus

$$\begin{aligned} J &\leq 2\|g\|_{L^\infty} \int_{\partial\mathbb{R}^n_+ - B(x^0,\delta)} K(x,y)\,dy \\ &\leq \frac{2^{n+2}\|g\|_{L^\infty} x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ - B(x^0,\delta)} |y - x^0|^{-n}\,dy \\ &\to 0 \quad \text{as } x_n \to 0^+. \end{aligned}$$

Combining this calculation with estimate (36), we deduce $|u(x)-g(x^0)| \leq 2\varepsilon$, provided $|x - x^0|$ is sufficiently small. $\square$

c. Green's function for a ball. To construct Green's function for the unit ball $B(0,1)$, we will again employ a kind of reflection, this time through the sphere $\partial B(0,1)$.

DEFINITION. *If $x \in \mathbb{R}^n - \{0\}$, the point*

$$\tilde{x} = \frac{x}{|x|^2}$$

is called the point dual *to x with respect to $\partial B(0,1)$. The mapping $x \mapsto \tilde{x}$ is* inversion *through the unit sphere $\partial B(0,1)$.*

We now employ inversion through the sphere to compute Green's function for the unit ball $U = B^0(0,1)$. Fix $x \in B^0(0,1)$. Remember that we must find a corrector function $\phi^x = \phi^x(y)$ solving

$$\begin{cases} \Delta\phi^x = 0 & \text{in } B^0(0,1) \\ \phi^x = \Phi(y-x) & \text{on } \partial B(0,1); \end{cases} \tag{37}$$

then Green's function will be

$$G(x,y) = \Phi(y-x) - \phi^x(y). \tag{38}$$

The idea now is to "invert the singularity" from $x \in B^0(0,1)$ to $\tilde{x} \notin B(0,1)$. Assume for the moment $n \geq 3$. Now the mapping $y \mapsto \Phi(y-\tilde{x})$ is harmonic for $y \neq \tilde{x}$. Thus $y \mapsto |x|^{2-n}\Phi(y-\tilde{x})$ is harmonic for $y \neq \tilde{x}$, and so

$$\phi^x(y) := \Phi(|x|(y-\tilde{x})) \tag{39}$$

is harmonic in U. Furthermore, if $y \in \partial B(0,1)$ and $x \neq 0$,

$$\begin{aligned} |x|^2|y-\tilde{x}|^2 &= |x|^2\left(|y|^2 - \frac{2y\cdot x}{|x|^2} + \frac{1}{|x|^2}\right) \\ &= |x|^2 - 2y\cdot x + 1 = |x-y|^2. \end{aligned}$$

Thus $(|x||y-\tilde{x}|)^{-(n-2)} = |x-y|^{-(n-2)}$. Consequently

$$\phi^x(y) = \Phi(y-x) \quad (y \in \partial B(0,1)), \tag{40}$$

as required.

DEFINITION. Green's function for the unit ball *is*

$$G(x,y) := \Phi(y-x) - \Phi(|x|(y-\tilde{x})) \quad (x,y \in B(0,1),\ x \neq y). \tag{41}$$

The same formula is valid for $n = 2$ as well.

Assume now u solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } B^0(0,1) \\ u = g & \text{in } \partial B(0,1). \end{cases} \tag{42}$$

Then using (30), we see

$$u(x) = -\int_{\partial B(0,1)} g(y) \frac{\partial G}{\partial \nu}(x,y)\, dS(y). \tag{43}$$

According to formula (41),

$$G_{y_i}(x,y) = \Phi_{y_i}(y-x) - \Phi(|x|(y-\tilde{x}))_{y_i}.$$

But

$$\Phi_{y_i}(y-x) = \frac{1}{n\alpha(n)} \frac{x_i - y_i}{|x-y|^n},$$

and furthermore

$$\Phi(|x|(y-\tilde{x}))_{y_i}. = \frac{-1}{n\alpha(n)} \frac{y_i|x|^2 - x_i}{(|x||y-\tilde{x}|)^n} = -\frac{1}{n\alpha(n)} \frac{y_i|x|^2 - x_i}{|x-y|^n}$$

if $y \in \partial B(0,1)$. Accordingly

$$\begin{aligned} \frac{\partial G}{\partial \nu}(x,y) &= \sum_{i=1}^n y_i G_{y_i}(x,y) \\ &= \frac{-1}{n\alpha(n)} \frac{1}{|x-y|^n} \sum_{i=1}^n y_i((y_i - x_i) - y_i|x|^2 + x_i) \\ &= \frac{-1}{n\alpha(n)} \frac{1-|x|^2}{|x-y|^n}. \end{aligned}$$

Hence formula (43) yields the representation formula

$$u(x) = \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n}\, dS(y).$$

Suppose now instead of (42) u solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } B^0(0,r) \\ u = g & \text{on } \partial B(0,r) \end{cases} \tag{44}$$

for $r > 0$. Then $\tilde{u}(x) = u(rx)$ solves (42), with $\tilde{g}(x) = g(rx)$ replacing g. We change variables to obtain *Poisson's formula*

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y) \quad (x \in B^0(0,r)). \tag{45}$$

The function

$$K(x,y) := \frac{r^2 - |x|^2}{n\alpha(n)r} \frac{1}{|x-y|^n} \quad (x \in B^0(0,r),\ y \in \partial B(0,r))$$

is *Poisson's kernel* for the ball $B(0,r)$.

We have established (45) under the assumption that a smooth solution of (44) exists. We next assert that this formula in fact gives a solution:

THEOREM 15 (Poisson's formula for ball). *Assume $g \in C(\partial B(0,r))$ and define u by* (45). *Then*

(i) $u \in C^\infty(B^0(0,r))$,

(ii) $\Delta u = 0$ *in* $B^0(0,r)$,

and

(iii) $\displaystyle\lim_{\substack{x \to x^0 \\ x \in B^0(0,r)}} u(x) = g(x^0)$ *for each point* $x^0 \in \partial B(0,r)$.

The proof is similar to that for Theorem 14 and is left as an exercise.

2.2.5. Energy methods.

Most of our analysis of harmonic functions thus far has depended upon fairly explicit representation formulas entailing the fundamental solution, Green's functions, etc. In this concluding subsection we illustrate some "energy" methods, which is to say techniques involving the L^2-norms of various expressions. These ideas foreshadow later theoretical developments in Parts II and III.

a. Uniqueness. Consider first the boundary-value problem

$$\begin{cases} -\Delta u = f & \text{in } U \\ \quad u = g & \text{on } \partial U. \end{cases} \tag{46}$$

We have already employed the maximum principle in §2.2.3 to show uniqueness, but now we set forth a simple alternative proof. Assume U is open, bounded, and ∂U is C^1.

THEOREM 16 (Uniqueness). *There exists at most one solution $u \in C^2(\bar{U})$ of* (46).

Proof. Assume $\tilde{u}$ is another solution and set $w := u - \tilde{u}$. Then $\Delta w = 0$ in U, and so an integration by parts shows

$$0 = -\int_U w \Delta w \, dx = \int_U |Dw|^2 \, dx.$$

Thus $Dw \equiv 0$ in U, and, since $w = 0$ on ∂U, we deduce $w = u - \tilde{u} \equiv 0$ in U. □

b. Dirichlet's principle. Next let us demonstrate that a solution of the boundary-value problem (46) for Poisson's equation can be characterized as the minimizer of an appropriate functional. For this, we define the *energy functional*

$$I[w] := \int_U \frac{1}{2}|Dw|^2 - wf \, dx,$$

w belonging to the *admissible set*

$$\mathcal{A} := \{w \in C^2(\bar{U}) \mid w = g \text{ on } \partial U\}.$$

THEOREM 17 (Dirichlet's principle). *Assume $u \in C^2(\bar{U})$ solves* (46). *Then*

$$I[u] = \min_{w \in \mathcal{A}} I[w]. \tag{47}$$

Conversely, if $u \in \mathcal{A}$ satisfies (47), *then u solves the boundary-value problem* (46).

In other words if $u \in \mathcal{A}$, the PDE $-\Delta u = f$ is equivalent to the statement that u minimizes the energy $I[\cdot]$.

Proof. 1. Choose $w \in \mathcal{A}$. Then (46) implies

$$0 = \int_U (-\Delta u - f)(u - w) \, dx.$$

An integration by parts yields

$$0 = \int_U Du \cdot D(u - w) - f(u - w) \, dx,$$

and there is no boundary term since $u - w = g - g \equiv 0$ on ∂U. Hence

$$\begin{aligned}\int_U |Du|^2 - uf\,dx &= \int_U Du \cdot Dw - wf\,dx \\ &\leq \int_U \frac{1}{2}|Du|^2\,dx + \int_U \frac{1}{2}|Dw|^2 - wf\,dx,\end{aligned}$$

where we employed the estimates

$$|Du \cdot Dw| \leq |Du||Dw| \leq \frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2,$$

following from the Cauchy–Schwarz and Cauchy inequalities (§B.2). Rearranging, we conclude

$$I[u] \leq I[w] \quad (w \in \mathcal{A}). \tag{48}$$

Since $u \in \mathcal{A}$, (47) follows from (48).

2. Now, conversely, suppose (47) holds. Fix any $v \in C_c^\infty(U)$ and write

$$i(\tau) := I[u + \tau v] \quad (\tau \in \mathbb{R}).$$

Since $u + \tau v \in \mathcal{A}$ for each τ, the scalar function $i(\cdot)$ has a minimum at zero, and thus

$$i'(0) = 0 \quad \left(' = \frac{d}{d\tau}\right),$$

provided this derivative exists. But

$$\begin{aligned}i(\tau) &= \int_U \frac{1}{2}|Du + \tau Dv|^2 - (u + \tau v)f\,dx \\ &= \int_U \frac{1}{2}|Du|^2 + \tau Du \cdot Dv + \frac{\tau^2}{2}|Dv|^2 - (u + \tau v)f\,dx.\end{aligned}$$

Consequently

$$0 = i'(0) = \int_U Du \cdot Dv - vf\,dx = \int_U (-\Delta u - f)v\,dx.$$

This identity is valid for each function $v \in C_c^\infty(U)$ and so $-\Delta u = f$ in U. □

Dirichlet's principle is an instance of the *calculus of variations* applied to Laplace's equation. See Chapter 8 for more.