Hence

$$\ddot{e}(t)e(t) \geq (\dot{e}(t))^2 \quad (0 \leq t \leq T). \tag{46}$$

2. Now if $e(t) = 0$ for all $0 \leq t \leq T$, we are done. Otherwise there exists an interval $[t_1, t_2] \subset [0, T]$, with

$$e(t) > 0 \quad \text{for } t_1 \leq t < t_2, \quad e(t_2) = 0. \tag{47}$$

3. Now write

$$f(t) := \log e(t) \quad (t_1 \leq t < t_2). \tag{48}$$

Then

$$\ddot{f}(t) = \frac{\ddot{e}(t)}{e(t)} - \frac{\dot{e}(t)^2}{e(t)^2} \geq 0 \quad \text{by (46)},$$

and so f is convex on the interval (t_1, t_2). Consequently if $0 < \tau < 1$, $t_1 < t < t_2$, we have

$$f((1-\tau)t_1 + \tau t) \leq (1-\tau)f(t_1) + \tau f(t).$$

Recalling (48), we deduce

$$e((1-\tau)t_1 + \tau t) \leq e(t_1)^{1-\tau} e(t)^\tau,$$

and so

$$0 \leq e((1-\tau)t_1 + \tau t_2) \leq e(t_1)^{1-\tau} e(t_2)^\tau \quad (0 < \tau < 1).$$

But in view of (47) this inequality implies $e(t) = 0$ for all times $t_1 \leq t \leq t_2$, a contradiction. □

2.4. WAVE EQUATION

In this section we investigate the *wave equation*

$$u_{tt} - \Delta u = 0 \tag{1}$$

and the *nonhomogeneous wave equation*

$$u_{tt} - \Delta u = f, \tag{2}$$

subject to appropriate initial and boundary conditions. Here $t > 0$ and $x \in U$, where $U \subset \mathbb{R}^n$ is open. The unknown is $u : \bar{U} \times [0, \infty) \to \mathbb{R}$, $u = u(x,t)$, and the Laplacian Δ is taken with respect to the spatial variables

$x = (x_1, \ldots, x_n)$. In (2) the function $f : U \times [0, \infty) \to \mathbb{R}$ is given. A common abbreviation is to write

$$\Box u := u_{tt} - \Delta u.$$

We shall discover that solutions of the wave equation behave quite differently than solutions of Laplace's equation or the heat equation. For example, these solutions are generally not C^∞, exhibit finite speed of propagation, etc.

Physical interpretation. The wave equation is a simplified model for a vibrating string ($n = 1$), membrane ($n = 2$), or elastic solid ($n = 3$). In these physical interpretations $u(x, t)$ represents the displacement in some direction of the point x at time $t \geq 0$.

Let V represent any smooth subregion of U. The acceleration within V is then

$$\frac{d^2}{dt^2} \int_V u \, dx = \int_V u_{tt} \, dx$$

and the net contact force is

$$- \int_{\partial V} \mathbf{F} \cdot \boldsymbol{\nu} \, dS,$$

where $\mathbf{F}$ denotes the force acting on V through ∂V and the mass density is taken to be unity. Newton's law asserts that the mass times the acceleration equals the net force:

$$\int_V u_{tt} \, dx = - \int_{\partial V} \mathbf{F} \cdot \boldsymbol{\nu} \, dS.$$

This identity obtains for each subregion V and so

$$u_{tt} = - \operatorname{div} \mathbf{F}.$$

For elastic bodies, $\mathbf{F}$ is a function of the displacement gradient Du, whence

$$u_{tt} + \operatorname{div} \mathbf{F}(Du) = 0.$$

For small Du, the linearization $\mathbf{F}(Du) \approx -aDu$ is often appropriate; and so

$$u_{tt} - a\Delta u = 0.$$

This is the wave equation if $a = 1$.

This physical interpretation strongly suggests it will be mathematically appropriate to specify *two* initial conditions, on the *displacement* u and the *velocity* u_t, at time $t = 0$.

2.4.1. Solution by spherical means.

We began §§2.2.1 and 2.3.1 by searching for certain scaling invariant solutions of Laplace's equation and the heat equation. For the wave equation however we will instead present the (reasonably) elegant method of solving (1) first for $n = 1$ directly and then for $n \geq 2$ by the method of spherical means.

a. Solution for $\mathbf{n = 1}$, d'Alembert's formula. We first focus our attention on the initial-value problem for the one-dimensional wave equation in all of $\mathbb{R}$:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g, \ u_t = h & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \tag{3}$$

where g, h are given. We desire to derive a formula for u in terms of g and h.

Let us first note that the PDE in (3) can be "factored", to read

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = u_{tt} - u_{xx} = 0. \tag{4}$$

Write

$$v(x,t) := \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t). \tag{5}$$

Then (4) says

$$v_t(x,t) + v_x(x,t) = 0 \quad (x \in \mathbb{R}, \ t > 0).$$

This is a transport equation with constant coefficients. Applying formula (3) from §2.1.1 (with $n = 1$, $b = 1$), we find

$$v(x,t) = a(x - t) \tag{6}$$

for $a(x) := v(x, 0)$. Combining now (4)–(6), we obtain

$$u_t(x,t) - u_x(x,t) = a(x - t) \quad \text{in } \mathbb{R} \times (0, \infty).$$

This is a nonhomogeneous transport equation; and so formula (5) from §2.1.2 (with $n = 1$, $b = -1$, $f(x,t) = a(x-t)$) implies for $b(x) := u(x, 0)$ that

$$\begin{aligned} u(x,t) &= \int_0^t a(x + (t - s) - s)\, ds + b(x + t) \\ &= \frac{1}{2} \int_{x-t}^{x+t} a(y)\, dy + b(x + t). \end{aligned} \tag{7}$$

We lastly invoke the initial conditions in (3) to compute a and b. The first initial condition in (3) gives

$$b(x) = g(x) \quad (x \in \mathbb{R}),$$

whereas the second initial condition and (5) imply

$$a(x) = v(x,0) = u_t(x,0) - u_x(x,0) = h(x) - g'(x) \quad (x \in \mathbb{R}).$$

Our substituting into (7) now yields

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} h(y) - g'(y)\,dy + g(x+t).$$

Hence

$$(8) \qquad u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy \quad (x \in \mathbb{R},\ t \geq 0).$$

This is *d'Alembert's formula.*

We have derived formula (8) assuming u is a (sufficiently smooth) solution of (3). We need to check that this really is a solution.

THEOREM 1 (Solution of wave equation, $n = 1$). *Assume* $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, *and define* u *by d'Alembert's formula* (8). *Then*

(i) $u \in C^2(\mathbb{R} \times [0,\infty))$,

(ii) $u_{tt} - u_{xx} = 0$ *in* $\mathbb{R} \times (0,\infty)$,

and

(iii) $\displaystyle\lim_{\substack{(x,t)\to(x^0,0)\\ t>0}} u(x,t) = g(x^0), \quad \lim_{\substack{(x,t)\to(x^0,0)\\ t>0}} u_t(x,t) = h(x^0)$

for each point $x^0 \in \mathbb{R}$.

The proof is a straightforward calculation.

Remarks. (i) In view of (8), our solution u has the form

$$u(x,t) = F(x+t) + G(x-t)$$

for appropriate functions F and G. Conversely any function of this form solves $u_{tt} - u_{xx} = 0$. Hence the general solution of the one-dimensional wave equation is a sum of the general solution of $u_t - u_x = 0$ and the general solution of $u_t + u_x = 0$. This is a consequence of the factorization (4). See Problem 19.

(ii) We see from (8) that if $g \in C^k$ and $h \in C^{k-1}$, then $u \in C^k$ but is not in general smoother. Thus the wave equation does *not* cause instantaneous smoothing of the initial data, as does the heat equation.

A reflection method. To illustrate a further application of d'Alembert's formula, let us next consider this initial/boundary-value problem on the half-line $\mathbb{R}_+ = \{x > 0\}$:

$$(9) \qquad \begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u = 0 & \text{on } \{x = 0\} \times (0,\infty), \end{cases}$$

where g, h are given, with $g(0) = h(0) = 0$.

We convert (9) into the form (3) by extending u, g, h to all of $\mathbb{R}$ by *odd reflection*. That is, we set

$$\tilde{u}(x,t) := \begin{cases} u(x,t) & (x \geq 0,\ t \geq 0) \\ -u(-x,t) & (x \leq 0,\ t \geq 0), \end{cases}$$
$$\tilde{g}(x) := \begin{cases} g(x) & (x \geq 0) \\ -g(-x) & (x \leq 0), \end{cases}$$
$$\tilde{h}(x) := \begin{cases} h(x) & (x \geq 0) \\ -h(-x) & (x \leq 0). \end{cases}$$

Then (9) becomes

$$\begin{cases} \tilde{u}_{tt} = \tilde{u}_{xx} & \text{in } \mathbb{R} \times (0,\infty) \\ \tilde{u} = \tilde{g},\ \tilde{u}_t = \tilde{h} & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Hence d'Alembert's formula (8) implies

$$\tilde{u}(x,t) = \frac{1}{2}[\tilde{g}(x+t) + \tilde{g}(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\,dy.$$

Recalling the definitions of $\tilde{u}, \tilde{g}, \tilde{h}$ above, we can transform this expression to read for $x \geq 0$, $t \geq 0$:

$$(10) \quad u(x,t) = \begin{cases} \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy & \text{if } x \geq t \geq 0 \\ \frac{1}{2}[g(x+t) - g(t-x)] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)\,dy & \text{if } 0 \leq x \leq t. \end{cases}$$

If $h \equiv 0$, we can understand formula (10) as saying that an initial displacement g splits into two parts, one moving to the right with speed one and the other to the left with speed one. The latter then reflects off the point $x = 0$, where the vibrating string is held fixed.

Note that our solution does not belong to C^2, unless $g''(0) = 0$. □

b. Spherical means. Now suppose $n \geq 2$, $m \geq 2$, and $u \in C^m(\mathbb{R}^n \times [0,\infty))$ solves the initial-value problem

$$(11) \qquad \begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g, \ u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

We intend to derive an explicit formula for u in terms of g, h. The plan will be to study first the average of u over certain spheres. These averages, taken as functions of the time t and the radius r, turn out to solve the Euler–Poisson–Darboux equation, a PDE which we can for odd n convert into the ordinary one-dimensional wave equation. Applying d'Alembert's formula, or more precisely its variant (10), eventually leads us to a formula for the solution.

NOTATION. (i) Let $x \in \mathbb{R}^n$, $t > 0$, $r > 0$. Define

$$(12) \qquad U(x;r,t) := ⨍_{\partial B(x,r)} u(y,t)\, dS(y),$$

the average of $u(\cdot, t)$ over the sphere $\partial B(x,r)$.

(ii) Similarly,

$$(13) \qquad \begin{cases} G(x;r) := ⨍_{\partial B(x,r)} g(y)\, dS(y) \\ H(x;r) := ⨍_{\partial B(x,r)} h(y)\, dS(y). \end{cases}$$

For fixed x, we hereafter regard U as a function of r and t and discover a partial differential equation that U solves:

LEMMA 1 (Euler–Poisson–Darboux equation). *Fix $x \in \mathbb{R}^n$, and let u satisfy* (11). *Then $U \in C^m(\bar{\mathbb{R}}_+ \times [0,\infty))$ and*

$$(14) \qquad \begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ U = G, \ U_t = H & \text{on } \mathbb{R}_+ \times \{t = 0\}. \end{cases}$$

The partial differential equation in (14) is the *Euler–Poisson–Darboux equation.* (Note that the term $U_{rr} + \frac{n-1}{r} U_r$ is the radial part of the Laplacian Δ in polar coordinates.)

Proof. 1. As in the proof of Theorem 2 in §2.2.2 we compute for $r > 0$

$$(15) \qquad U_r(x;r,t) = \frac{r}{n} ⨍_{B(x,r)} \Delta u(y,t)\, dy.$$

From this equality we deduce $\lim_{r\to 0^+} U_r(x;r,t) = 0$. We next differentiate (15), to discover after some computations that

$$U_{rr}(x;r,t) = \fint_{\partial B(x,r)} \Delta u \, dS + \left(\frac{1}{n} - 1\right) \fint_{B(x,r)} \Delta u \, dy. \tag{16}$$

Thus $\lim_{r\to 0^+} U_{rr}(x;r,t) = \frac{1}{n}\Delta u(x,t)$. Using formula (16), we can similarly compute U_{rrr}, etc., and so verify that $U \in C^m(\bar{\mathbb{R}}_+ \times [0,\infty))$.

2. Continuing the calculation above, we see from (15) that

$$\begin{aligned} U_r &= \frac{r}{n} \fint_{B(x,r)} u_{tt} \, dy \quad \text{by (11)} \\ &= \frac{1}{n\alpha(n)} \frac{1}{r^{n-1}} \int_{B(x,r)} u_{tt} \, dy. \end{aligned}$$

Thus

$$r^{n-1} U_r = \frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt} \, dy,$$

and so

$$\begin{aligned} \left(r^{n-1} U_r\right)_r &= \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt} \, dS \\ &= r^{n-1} \fint_{\partial B(x,r)} u_{tt} \, dS = r^{n-1} U_{tt}. \end{aligned}$$ □

c. Solution for n = 3, 2, Kirchhoff's and Poisson's formulas. The plan in the ensuing subsections will be to transform the Euler–Poisson–Darboux equation (14) into the usual one-dimensional wave equation. As the full procedure is rather complicated, we pause here to handle the simpler cases $n = 3, 2$, in that order.

Solution for n = 3. Let us therefore hereafter take $n = 3$, and suppose $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ solves the initial-value problem (11). We recall the definitions (12), (13) of U, G, H and then set

$$\tilde{U} := rU, \tag{17}$$

$$\tilde{G} := rG, \ \tilde{H} := rH. \tag{18}$$

We now assert that $\tilde{U}$ solves

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ \tilde{U} = \tilde{G}, \ \tilde{U}_t = \tilde{H} & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ \tilde{U} = 0 & \text{on } \{r = 0\} \times (0,\infty). \end{cases} \tag{19}$$

Indeed

$$\begin{aligned}\tilde{U}_{tt} &= rU_{tt}\\ &= r\left[U_{rr} + \frac{2}{r}U_r\right] \quad \text{by (14), with } n = 3\\ &= rU_{rr} + 2U_r = (U + rU_r)_r = \tilde{U}_{rr}.\end{aligned}$$

Notice also that $\tilde{G}_{rr}(0) = 0$. Applying formula (10) to (19), we find for $0 \leq r \leq t$

$$\tilde{U}(x;r,t) = \frac{1}{2}[\tilde{G}(r+t) - \tilde{G}(t-r)] + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(y)\,dy. \tag{20}$$

Since (12) implies $u(x,t) = \lim_{r\to 0^+} U(x;r,t)$, we conclude from (17), (18), (20) that

$$\begin{aligned}u(x,t) &= \lim_{r\to 0^+} \frac{\tilde{U}(x;r,t)}{r}\\ &= \lim_{r\to 0^+}\left[\frac{\tilde{G}(t+r) - \tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(y)\,dy\right]\\ &= \tilde{G}'(t) + \tilde{H}(t).\end{aligned}$$

Owing then to (13), we deduce

$$u(x,t) = \frac{\partial}{\partial t}\left(t ⨍_{\partial B(x,t)} g\,dS\right) + t ⨍_{\partial B(x,t)} h\,dS. \tag{21}$$

But

$$⨍_{\partial B(x,t)} g(y)\,dS(y) = ⨍_{\partial B(0,1)} g(x+tz)\,dS(z);$$

and so

$$\begin{aligned}\frac{\partial}{\partial t}\left(⨍_{\partial B(x,t)} g\,dS\right) &= ⨍_{\partial B(0,1)} Dg(x+tz)\cdot z\,dS(z)\\ &= ⨍_{\partial B(x,t)} Dg(y)\cdot\left(\frac{y-x}{t}\right)\,dS(y).\end{aligned}$$

Returning to (21), we therefore conclude

$$u(x,t) = ⨍_{\partial B(x,t)} th(y) + g(y) + Dg(y)\cdot(y-x)\,dS(y) \quad (x \in \mathbb{R}^3,\ t > 0). \tag{22}$$

This is *Kirchhoff's formula* for the solution of the initial-value problem (11) in three dimensions.

Solution for n = 2. No transformation like (17) works to convert the Euler–Poisson–Darboux equation into the one-dimensional wave equation when $n = 2$. Instead we will take the initial-value problem (11) for $n = 2$ and simply regard it as a problem for $n = 3$, in which the third spatial variable x_3 does not appear.

Indeed, assuming $u \in C^2(\mathbb{R}^2 \times [0,\infty))$ solves (11) for $n = 2$, let us write

$$\bar{u}(x_1, x_2, x_3, t) := u(x_1, x_2, t). \tag{23}$$

Then (11) implies

$$\begin{cases} \bar{u}_{tt} - \Delta \bar{u} = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ \bar{u} = \bar{g},\ \bar{u}_t = \bar{h} & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases} \tag{24}$$

for

$$\bar{g}(x_1, x_2, x_3) := g(x_1, x_2),\ \bar{h}(x_1, x_2, x_3) := h(x_1, x_2).$$

If we write $x = (x_1, x_2) \in \mathbb{R}^2$ and $\bar{x} = (x_1, x_2, 0) \in \mathbb{R}^3$, then (24) and Kirchhoff's formula (in the form (21)) imply

$$\begin{aligned} u(x,t) &= \bar{u}(\bar{x},t) \\ &= \frac{\partial}{\partial t}\left(t \fint_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} \right) + t \fint_{\partial \bar{B}(\bar{x},t)} \bar{h}\, d\bar{S}, \end{aligned} \tag{25}$$

where $\bar{B}(\bar{x},t)$ denotes the ball in $\mathbb{R}^3$ with center $\bar{x}$, radius $t > 0$ and where $d\bar{S}$ denotes two-dimensional surface measure on $\partial \bar{B}(\bar{x},t)$. We simplify (25) by observing

$$\begin{aligned} \fint_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} &= \frac{1}{4\pi t^2} \int_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} \\ &= \frac{2}{4\pi t^2} \int_{B(x,t)} g(y)(1 + |D\gamma(y)|^2)^{1/2}\, dy, \end{aligned}$$

where $\gamma(y) = (t^2 - |y-x|^2)^{1/2}$ for $y \in B(x,t)$. The factor "2" enters since $\partial \bar{B}(\bar{x},t)$ consists of two hemispheres. Observe that $(1 + |D\gamma|^2)^{1/2} = t(t^2 - |y-x|^2)^{-1/2}$. Therefore

$$\begin{aligned} \fint_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} &= \frac{1}{2\pi t} \int_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}}\, dy \\ &= \frac{t}{2} \fint_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}}\, dy. \end{aligned}$$

Consequently formula (25) becomes

$$\text{(26)}\qquad \begin{aligned} u(x,t) &= \frac{1}{2}\frac{\partial}{\partial t}\left(t^2 \fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right) \\ &\quad + \frac{t^2}{2}\fint_{B(x,t)} \frac{h(y)}{(t^2-|y-x|^2)^{1/2}}\,dy. \end{aligned}$$

But

$$t^2 \fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy = t\fint_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{1/2}}\,dz,$$

and so

$$\begin{aligned} &\frac{\partial}{\partial t}\left(t^2 \fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right) \\ &= \fint_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{1/2}}\,dz + t\fint_{B(0,1)} \frac{Dg(x+tz)\cdot z}{(1-|z|^2)^{1/2}}\,dz \\ &= t\fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy + t\fint_{B(x,t)} \frac{Dg(y)\cdot(y-x)}{(t^2-|y-x|^2)^{1/2}}\,dy. \end{aligned}$$

Hence we can rewrite (26) and obtain the relation

$$\text{(27)}\qquad u(x,t) = \frac{1}{2}\fint_{B(x,t)} \frac{tg(y)+t^2h(y)+tDg(y)\cdot(y-x)}{(t^2-|y-x|^2)^{1/2}}\,dy$$

for $x \in \mathbb{R}^2$, $t > 0$. This is *Poisson's formula* for the solution of the initial-value problem (11) in two dimensions.

The trick of solving the problem for $n = 3$ first and then dropping to $n = 2$ is the *method of descent*.

d. Solution for odd n. In this subsection we solve the Euler–Poisson–Darboux PDE for odd $n \geq 3$. We first record some technical facts.

LEMMA 2 (Some useful identities). *Let $\phi : \mathbb{R} \to \mathbb{R}$ be C^{k+1}. Then for $k = 1, 2, \dots$*

(i) $\left(\frac{d^2}{dr^2}\right)\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\frac{d\phi}{dr}(r)\right)$,

(ii) $\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{d^j\phi}{dr^j}(r)$,
where the constants β_j^k $(j = 0, \dots, k-1)$ are independent of ϕ.

Furthermore,

(iii) $\beta_0^k = 1\cdot 3\cdot 5\cdots(2k-1)$.

The proof by induction is left as an exercise.

Now assume

$$n \geq 3 \text{ is an odd integer}$$

and set

$$n = 2k + 1 \quad (k \geq 1).$$

Henceforth suppose $u \in C^{k+1}(\mathbb{R}^n \times [0, \infty))$ solves the initial-value problem (11). Then the function U defined by (12) is C^{k+1}.

NOTATION. We write

$$(28) \qquad \begin{cases} \tilde{U}(r,t) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1}U(x;r,t)\right) \\ \tilde{G}(r) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1}G(x;r)\right) \\ \tilde{H}(r) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1}H(x;r)\right) \end{cases} \qquad (r > 0,\ t \geq 0).$$

Then

$$(29) \qquad \tilde{U}(r,0) = \tilde{G}(r),\ \tilde{U}_t(r,0) = \tilde{H}(r).$$

Next we combine Lemma 1 and the identities provided by Lemma 2 to demonstrate that the transformation (28) of U into $\tilde{U}$ in effect converts the Euler–Poisson–Darboux equation into the wave equation.

LEMMA 3 ($\tilde{U}$ solves the one-dimensional wave equation). *We have*

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ \tilde{U} = \tilde{G},\ \tilde{U}_t = \tilde{H} & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ \tilde{U} = 0 & \text{on } \{r = 0\} \times (0,\infty). \end{cases}$$

Proof. If $r > 0$,

$$\begin{aligned} \tilde{U}_{rr} &= \left(\frac{\partial^2}{\partial r^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(r^{2k-1}U) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k}(r^{2k}U_r) \qquad \text{by Lemma 2(i)} \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}[r^{2k-1}U_{rr} + 2kr^{2k-2}U_r] \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}\left(U_{rr} + \frac{n-1}{r}U_r\right)\right] \qquad (n = 2k+1) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(r^{2k-1}U_{tt}) = \tilde{U}_{tt}, \end{aligned}$$

the next-to-last equality holding according to (14). Using Lemma 2(ii) we conclude as well that $\tilde{U} = 0$ on $\{r = 0\}$. □

In view of Lemma 3, (29), and formula (10), we conclude for $0 \leq r \leq t$ that

$$\tilde{U}(r,t) = \frac{1}{2}[\tilde{G}(r+t) - \tilde{G}(t-r)] + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(y)\,dy \tag{30}$$

for all $r \in \mathbb{R}$, $t \geq 0$. But recall $u(x,t) = \lim_{r\to 0} U(x;r,t)$. Furthermore Lemma 2(ii) asserts

$$\begin{aligned}\tilde{U}(r,t) &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(r^{2k-1}U(x;r,t))\\ &= \sum_{j=0}^{k-1} \beta_j^k r^{j+1} \frac{\partial^j}{\partial r^j} U(x;r,t),\end{aligned}$$

and so

$$\lim_{r\to 0} \frac{\tilde{U}(r,t)}{\beta_0^k r} = \lim_{r\to 0} U(x;r,t) = u(x,t).$$

Thus (30) implies

$$\begin{aligned}u(x,t) &= \frac{1}{\beta_0^k}\lim_{r\to 0}\left[\frac{\tilde{G}(t+r) - \tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r}\tilde{H}(y)\,dy\right]\\ &= \frac{1}{\beta_0^k}[\tilde{G}'(t) + \tilde{H}(t)].\end{aligned}$$

Finally then, since $n = 2k+1$, (30) and Lemma 2(iii) yield this representation formula:

$$\left\{\begin{aligned} u(x,t) = \frac{1}{\gamma_n}\Bigg[&\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}⨍_{\partial B(x,t)} g\,dS\right)\\ &+ \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}⨍_{\partial B(x,t)} h\,dS\right)\Bigg]\\ &\text{where } n \text{ is } \textit{odd} \text{ and } \gamma_n = 1\cdot 3\cdot 5\cdots(n-2),\end{aligned}\right. \tag{31}$$

for $x \in \mathbb{R}^n$, $t > 0$.

We note that $\gamma_3 = 1$, and so (31) agrees for $n = 3$ with (21) and thus with Kirchhoff's formula (22).

It remains to check that formula (31) really provides a solution of (11).