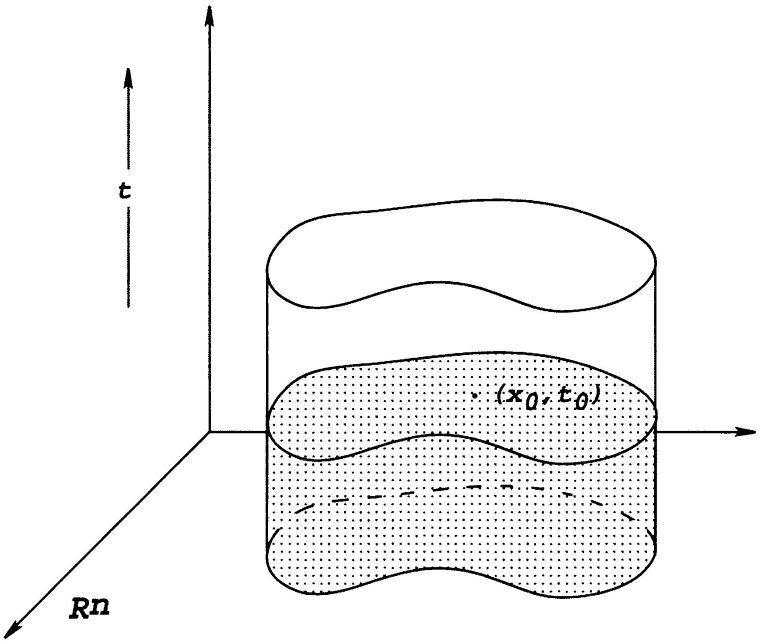


Strong maximum principle for the heat equation

2.3.3. Properties of solutions.

a. Strong maximum principle, uniqueness. First we employ the mean-value property to give a quick proof of the strong maximum principle.

THEOREM 4 (Strong maximum principle for the heat equation). *Assume* $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ *solves the heat equation in* U_T.

(i) *Then*

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u.$$

(ii) *Furthermore, if* U *is connected and there exists a point* $(x_0, t_0) \in U_T$ *such that*

$$u(x_0, t_0) = \max_{\bar{U}_T} u,$$

then

$$u \text{ is constant in } \bar{U}_{t_0}.$$

Assertion (i) is the *maximum principle* for the heat equation and (ii) is the *strong maximum principle*. Similar assertions are valid with "min" replacing "max".

Interpretation. *So if u attains its maximum (or minimum) at an interior point, then u is constant at all earlier times.* This accords with our strong intuitive understanding of the variable t as denoting time: the solution will be constant on the time interval $[0, t_0]$ provided the initial and boundary conditions are constant. However, the solution may change at times $t > t_0$, provided the boundary conditions alter after t_0. The solution will however not respond to changes in boundary conditions until these changes happen.

Take note that whereas all this is obvious on intuitive, physical grounds, such insights do not constitute a proof. The task is to *deduce* such behavior from the PDE.

Proof. 1. Suppose there exists a point $(x_0, t_0) \in U_T$ with $u(x_0, t_0) = M := \max_{\bar{U}_T} u$. Then for all sufficiently small $r > 0$, $E(x_0, t_0; r) \subset U_T$; and we employ the mean-value property to deduce

$$M = u(x_0, t_0) = \frac{1}{4r^n} \iint_{E(x_0,t_0;\, r)} u(y,s) \frac{|x_0 - y|^2}{(t_0 - s)^2}\, dyds \leq M,$$

since

$$1 = \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} \frac{|x_0 - y|^2}{(t_0 - s)^2}\, dyds.$$

Equality holds only if u is identically equal to M within $E(x_0, t_0; r)$. Consequently

$$u(y, s) = M \qquad \text{for all } (y, s) \in E(x_0, t_0; r).$$

Draw any line segment L in U_T connecting (x_0, t_0) with some other point $(y_0, s_0) \in U_T$, with $s_0 < t_0$. Consider

$$r_0 := \min\{s \geq s_0 \mid u(x,t) = M \text{ for all points } (x,t) \in L,\ s \leq t \leq t_0\}.$$

Since u is continuous, the minimum is attained. Assume $r_0 > s_0$. Then $u(z_0, r_0) = M$ for some point (z_0, r_0) on $L \cap U_T$ and so $u \equiv M$ on $E(z_0, r_0; r)$ for all sufficiently small $r > 0$. Since $E(z_0, r_0; r)$ contains $L \cap \{r_0 - \sigma \leq t \leq r_0\}$ for some small $\sigma > 0$, we have a contradiction. Thus $r_0 = s_0$, and hence $u \equiv M$ on L.

2. Now fix any point $x \in U$ and any time $0 \leq t < t_0$. There exist points $\{x_0, x_1, \ldots, x_m = x\}$ such that the line segments in $\mathbb{R}^n$ connecting x_{i-1} to x_i lie in U for $i = 1, \ldots, m$. (This follows since the set of points in U which can be so connected to x_0 by a polygonal path is nonempty, open and relatively closed in U.) Select times $t_0 > t_1 > \cdots > t_m = t$. Then the line segments in $\mathbb{R}^{n+1}$ connecting (x_{i-1}, t_{i-1}) to (x_i, t_i) $(i = 1, \ldots, m)$ lie in U_T. According to step 1, $u \equiv M$ on each such segment and so $u(x, t) = M$. □

Infinite propagation speed again. The strong maximum principle implies that if U is connected and $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ satisfies

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0, T] \\ u = g & \text{on } U \times \{t = 0\} \end{cases}$$

where $g \geq 0$, then u is positive *everywhere* within U_T if g is positive *somewhere* on U. This is another illustration of infinite propagation speed for disturbances.

An important application of the maximum principle is the following uniqueness assertion.

THEOREM 5 (Uniqueness on bounded domains). *Let $g \in C(\Gamma_T)$, $f \in C(U_T)$. Then there exists at most one solution $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ of the initial/boundary-value problem*

$$\text{(22)} \qquad \begin{cases} u_t - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T. \end{cases}$$

Proof. If u and $\tilde{u}$ are two solutions of (22), apply Theorem 4 to $w := \pm(u - \tilde{u})$. □

We next extend our uniqueness assertion to the *Cauchy problem*, that is, the initial-value problem for $U = \mathbb{R}^n$. As we are no longer on a bounded region, we must introduce some control on the behavior of solutions for large $|x|$.

THEOREM 6 (Maximum principle for the Cauchy problem). *Suppose $u \in C_1^2(\mathbb{R}^n \times (0, T]) \cap C(\mathbb{R}^n \times [0, T])$ solves*

$$\text{(23)} \qquad \begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, T) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

and satisfies the growth estimate

$$\text{(24)} \qquad u(x, t) \leq A e^{a|x|^2} \quad (x \in \mathbb{R}^n,\ 0 \leq t \leq T)$$

for constants $A, a > 0$. Then

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g.$$

Proof. 1. First assume

$$4aT < 1, \tag{25}$$

in which case

$$4a(T+\varepsilon) < 1 \tag{26}$$

for some $\varepsilon > 0$. Fix $y \in \mathbb{R}^n$, $\mu > 0$, and define

$$v(x,t) := u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}} \quad (x \in \mathbb{R}^n,\ t > 0).$$

A direct calculation (cf. §2.3.1) shows

$$v_t - \Delta v = 0 \quad \text{in } \mathbb{R}^n \times (0,T].$$

Fix $r > 0$ and set $U := B^0(y,r)$, $U_T = B^0(y,r) \times (0,T]$. Then according to Theorem 4,

$$\max_{\bar{U}_T} v = \max_{\Gamma_T} v. \tag{27}$$

2. Now if $x \in \mathbb{R}^n$,

$$\begin{aligned} v(x,0) &= u(x,0) - \frac{\mu}{(T+\varepsilon)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon)}} \\ &\leq u(x,0) = g(x); \end{aligned} \tag{28}$$

and if $|x-y| = r$, $0 \leq t \leq T$, then

$$\begin{aligned} v(x,t) &= u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \\ &\leq Ae^{a|x|^2} - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \qquad \text{by (24)} \\ &\leq Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon)}}. \end{aligned}$$

Now according to (26), $\frac{1}{4(T+\varepsilon)} = a+\gamma$ for some $\gamma > 0$. Thus we may continue the calculation above to find

$$v(x,t) \leq Ae^{a(|y|+r)^2} - \mu(4(a+\gamma))^{n/2} e^{(a+\gamma)r^2} \leq \sup_{\mathbb{R}^n} g, \tag{29}$$

for r selected sufficiently large. Thus (27)–(29) imply

$$v(y,t) \leq \sup_{\mathbb{R}^n} g$$

for all $y \in \mathbb{R}^n$, $0 \leq t \leq T$, provided (25) is valid. Let $\mu \to 0$.

3. In the general case that (25) fails, we repeatedly apply the result above on the time intervals $[0,T_1]$, $[T_1, 2T_1,]$, etc., for $T_1 = \frac{1}{8a}$. □

THEOREM 7 (Uniqueness for Cauchy problem). *Let* $g \in C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n \times [0,T])$. *Then there exists at most one solution* $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ *of the initial-value problem*

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{30}$$

satisfying the growth estimate

$$|u(x,t)| \leq A e^{a|x|^2} \quad (x \in \mathbb{R}^n,\ 0 \leq t \leq T) \tag{31}$$

for constants A, $a > 0$.

Proof. If u and $\tilde{u}$ both satisfy (30), (31), we apply Theorem 6 to $w := \pm(u - \tilde{u})$. □

Nonphysical solutions. There are in fact infinitely many solutions of

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}; \end{cases} \tag{32}$$

see for instance John [**J2,** Chapter 7]. Each of these solutions besides $u \equiv 0$ grows very rapidly as $|x| \to \infty$.

There is an interesting point here: although $u \equiv 0$ is certainly the "physically correct" solution of (32), this initial-value problem in fact admits other, "nonphysical", solutions. Theorem 7 provides a criterion which excludes the "wrong" solutions. We will encounter somewhat analogous situations in our study of Hamilton–Jacobi equations and conservation laws, in Chapters 3, 10 and 11.

b. Regularity. We next demonstrate that solutions of the heat equation are automatically smooth.

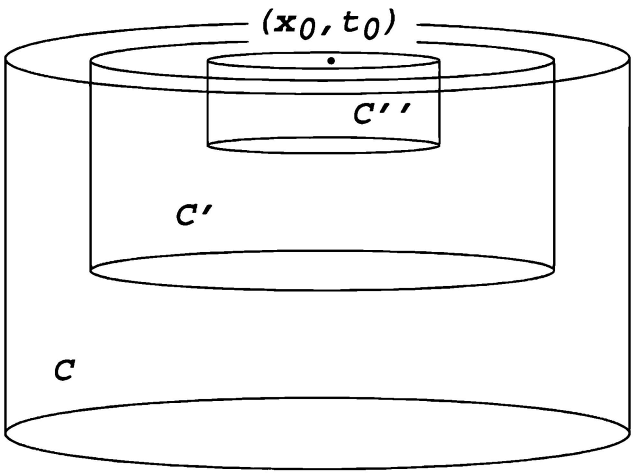
THEOREM 8 (Smoothness). *Suppose* $u \in C_1^2(U_T)$ *solves the heat equation in* U_T. *Then*

$$u \in C^\infty(U_T).$$

This regularity assertion is valid even if u attains nonsmooth boundary values on Γ_T.

Proof. 1. Recall from §A.2 that we write

$$C(x,t;r) = \{(y,s) \mid |x-y| \leq r,\ t - r^2 \leq s \leq t\}$$



to denote the closed circular cylinder of radius r, height r^2, and top center point (x,t).

Fix $(x_0,t_0) \in U_T$ and choose $r>0$ so small that $C := C(x_0,t_0;r) \subset U_T$. Define also the smaller cylinders $C' := C(x_0,t_0;\frac{3}{4}r)$, $C'' := C(x_0,t_0;\frac{1}{2}r)$, which have the same top center point (x_0,t_0).

Choose a smooth cutoff function $\zeta = \zeta(x,t)$ such that

$$\begin{cases} 0 \le \zeta \le 1, \ \zeta \equiv 1 \text{ on } C', \\ \zeta \equiv 0 \text{ near the parabolic boundary of } C. \end{cases}$$

Extend $\zeta \equiv 0$ in $(\mathbb{R}^n \times [0,t_0]) - C$.

2. Assume temporarily that $u \in C^\infty(U_T)$ and set

$$v(x,t) := \zeta(x,t)u(x,t) \quad (x \in \mathbb{R}^n,\ 0 \le t \le t_0).$$

Then

$$v_t = \zeta u_t + \zeta_t u, \ \ \Delta v = \zeta \Delta u + 2D\zeta \cdot Du + u\Delta\zeta.$$

Consequently

$$v = 0 \quad \text{on } \mathbb{R}^n \times \{t=0\}, \tag{33}$$

and

$$v_t - \Delta v = \zeta_t u - 2D\zeta \cdot Du - u\Delta\zeta =: \tilde{f} \tag{34}$$

in $\mathbb{R}^n \times (0,t_0)$. Now set

$$\tilde{v}(x,t) := \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\tilde{f}(y,s)\,dyds.$$

According to Theorem 2

$$\begin{cases} \tilde{v}_t - \Delta\tilde{v} = \tilde{f} & \text{in } \mathbb{R}^n \times (0,t_0) \\ \tilde{v} = 0 & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{35}$$

Since $|v|, |\tilde{v}| \le A$ for some constant A, Theorem 7 implies $v \equiv \tilde{v}$; that is,

$$(36) \qquad v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\tilde{f}(y,s)\, dyds.$$

Now suppose $(x,t) \in C''$. As $\zeta \equiv 0$ off the cylinder C, (34) and (36) imply

$$u(x,t) = \iint_C \Phi(x-y,t-s)[(\zeta_s(y,s) - \Delta\zeta(y,s))u(y,s) - 2D\zeta(y,s)\cdot Du(y,s)]\, dyds.$$

Note in this equation that the expression in the square brackets vanishes in some region *near* the singularity of Φ. Integrate the last term by parts:

$$(37) \qquad u(x,t) = \iint_C [\Phi(x-y,t-s)(\zeta_s(y,s) + \Delta\zeta(y,s)) + 2D_y\Phi(x-y,t-s)\cdot D\zeta(y,s)]u(y,s)\, dyds.$$

We have proved this formula assuming $u \in C^\infty$. If u satisfies only the hypotheses of the theorem, we derive (37) with $u^\varepsilon = \eta_\varepsilon * u$ replacing u, η_ε being the standard mollifier in the variables x and t, and let $\varepsilon \to 0$.

3. Formula (37) has the form

$$(38) \qquad u(x,t) = \iint_C K(x,t,y,s)u(y,s)\, dyds \quad ((x,t) \in C''),$$

where

$$K(x,t,y,s) = 0 \quad \text{for all points } (y,s) \in C',$$

since $\zeta \equiv 1$ on C'. Note also K is smooth on $C - C'$. In view of expression (38), we see u is C^∞ within $C'' = C(x_0, t_0; \frac{1}{2}r)$. □

c. Local estimates for solutions of the heat equation. Let us now record some estimates on the derivatives of solutions to the heat equation, paying attention to the differences between derivatives with respect to x_i $(i = 1, \dots, n)$ and with respect to t.

THEOREM 9 (Estimates on derivatives). *There exists for each pair of integers $k, l = 0, 1, \dots$ a constant $C_{k,l}$ such that*

$$\max_{C(x,t;\,r/2)} |D_x^k D_t^l u| \le \frac{C_{kl}}{r^{k+2l+n+2}} \|u\|_{L^1(C(x,t;r))}$$

for all cylinders $C(x,t;r/2) \subset C(x,t;r) \subset U_T$ and all solutions u of the heat equation in U_T.

Proof. 1. Fix some point in U_T. Upon shifting the coordinates, we may as well assume the point is $(0,0)$. Suppose first that the cylinder $C(1) := C(0,0;1)$ lies in U_T. Let $C\left(\frac{1}{2}\right) := C\left(0,0;\frac{1}{2}\right)$. Then, as in the proof of Theorem 8,

$$u(x,t) = \iint_{C(1)} K(x,t,y,s)u(y,s)\,dyds \quad ((x,t) \in C(\tfrac{1}{2}))$$

for some smooth function K. Consequently

$$\begin{aligned} |D_x^k D_t^l u(x,t)| &\leq \iint_{C(1)} |D_t^l D_x^k K(x,t,y,s)||u(y,s)|\,dyds \\ &\leq C_{kl}\|u\|_{L^1(C(1))} \end{aligned} \tag{39}$$

for some constant C_{kl}.

2. Now suppose the cylinder $C(r) := C(0,0;r)$ lies in U_T. Let $C(r/2) = C(0,0;r/2)$. We rescale by defining

$$v(x,t) := u(rx, r^2t).$$

Then $v_t - \Delta v = 0$ in the cylinder $C(1)$. According to (39),

$$|D_x^k D_t^l v(x,t)| \leq C_{kl}\|v\|_{L^1(C(1))} \quad ((x,t) \in C(\tfrac{1}{2})).$$

But $D_x^k D_t^l v(x,t) = r^{2l+k} D_x^k D_t^l u(rx, r^2t)$ and $\|v\|_{L^1(C(1))} = \frac{1}{r^{n+2}}\|u\|_{L^1(C(r))}$. Therefore

$$\max_{C(r/2)} |D_x^k D_t^l u| \leq \frac{C_{kl}}{r^{2l+k+n+2}}\|u\|_{L^1(C(r))}. \qquad \square$$

If u solves the heat equation within U_T, then for each time $0 < t \leq T$, the mapping $x \mapsto u(x,t)$ is analytic. (See Mikhailov [**M**].) However the mapping $t \mapsto u(x,t)$ is not in general analytic.

2.3.4. Energy methods.

a. Uniqueness. We investigate again the initial/boundary-value problem

$$\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T. \end{cases} \tag{40}$$

We earlier invoked the maximum principle to show uniqueness and now—by analogy with §2.2.5—provide an alternative argument based upon integration by parts. We assume as usual that $U \subset \mathbb{R}^n$ is open and bounded and that ∂U is C^1. The terminal time $T > 0$ is given.

THEOREM 10 (Uniqueness). *There exists only one solution* $u \in C_1^2(\bar{U}_T)$ *of the initial/boundary-value problem* (40).

Proof. 1. If $\tilde{u}$ is another solution, $w := u - \tilde{u}$ solves

$$\begin{cases} w_t - \Delta w = 0 & \text{in } U_T \\ w = 0 & \text{on } \Gamma_T. \end{cases} \tag{41}$$

2. Set

$$e(t) := \int_U w^2(x,t)\,dx \quad (0 \le t \le T).$$

Then

$$\begin{aligned} \dot{e}(t) &= 2\int_U w w_t\,dx \qquad \left(\dot{} = \frac{d}{dt}\right) \\ &= 2\int_U w\Delta w\,dx \\ &= -2\int_U |Dw|^2\,dx \le 0, \end{aligned}$$

and so

$$e(t) \le e(0) = 0 \quad (0 \le t \le T).$$

Consequently $w = u - \tilde{u} \equiv 0$ in U_T. □

Observe that the foregoing is a time-dependent variant of the proof of Theorem 16 in §2.2.5.

b. Backwards uniqueness. A rather more subtle question asks about uniqueness *backwards in time* for the heat equation. For this, suppose u and $\tilde{u}$ are both smooth solutions of the heat equation in U_T, with the same boundary conditions on ∂U:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T \\ u = g & \text{on } \partial U \times [0,T], \end{cases} \tag{42}$$

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = 0 & \text{in } U_T \\ \tilde{u} = g & \text{on } \partial U \times [0,T], \end{cases} \tag{43}$$

for some function g. Note carefully that we are *not* supposing $u = \tilde{u}$ at time $t = 0$.

THEOREM 11 (Backwards uniqueness). *Suppose* $u, \tilde{u} \in C^2(\bar{U}_T)$ *solve* (42), (43). *If*

$$u(x,T) = \tilde{u}(x,T) \quad (x \in U),$$

then

$$u \equiv \tilde{u} \quad \text{within } U_T.$$

In other words, if two temperature distributions on U agree at some time $T > 0$ and have had the same boundary values for times $0 \leq t \leq T$, then these temperatures must have been identically equal within U at all earlier times. This is not at all obvious.

Proof. 1. Write $w := u - \tilde{u}$ and, as in the proof of Theorem 10, set

$$e(t) := \int_U w^2(x,t)\,dx \quad (0 \leq t \leq T).$$

As before

$$\dot{e}(t) = -2\int_U |Dw|^2\,dx \quad \left(\dot{} = \frac{d}{dt}\right). \tag{44}$$

Furthermore

$$\begin{aligned} \ddot{e}(t) &= -4\int_U Dw \cdot Dw_t\,dx \\ &= 4\int_U \Delta w w_t\,dx \\ &= 4\int_U (\Delta w)^2\,dx \quad \text{by (41).} \end{aligned} \tag{45}$$

Now since $w = 0$ on ∂U,

$$\begin{aligned} \int_U |Dw|^2\,dx &= -\int_U w\Delta w\,dx \\ &\leq \left(\int_U w^2\,dx\right)^{1/2} \left(\int_U (\Delta w)^2\,dx\right)^{1/2}. \end{aligned}$$

Thus (44) and (45) imply

$$\begin{aligned} (\dot{e}(t))^2 &= 4\left(\int_U |Dw|^2\,dx\right)^2 \\ &\leq \left(\int_U w^2\,dx\right)\left(4\int_U (\Delta w)^2\,dx\right) \\ &= e(t)\ddot{e}(t). \end{aligned}$$

Hence

$$\ddot{e}(t)e(t) \geq (\dot{e}(t))^2 \quad (0 \leq t \leq T). \tag{46}$$

2. Now if $e(t) = 0$ for all $0 \leq t \leq T$, we are done. Otherwise there exists an interval $[t_1, t_2] \subset [0, T]$, with

$$e(t) > 0 \quad \text{for } t_1 \leq t < t_2, \;\; e(t_2) = 0. \tag{47}$$

3. Now write

$$f(t) := \log e(t) \quad (t_1 \leq t < t_2). \tag{48}$$

Then

$$\ddot{f}(t) = \frac{\ddot{e}(t)}{e(t)} - \frac{\dot{e}(t)^2}{e(t)^2} \geq 0 \quad \text{by (46)},$$

and so f is convex on the interval (t_1, t_2). Consequently if $0 < \tau < 1$, $t_1 < t < t_2$, we have

$$f((1-\tau)t_1 + \tau t) \leq (1-\tau)f(t_1) + \tau f(t).$$

Recalling (48), we deduce

$$e((1-\tau)t_1 + \tau t) \leq e(t_1)^{1-\tau} e(t)^{\tau},$$

and so

$$0 \leq e((1-\tau)t_1 + \tau t_2) \leq e(t_1)^{1-\tau} e(t_2)^{\tau} \quad (0 < \tau < 1).$$

But in view of (47) this inequality implies $e(t) = 0$ for all times $t_1 \leq t \leq t_2$, a contradiction. □

2.4. WAVE EQUATION

In this section we investigate the *wave equation*

$$u_{tt} - \Delta u = 0 \tag{1}$$

and the *nonhomogeneous wave equation*

$$u_{tt} - \Delta u = f, \tag{2}$$

subject to appropriate initial and boundary conditions. Here $t > 0$ and $x \in U$, where $U \subset \mathbb{R}^n$ is open. The unknown is $u : \bar{U} \times [0, \infty) \to \mathbb{R}$, $u = u(x, t)$, and the Laplacian Δ is taken with respect to the spatial variables