2.3. HEAT EQUATION

Next we study the *heat equation*

$$u_t - \Delta u = 0 \tag{1}$$

and the *nonhomogeneous heat equation*

$$u_t - \Delta u = f, \tag{2}$$

subject to appropriate initial and boundary conditions. Here $t > 0$ and $x \in U$, where $U \subset \mathbb{R}^n$ is open. The unknown is $u : \bar{U} \times [0, \infty) \to \mathbb{R}$, $u = u(x, t)$, and the Laplacian Δ is taken with respect to the spatial variables $x = (x_1, \dots, x_n)$: $\Delta u = \Delta_x u = \sum_{i=1}^n u_{x_i x_i}$. In (2) the function $f : U \times [0, \infty) \to \mathbb{R}$ is given.

A guiding principle is that any assertion about harmonic functions yields an analogous (but more complicated) statement about solutions of the heat equation. Accordingly our development will largely parallel the corresponding theory for Laplace's equation.

Physical interpretation. The heat equation, also known as the *diffusion equation*, describes in typical applications the evolution in time of the density u of some quantity such as heat, chemical concentration, etc. If $V \subset U$ is any smooth subregion, the rate of change of the total quantity within V equals the negative of the net flux through ∂V:

$$\frac{d}{dt} \int_V u \, dx = - \int_{\partial V} \mathbf{F} \cdot \boldsymbol{\nu} \, dS,$$

$\mathbf{F}$ being the flux density. Thus

$$u_t = - \operatorname{div} \mathbf{F}, \tag{3}$$

as V was arbitrary. In many situations $\mathbf{F}$ is proportional to the gradient of u but points in the opposite direction (since the flow is from regions of higher to lower concentration):

$$\mathbf{F} = -aDu \quad (a > 0).$$

Substituting into (3), we obtain the PDE

$$u_t = a \operatorname{div}(Du) = a\Delta u,$$

which for $a = 1$ is the heat equation.

The heat equation appears as well in the study of Brownian motion.

2.3.1. Fundamental solution.

a. Derivation of the fundamental solution. As noted in §2.2.1 an important first step in studying any PDE is often to come up with some specific solutions.

We observe that the heat equation involves one derivative with respect to the time variable t, but two derivatives with respect to the space variables x_i $(i = 1, \dots, n)$. Consequently we see that if u solves (1), then so does $u(\lambda x, \lambda^2 t)$ for $\lambda \in \mathbb{R}$. This scaling indicates the ratio $\frac{r^2}{t}$ $(r = |x|)$ is important for the heat equation and suggests that we search for a solution of (1) having the form $u(x,t) = v(\frac{r^2}{t}) = v(\frac{|x|^2}{t})$ $(t > 0,\ x \in \mathbb{R}^n)$, for some function v as yet undetermined.

Although this approach eventually leads to what we want (see Problem 13), it is quicker to seek a solution u having the special structure

$$u(x,t) = \frac{1}{t^\alpha} v\Big(\frac{x}{t^\beta}\Big) \quad (x \in \mathbb{R}^n,\ t > 0), \tag{4}$$

where the constants α, β and the function $v : \mathbb{R}^n \to \mathbb{R}$ must be found. We come to (4) if we look for a solution u of the heat equation invariant under the *dilation scaling*

$$u(x,t) \mapsto \lambda^\alpha u(\lambda^\beta x, \lambda t).$$

That is, we ask that

$$u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$$

for all $\lambda > 0$, $x \in \mathbb{R}^n$, $t > 0$. Setting $\lambda = t^{-1}$, we derive (4) for $v(y) := u(y,1)$.

Let us insert (4) into (1) and thereafter compute

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot Dv(y) + t^{-(\alpha+2\beta)} \Delta v(y) = 0 \tag{5}$$

for $y := t^{-\beta} x$. In order to transform (5) into an expression involving the variable y alone, we take $\beta = \frac{1}{2}$. Then the terms with t are identical, and so (5) reduces to

$$\alpha v + \frac{1}{2} y \cdot Dv + \Delta v = 0. \tag{6}$$

We simplify further by guessing v to be radial; that is, $v(y) = w(|y|)$ for some $w : \mathbb{R} \to \mathbb{R}$. Thereupon (6) becomes

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0,$$

for $r = |y|$, $' = \frac{d}{dr}$. Now if we set $\alpha = \frac{n}{2}$, this simplifies to read

$$(r^{n-1}w')' + \frac{1}{2}(r^n w)' = 0.$$

Thus

$$r^{n-1}w' + \frac{1}{2}r^n w = a$$

for some constant a. Assuming $\lim_{r\to\infty} w,\ w' = 0$, we conclude $a = 0$, whence

$$w' = -\frac{1}{2}rw.$$

But then for some constant b

(7) $$w = be^{-\frac{r^2}{4}}.$$

Combining (4), (7) and our choices for α, β, we conclude that $\frac{b}{t^{n/2}}e^{-\frac{|x|^2}{4t}}$ solves the heat equation (1).

This computation motivates the following

DEFINITION. *The function*

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{|x|^2}{4t}} & (x \in \mathbb{R}^n,\ t > 0) \\ 0 & (x \in \mathbb{R}^n,\ t < 0) \end{cases}$$

is called the fundamental solution of the heat equation.

Notice that Φ is singular at the point $(0,0)$. We will sometimes write $\Phi(x,t) = \Phi(|x|,t)$ to emphasize that the fundamental solution is radial in the variable x. The choice of the normalizing constant $(4\pi)^{-n/2}$ is dictated by the following

LEMMA (Integral of fundamental solution). *For each time $t > 0$,*

$$\int_{\mathbb{R}^n} \Phi(x,t)\, dx = 1.$$

Proof. We calculate

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi(x,t)\, dx &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\, dx \\ &= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2}\, dz \\ &= \frac{1}{\pi^{n/2}} \prod_{i=1}^{n} \int_{-\infty}^{\infty} e^{-z_i^2}\, dz_i = 1.\end{aligned}$$

□

A different derivation of the fundamental solution of the heat equation appears in §4.3.1.

b. Initial-value problem. We now employ Φ to fashion a solution to the *initial-value* (or *Cauchy*) *problem*

$$\left\{\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t=0\}. \end{aligned}\right. \tag{8}$$

Let us note that the function $(x,t) \mapsto \Phi(x,t)$ solves the heat equation away from the singularity at $(0,0)$, and thus so does $(x,t) \mapsto \Phi(x-y,t)$ for each fixed $y \in \mathbb{R}^n$. Consequently the convolution

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\,dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy \quad (x \in \mathbb{R}^n,\ t>0) \end{aligned} \tag{9}$$

should also be a solution.

THEOREM 1 (Solution of initial-value problem). *Assume* $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, *and define* u *by* (9). *Then*

(i) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$,

(ii) $u_t(x,t) - \Delta u(x,t) = 0 \quad (x \in \mathbb{R}^n,\ t>0)$,

and

(iii) $\displaystyle\lim_{\substack{(x,t)\to(x^0,0) \\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = g(x^0)$ *for each point* $x^0 \in \mathbb{R}^n$.

Proof. 1. Since the function $\frac{1}{t^{n/2}}e^{-\frac{|x|^2}{4t}}$ is infinitely differentiable, with uniformly bounded derivatives of all orders, on $\mathbb{R}^n \times [\delta,\infty)$ for each $\delta > 0$, we see that $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$. Furthermore

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \int_{\mathbb{R}^n} [(\Phi_t - \Delta_x \Phi)(x-y,t)]g(y)\,dy \\ &= 0 \quad (x \in \mathbb{R}^n,\ t>0), \end{aligned} \tag{10}$$

since Φ itself solves the heat equation.

2. Fix $x^0 \in \mathbb{R}^n$, $\varepsilon > 0$. Choose $\delta > 0$ such that

$$|g(y) - g(x^0)| < \varepsilon \quad \text{if } |y - x^0| < \delta,\ y \in \mathbb{R}^n. \tag{11}$$

Then if $|x - x^0| < \frac{\delta}{2}$, we have, according to the lemma,

$$\begin{aligned}
|u(x,t) - g(x^0)| &= \left|\int_{\mathbb{R}^n} \Phi(x-y,t)[g(y) - g(x^0)]\,dy\right| \\
&\leq \int_{B(x^0,\delta)} \Phi(x-y,t)|g(y) - g(x^0)|\,dy \\
&\quad + \int_{\mathbb{R}^n - B(x^0,\delta)} \Phi(x-y,t)|g(y) - g(x^0)|\,dy \\
&=: I + J.
\end{aligned}$$

Now

$$I \leq \varepsilon \int_{\mathbb{R}^n} \Phi(x-y,t)\,dy = \varepsilon,$$

owing to (11) and the lemma. Furthermore, if $|x - x^0| \leq \frac{\delta}{2}$ and $|y - x^0| \geq \delta$, then

$$|y - x^0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x^0|.$$

Thus $|y - x| \geq \frac{1}{2}|y - x^0|$. Consequently

$$\begin{aligned}
J &\leq 2\|g\|_{L^\infty} \int_{\mathbb{R}^n - B(x^0,\delta)} \Phi(x-y,t)\,dy \\
&\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n - B(x^0,\delta)} e^{-\frac{|x-y|^2}{4t}}\,dy \\
&\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n - B(x^0,\delta)} e^{-\frac{|y-x^0|^2}{16t}}\,dy \\
&= C \int_{\mathbb{R}^n - B(x^0,\delta/\sqrt{t})} e^{-\frac{|z|^2}{16}}\,dz \to 0 \quad \text{as } t \to 0^+.
\end{aligned}$$

Hence if $|x - x^0| < \frac{\delta}{2}$ and $t > 0$ is small enough, $|u(x,t) - g(x^0)| < 2\varepsilon$. □

Interpretation of fundamental solution. In view of Theorem 1 we sometimes write

$$\begin{cases} \Phi_t - \Delta\Phi = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \Phi = \delta_0 & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

δ_0 denoting the Dirac measure on $\mathbb{R}^n$ giving unit mass to the point 0.

Infinite propagation speed. Notice that if g is bounded, continuous, $g \geq 0$, $g \not\equiv 0$, then

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy$$

is in fact positive for *all* points $x \in \mathbb{R}^n$ and times $t > 0$. We interpret this observation by saying the heat equation forces *infinite propagation speed* for disturbances. If the initial temperature is nonnegative and is positive somewhere, the temperature at any later time (no matter how small) is everywhere positive. (We will learn in §2.4.3 that the wave equation in contrast supports finite propagation speed for disturbances.)

c. Nonhomogeneous problem. Now let us turn our attention to the *nonhomogeneous* initial-value problem

$$\text{(12)} \qquad \begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ \qquad\quad u = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

How can we produce a formula for the solution? If we recall the motivation leading up to (9), we should note further that the mapping $(x,t) \mapsto \Phi(x-y, t-s)$ is a solution of the heat equation (for given $y \in \mathbb{R}^n$, $0 < s < t$). Now for fixed s, the function

$$u = u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy$$

solves

$$\text{(12}_s\text{)} \qquad \begin{cases} u_t(\cdot;s) - \Delta u(\cdot;s) = 0 & \text{in } \mathbb{R}^n \times (s,\infty) \\ \qquad\qquad u(\cdot;s) = f(\cdot,s) & \text{on } \mathbb{R}^n \times \{t = s\}, \end{cases}$$

which is just an initial-value problem of the form (8), with the starting time $t = 0$ replaced by $t = s$ and g replaced by $f(\cdot,s)$. Thus $u(\cdot;s)$ is certainly not a solution of (12).

However *Duhamel's principle** asserts that we can build a solution of (12) out of the solutions of (12_s), by integrating with respect to s. The idea is to consider

$$u(x,t) = \int_0^t u(x,t;s)\, ds \quad (x \in \mathbb{R}^n,\ t \geq 0).$$

Rewriting, we have

$$\text{(13)} \qquad \begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dyds \\ &= \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\, dyds, \end{aligned}$$

for $x \in \mathbb{R}^n$, $t > 0$.

To confirm that formula (13) works, let us for simplicity assume $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ and f has compact support.

*Duhamel's principle has wide applicability to linear ODE and PDE and does not depend on the specific structure of the heat equation. It yields, for example, the solution of the nonhomogeneous transport equation, obtained by different means in §2.1.2. We will invoke Duhamel's principle for the wave equation in §2.4.2.

THEOREM 2 (Solution of nonhomogeneous problem). *Define u by* (13). *Then*

(i) $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$,

(ii) $u_t(x,t) - \Delta u(x,t) = f(x,t) \quad (x \in \mathbb{R}^n,\, t > 0)$,

and

(iii) $\displaystyle\lim_{\substack{(x,t)\to(x^0,0)\\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = 0$ *for each point* $x^0 \in \mathbb{R}^n$.

Proof. 1. Since Φ has a singularity at $(0,0)$, we cannot directly justify differentiating under the integral sign. We instead proceed somewhat as in the proof of Theorem 1 in §2.2.1.

First we change variables, to write

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,t-s)\, dyds.$$

As $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ has compact support and $\Phi = \Phi(y,s)$ is smooth near $s = t > 0$, we compute

$$\begin{aligned} u_t(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_t(x-y,t-s)\, dyds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy \end{aligned}$$

and

$$u_{x_i x_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_{x_i x_j}(x-y,t-s)\, dyds \quad (i,j = 1,\dots,n).$$

Thus $u_t, D_x^2 u$, and likewise $u, D_x u$, belong to $C(\mathbb{R}^n \times (0,\infty))$.

2. We then calculate

(14)

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\Big[\Big(\frac{\partial}{\partial t} - \Delta_x\Big) f(x-y,t-s)\Big]\, dyds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy \\ &= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\Big[\Big(-\frac{\partial}{\partial s} - \Delta_y\Big) f(x-y,t-s)\Big]\, dyds \\ &\quad + \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)\Big[\Big(-\frac{\partial}{\partial s} - \Delta_y\Big) f(x-y,t-s)\Big]\, dyds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy. \\ &=: I_\varepsilon + J_\varepsilon + K. \end{aligned}$$

Now

$$\text{(15)} \qquad |J_\varepsilon| \le (\|f_t\|_{L^\infty} + \|D^2 f\|_{L^\infty}) \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)\, dyds \le \varepsilon C,$$

by the lemma. Integrating by parts, we also find

$$\text{(16)} \qquad \begin{aligned} I_\varepsilon &= \int_\varepsilon^t \int_{\mathbb{R}^n} \big[\big(\frac{\partial}{\partial s} - \Delta_y\big)\Phi(y,s)\big] f(x-y,t-s)\, dyds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\, dy \\ &\quad - \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy \\ &= \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\, dy - K, \end{aligned}$$

since Φ solves the heat equation. Combining (14)–(16), we ascertain

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\, dy \\ &= f(x,t) \quad (x \in \mathbb{R}^n,\ t > 0), \end{aligned}$$

the limit as $\varepsilon \to 0$ being computed as in the proof of Theorem 1. Finally note $\|u(\cdot,t)\|_{L^\infty} \le t\|f\|_{L^\infty} \to 0$. □

Solution of homogeneous problem with general initial data. We can of course combine Theorems 1 and 2 to discover that

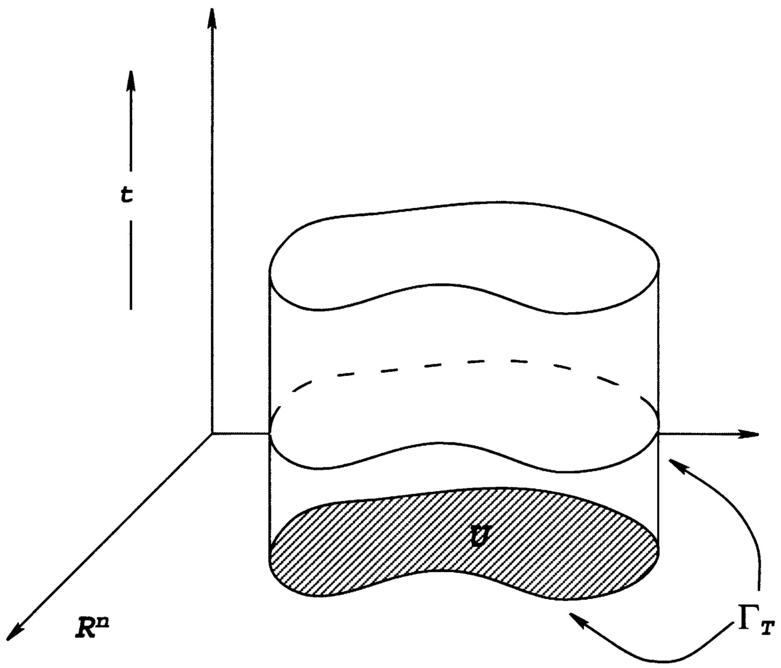
$$\text{(17)} \quad u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dyds$$

is, under the hypotheses on g and f as above, a solution of

$$\text{(18)} \qquad \begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ \qquad\quad u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

2.3.2. Mean-value formula.

First we recall some useful notation from §A.2. Assume $U \subset \mathbb{R}^n$ is open and bounded, and fix a time $T > 0$.



The region $\mathbf{U}_T$

DEFINITIONS.

(i) *We define the* parabolic cylinder

$$U_T := U \times (0, T].$$

(ii) *The* parabolic boundary *of U_T is*

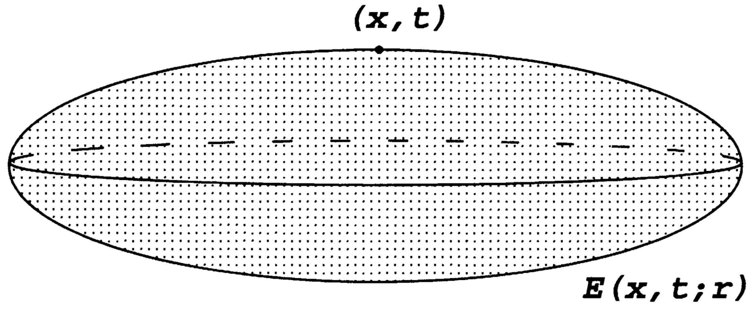
$$\Gamma_T := \bar{U}_T - U_T.$$

We interpret U_T as being the *parabolic interior* of $\bar{U} \times [0, T]$: note carefully that U_T includes the top $U \times \{t = T\}$. The parabolic boundary Γ_T comprises the bottom and vertical sides of $U \times [0, T]$, but not the top.

We want next to derive a kind of analogue to the mean-value property for harmonic functions, as discussed in §2.2.2. There is no such simple formula. However let us observe that for fixed x the spheres $\partial B(x, r)$ are level sets of the fundamental solution $\Phi(x-y)$ for Laplace's equation. This suggests that perhaps for fixed (x, t) the level sets of fundamental solution $\Phi(x-y, t-s)$ for the heat equation may be relevant.

DEFINITION. *For fixed $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $r > 0$, we define*

$$E(x, t; r) := \left\{ (y, s) \in \mathbb{R}^{n+1} \mid s \leq t,\ \Phi(x-y, t-s) \geq \frac{1}{r^n} \right\}.$$



A "heat ball"

This is a region in space-time, the boundary of which is a level set of $\Phi(x-y, t-s)$. Note that the point (x,t) is at the center of the top. $E(x,t;r)$ is sometimes called a "heat ball".

THEOREM 3 (A mean-value property for the heat equation). *Let $u \in C_1^2(U_T)$ solve the heat equation. Then*

$$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t;\,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}\, dyds \tag{19}$$

for each $E(x,t;r) \subset U_T$.

Formula (19) is a sort of analogue for the heat equation of the mean-value formulas for Laplace's equation. Observe that the right-hand side involves only $u(y,s)$ for times $s \le t$. This is reasonable, as the value $u(x,t)$ should not depend upon future times.

Proof. Shift the space and time coordinates so that $x = 0$ and $t = 0$. Upon mollifying if necessary, we may assume u is smooth. Write $E(r) = E(0,0;r)$ and set

$$\begin{aligned} \phi(r) &:= \frac{1}{r^n} \iint_{E(r)} u(y,s) \frac{|y|^2}{s^2}\, dyds \\ &= \iint_{E(1)} u(ry, r^2 s) \frac{|y|^2}{s^2}\, dyds. \end{aligned} \tag{20}$$

We compute

$$\begin{aligned} \phi'(r) &= \iint_{E(1)} \sum_{i=1}^n u_{y_i} y_i \frac{|y|^2}{s^2} + 2ru_s \frac{|y|^2}{s}\, dyds \\ &= \frac{1}{r^{n+1}} \iint_{E(r)} \sum_{i=1}^n u_{y_i} y_i \frac{|y|^2}{s^2} + 2u_s \frac{|y|^2}{s}\, dyds \\ &=: A + B. \end{aligned}$$

Also, let us introduce the useful function

$$\psi := -\frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s} + n\log r \tag{21}$$

and observe $\psi = 0$ on $\partial E(r)$, since $\Phi(y,-s) = r^{-n}$ on $\partial E(r)$. We utilize (21) to write

$$\begin{aligned} B &= \frac{1}{r^{n+1}} \iint_{E(r)} 4u_s \sum_{i=1}^n y_i \psi_{y_i}\, dyds \\ &= -\frac{1}{r^{n+1}} \iint_{E(r)} 4nu_s\psi + 4\sum_{i=1}^n u_{sy_i} y_i \psi\, dyds; \end{aligned}$$

there is no boundary term since $\psi = 0$ on $\partial E(r)$. Integrating by parts with respect to s, we discover

$$\begin{aligned} B &= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi + 4\sum_{i=1}^n u_{y_i} y_i \psi_s\, dyds \\ &= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi + 4\sum_{i=1}^n u_{y_i} y_i \left(-\frac{n}{2s} - \frac{|y|^2}{4s^2}\right) dyds \\ &= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi - \frac{2n}{s}\sum_{i=1}^n u_{y_i} y_i\, dyds - A. \end{aligned}$$

Consequently, since u solves the heat equation,

$$\begin{aligned} \phi'(r) &= A + B \\ &= \frac{1}{r^{n+1}} \iint_{E(r)} -4n\Delta u\psi - \frac{2n}{s}\sum_{i=1}^n u_{y_i} y_i\, dyds \\ &= \sum_{i=1}^n \frac{1}{r^{n+1}} \iint_{E(r)} 4nu_{y_i}\psi_{y_i} - \frac{2n}{s} u_{y_i} y_i\, dyds \\ &= 0, \quad \text{according to (21).} \end{aligned}$$

Thus ϕ is constant, and therefore

$$\phi(r) = \lim_{t\to 0}\phi(t) = u(0,0)\left(\lim_{t\to 0}\frac{1}{t^n}\iint_{E(t)} \frac{|y|^2}{s^2}\, dyds\right) = 4u(0,0),$$

as

$$\frac{1}{t^n}\iint_{E(t)} \frac{|y|^2}{s^2}\, dyds = \iint_{E(1)} \frac{|y|^2}{s^2}\, dyds = 4.$$

We omit the details of this last computation. □