**Example 5.** Let $H(p) = |p|^2$, $h \equiv 0$ in Example 3 above. Then

$$u'(x,t;a) = x \cdot a - t|a|^2.$$

We calculate the envelope by setting $D_a u' = x - 2ta = 0$. Hence $a = \frac{x}{2t}$, and so

$$v'(x,t) = x \cdot \frac{x}{2t} - t\left|\frac{x}{2t}\right|^2 = \frac{|x|^2}{4t} \quad (x \in \mathbb{R}^n,\ t > 0)$$

solves the Hamilton–Jacobi equation $v'_t + |Dv'|^2 = 0$. □

**Remark.** It is tempting to believe that once we can find as above a solution of (1) depending on an arbitrary function $h$, we have found all the solutions of (1). But this need not be so. Suppose our PDE has the structure

$$F(Du,u,x) = F_1(Du,u,x)F_2(Du,u,x) = 0.$$

If $u_1(x,a)$ is a complete integral of the PDE $F_1(Du,u,x) = 0$ and we succeed in finding a general integral corresponding to any function $h$, we will still have missed all the solutions of the PDE $F_2(Du,u,x) = 0$.

## 3.2. CHARACTERISTICS

### 3.2.1. Derivation of characteristic ODE.

We return to our basic nonlinear first-order PDE

$$F(Du,u,x) = 0 \quad \text{in } U, \tag{1}$$

subject now to the boundary condition

$$u = g \quad \text{on } \Gamma, \tag{2}$$

where $\Gamma \subseteq \partial U$ and $g : \Gamma \to \mathbb{R}$ are given. We hereafter suppose that $F, g$ are smooth functions.

We develop next the method of *characteristics*, which solves (1), (2) by converting the PDE into an appropriate system of ODE. This is the plan. Suppose $u$ solves (1), (2) and fix any point $x \in U$. We would like to calculate $u(x)$ by finding some curve lying within $U$, connecting $x$ with a point $x^0 \in \Gamma$ and along which we can compute $u$. Since (2) says $u = g$ on $\Gamma$, we know the value of $u$ at the one end $x^0$. We hope then to be able to calculate $u$ all along the curve, and so in particular at $x$.

**Finding the characteristic ODE.** How can we choose a path in $U$ so all this will work? Let us suppose the curve is described parametrically by the function $\mathbf{x}(s) = (x^1(s), \dots, x^n(s))$, the parameter $s$ lying in some subinterval $I \subseteq \mathbb{R}$. Assuming $u$ is a $C^2$ solution of (1), we define also

$$z(s) := u(\mathbf{x}(s)). \tag{3}$$

In addition, set

$$\mathbf{p}(s) := Du(\mathbf{x}(s)); \tag{4}$$

that is, $\mathbf{p}(s) = (p^1(s), \dots, p^n(s))$, where

$$p^i(s) = u_{x_i}(\mathbf{x}(s)) \quad (i = 1, \dots, n). \tag{5}$$

So $z(\cdot)$ gives the values of $u$ along the curve and $\mathbf{p}(\cdot)$ records the values of the gradient $Du$. We must choose the function $\mathbf{x}(\cdot)$ in such a way that we can compute $z(\cdot)$ and $\mathbf{p}(\cdot)$.

For this, first differentiate (5):

$$\dot{p}^i(s) = \sum_{j=1}^n u_{x_i x_j}(\mathbf{x}(s))\dot{x}^j(s) \quad \left(\cdot = \frac{d}{ds}\right). \tag{6}$$

This expression is not too promising, since it involves the second derivatives of $u$. On the other hand, we can also differentiate the PDE (1) with respect to $x_i$:

$$\sum_{j=1}^n F_{p_j}(Du, u, x)u_{x_j x_i} + F_z(Du, u, x)u_{x_i} + F_{x_i}(Du, u, x) = 0. \tag{7}$$

We are able to employ this identity to get rid of the second derivative terms in (6), provided we first set

$$\dot{x}^j(s) = F_{p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)) \quad (j = 1, \dots, n). \tag{8}$$

Assuming now (8) holds, we evaluate (7) at $x = \mathbf{x}(s)$, obtaining thereby from (3), (4) the identity:

$$\sum_{j=1}^n F_{p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s))u_{x_i x_j}(\mathbf{x}(s)) + F_z(\mathbf{p}(s), z(s), \mathbf{x}(s))p^i(s) + F_{x_i}(\mathbf{p}(s), z(s), \mathbf{x}(s)) = 0.$$

Substitute this expression and (8) into (6):

$$\tag{9} \begin{aligned} \dot{p}^i(s) = -F_{x_i}(\mathbf{p}(s), z(s), \mathbf{x}(s)) \\ - F_z(\mathbf{p}(s), z(s), \mathbf{x}(s))p^i(s) \quad (i = 1, \dots, n). \end{aligned}$$

Finally we differentiate (3):

$$\tag{10} \dot{z}(s) = \sum_{j=1}^{n} u_{x_j}(\mathbf{x}(s))\dot{x}^j(s) = \sum_{j=1}^{n} p^j(s)F_{p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)),$$

the second equality holding by (5) and (8).

**The characteristic equations.** We summarize by rewriting equations (8)–(10) in vector notation:

$$\tag{11} \begin{cases} \text{(a) } \dot{\mathbf{p}}(s) = -D_xF(\mathbf{p}(s), z(s), \mathbf{x}(s)) - D_zF(\mathbf{p}(s), z(s), \mathbf{x}(s))\mathbf{p}(s) \\ \text{(b) } \dot{z}(s) = D_pF(\mathbf{p}(s), z(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) \\ \text{(c) } \dot{\mathbf{x}}(s) = D_pF(\mathbf{p}(s), z(s), \mathbf{x}(s)). \end{cases}$$

Furthermore,

$$\tag{12} F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \equiv 0.$$

These identities hold for $s \in I$.

The important system (11) of $2n + 1$ first-order ODE comprises the *characteristic equations* of the nonlinear first-order PDE (1). The functions $\mathbf{p}(\cdot) = (p^1(\cdot), \dots, p^n(\cdot))$, $z(\cdot)$, $\mathbf{x}(\cdot) = (x^1(\cdot), \dots, x^n(\cdot))$ are called the *characteristics.* We will sometimes refer to $\mathbf{x}(\cdot)$ as the *projected characteristic*: it is the projection of the full characteristics $(\mathbf{p}(\cdot), z(\cdot), \mathbf{x}(\cdot)) \subset \mathbb{R}^{2n+1}$ onto the physical region $U \subset \mathbb{R}^n$.

We have proved:

**THEOREM 1** (Structure of characteristic ODE). *Let $u \in C^2(U)$ solve the nonlinear, first-order partial differential equation* (1) *in $U$. Assume $\mathbf{x}(\cdot)$ solves the* ODE (11)(c), *where $\mathbf{p}(\cdot) = Du(\mathbf{x}(\cdot))$, $z(\cdot) = u(\mathbf{x}(\cdot))$. Then $\mathbf{p}(\cdot)$ solves the* ODE (11)(a) *and $z(\cdot)$ solves the* ODE (11)(b), *for those $s$ such that $\mathbf{x}(s) \in U$.*

We still need to discover appropriate initial conditions for the system of ODE (11), in order that this theorem be useful. We accomplish this in §3.2.3 below.

**Remark.** The characteristic ODE are truly remarkable in that they form an exact system of equations for $\mathbf{x}(\cdot)$, $z(\cdot) = u(\mathbf{x}(\cdot))$, and $\mathbf{p}(\cdot) = Du(\mathbf{x}(\cdot))$, whenever $u$ is a smooth solution of the general nonlinear PDE (1). The key step in the derivation is our setting $\dot{\mathbf{x}} = D_pF$, so that—as explained above—the terms involving second derivatives drop out. We thereby obtain *closure* and in particular are not forced to introduce ODE for the second and higher derivatives of $u$.

### 3.2.2. Examples.

Before continuing our investigation of the characteristic equations (11), we pause to consider some special cases for which the structure of these equations is especially simple. We illustrate as well how we can sometimes actually solve the characteristic ODE and thereby explicitly compute solutions of certain first-order PDE, subject to appropriate boundary conditions.

**a. F linear.** Consider first the situation that our PDE (1) is linear and homogeneous and thus has the form

$$F(Du, u, x) = \mathbf{b}(x) \cdot Du(x) + c(x)u(x) = 0 \quad (x \in U). \tag{13}$$

Then $F(p, z, x) = \mathbf{b}(x) \cdot p + c(x)z$, and so

$$D_pF = \mathbf{b}(x).$$

In this circumstance equation (11)(c) becomes

$$\dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s)), \tag{14}$$

an ODE involving only the function $\mathbf{x}(\cdot)$. Furthermore equation (11)(b) becomes

$$\dot{z}(s) = \mathbf{b}(\mathbf{x}(s)) \cdot \mathbf{p}(s). \tag{15}$$

Then equation (12) simplifies (15), yielding

$$\dot{z}(s) = -c(\mathbf{x}(s))z(s). \tag{16}$$

This ODE is linear in $z(\cdot)$, once we know the function $\mathbf{x}(\cdot)$ by solving (14). In summary,

$$\begin{cases} \text{(a)} \quad \dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s)) \\ \text{(b)} \quad \dot{z}(s) = -c(\mathbf{x}(s))z(s) \end{cases} \tag{17}$$

comprise the characteristic equations for the linear, first-order PDE (13). (We will see later that the equation for $\mathbf{p}(\cdot)$ is not needed.) □

**Example 1.** We demonstrate the utility of equations (17) by explicitly solving the problem

$$\text{(18)}\qquad \begin{cases} x_1 u_{x_2} - x_2 u_{x_1} = u & \text{in } U \\ u = g & \text{on } \Gamma, \end{cases}$$

where $U$ is the quadrant $\{x_1 > 0, x_2 > 0\}$ and $\Gamma = \{x_1 > 0, x_2 = 0\} \subseteq \partial U$. The PDE in (18) is of the form (12), for $\mathbf{b} = (-x_2, x_1)$ and $c = -1$. Thus the equations (17) read

$$\text{(19)}\qquad \begin{cases} \dot{x}^1 = -x^2, \ \dot{x}^2 = x^1 \\ \dot{z} \ = z. \end{cases}$$

Accordingly we have

$$\begin{cases} x^1(s) = x^0 \cos s, \ x^2(s) = x^0 \sin s \\ z(s) = z^0 e^s = g(x^0)\, e^s, \end{cases}$$

where $x^0 \geq 0$, $0 \leq s \leq \frac{\pi}{2}$. Fix a point $(x_1, x_2) \in U$. We select $s > 0$, $x^0 > 0$ so that $(x_1, x_2) = (x^1(s), x^2(s)) = (x^0 \cos s, x^0 \sin s)$. That is, $x^0 = (x_1^2 + x_2^2)^{1/2}$, $s = \arctan\left(\frac{x_2}{x_1}\right)$. Therefore

$$u(x) = u(x^1(s), x^2(s)) = z(s) = g(x^0)\, e^s = g((x_1^2 + x_2^2)^{1/2})\, e^{\arctan\left(\frac{x_2}{x_1}\right)}.$$

□

**b. F quasilinear.** The partial differential equation (1) is quasilinear should it have the form

$$\text{(20)}\qquad F(Du, u, x) = \mathbf{b}(x, u(x)) \cdot Du(x) + c(x, u(x)) = 0.$$

In this circumstance $F(p, z, x) = \mathbf{b}(x, z) \cdot p + c(x, z)$, whence

$$D_p F = \mathbf{b}(x, z).$$

Hence equation (11)(c) reads

$$\dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s), z(s)),$$

and (11)(b) becomes

$$\dot{z}(s) = \mathbf{b}(\mathbf{x}(s), z(s)) \cdot \mathbf{p}(s) = -c(\mathbf{x}(s), z(s)), \qquad \text{by (12)}.$$

Consequently

$$\text{(21)}\qquad \begin{cases} \text{(a)} \quad \dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s), z(s)) \\ \text{(b)} \quad \dot{z}(s) = -c(\mathbf{x}(s), z(s)) \end{cases}$$

are the characteristic equations for the quasilinear first-order PDE (20). (Once again we do not require the equation for $\mathbf{p}(\cdot)$.) □

**Example 2.** The characteristic ODE (21) are in general difficult to solve, and so we work out in this example the simpler case of a boundary-value problem for a semilinear PDE:

$$\begin{cases} u_{x_1} + u_{x_2} = u^2 & \text{in } U \\ u = g & \text{on } \Gamma. \end{cases} \tag{22}$$

Now $U$ is the half-space $\{x_2 > 0\}$ and $\Gamma = \{x_2 = 0\} = \partial U$. Here $\mathbf{b} = (1,1)$ and $c = -z^2$. Then (21) becomes

$$\begin{cases} \dot{x}^1 = 1, \ \dot{x}^2 = 1 \\ \dot{z} = z^2. \end{cases}$$

Consequently

$$\begin{cases} x^1(s) = x^0 + s, \ x^2(s) = s \\ z(s) = \frac{z^0}{1-sz^0} = \frac{g(x^0)}{1-sg(x^0)}, \end{cases}$$

where $x^0 \in \mathbb{R}$, $s \geq 0$, provided the denominator is not zero.

Fix a point $(x_1, x_2) \in U$. We select $s > 0$ and $x^0 \in \mathbb{R}$ so that $(x_1, x_2) = (x^1(s), x^2(s)) = (x^0 + s, s)$; that is, $x^0 = x_1 - x_2$, $s = x_2$. Then

$$u(x) = u(x^1(s), x^2(s)) = z(s) = \frac{g(x^0)}{1 - sg(x^0)} = \frac{g(x_1 - x_2)}{1 - x_2 g(x_1 - x_2)}.$$

This solution of course makes sense only if $1 - x_2 g(x_1 - x_2) \neq 0$. □

**c. F fully nonlinear.** In the general case, we must integrate the full characteristic equations (11), if possible.

**Example 3.** Consider the fully nonlinear problem

$$\begin{cases} u_{x_1} u_{x_2} = u & \text{in } U \\ u = x_2^2 & \text{on } \Gamma, \end{cases} \tag{23}$$

where $U = \{x_1 > 0\}$, $\Gamma = \{x_1 = 0\} = \partial U$. Here $F(p, z, x) = p_1 p_2 - z$, and hence the characteristic ODE (11) become

$$\begin{cases} \dot{p}^1 = p^1, \ \dot{p}^2 = p^2 \\ \dot{z} = 2p^1 p^2 \\ \dot{x}^1 = p^2, \ \dot{x}^2 = p^1. \end{cases}$$

We integrate these equations to find

$$\begin{cases} x^1(s) = p_2^0(e^s - 1), \ x^2(s) = x^0 + p_1^0(e^s - 1) \\ z(s) = z^0 + p_1^0 p_2^0 (e^{2s} - 1) \\ p^1(s) = p_1^0 e^s, \ p^2(s) = p_2^0 e^s, \end{cases}$$

where $x^0 \in \mathbb{R}$, $s \in \mathbb{R}$, and $z^0 = (x^0)^2$.

We must determine $p^0 = (p_1^0, p_2^0)$. Since $u = x_2^2$ on $\Gamma$, $p_2^0 = u_{x_2}(0, x^0) = 2x^0$. Furthermore the PDE $u_{x_1}u_{x_2} = u$ itself implies $p_1^0 p_2^0 = z^0 = (x^0)^2$, and so $p_1^0 = \frac{x^0}{2}$. Consequently the formulas above become

$$\begin{cases} x^1(s) = 2x^0(e^s - 1), x^2(s) = \frac{x^0}{2}(e^s + 1) \\ z(s) = (x^0)^2 e^{2s} \\ p^1(s) = \frac{x^0}{2}\, e^s,\ p^2(s) = 2x^0 e^s. \end{cases}$$

Fix a point $(x_1, x_2) \in U$. Select $s$ and $x^0$ so that $(x_1, x_2) = (x^1(s), x^2(s)) = (2x^0(e^s - 1), \frac{x^0}{2}(e^s + 1))$. This equality implies $x^0 = \frac{4x_2 - x_1}{4}$, $e^s = \frac{x_1 + 4x_2}{4x_2 - x_1}$; and so

$$u(x) = u(x^1(s), x^2(s)) = z(s) = (x^0)^2 e^{2s} = \frac{(x_1 + 4x_2)^2}{16}.$$

□

### 3.2.3. Boundary conditions.

We return now to developing the general theory and intend in the section following to invoke the characteristic ODE (11) actually to solve the boundary-value problem (1), (2), at least in a small region near an appropriate portion $\Gamma$ of $\partial U$.

**a. Straightening the boundary.** To simplify subsequent calculations, it is convenient first to change variables, so as to "flatten out" part of the boundary $\partial U$. To accomplish this, we first fix any point $x^0 \in \partial U$. Then utilizing the notation from §C.1, we find smooth mappings $\mathbf{\Phi}, \mathbf{\Psi} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathbf{\Psi} = \mathbf{\Phi}^{-1}$ and $\mathbf{\Phi}$ straightens out $\partial U$ near $x^0$. (See the illustration in §C.1.)

Given any function $u : U \to \mathbb{R}$, let us write $V := \mathbf{\Phi}(U)$ and set

$$v(y) := u(\mathbf{\Psi}(y)) \quad (y \in V). \tag{24}$$

Then

$$u(x) = v(\mathbf{\Phi}(x)) \quad (x \in U). \tag{25}$$

Now suppose that $u$ is a $C^1$ solution of our boundary-value problem (1), (2) in $U$. What PDE does $v$ then satisfy in $V$?

According to (25), we see

$$u_{x_i}(x) = \sum_{k=1}^{n} v_{y_k}(\mathbf{\Phi}(x))\Phi^k_{x_i}(x) \quad (i = 1, \ldots, n);$$

that is,

$$Du(x) = Dv(y)D\mathbf{\Phi}(x).$$

Thus (1) implies

$$F(Dv(y)D\mathbf{\Phi}(\mathbf{\Psi}(y)), v(y), \mathbf{\Psi}(y)) = F(Du(x), u(x), x) = 0. \tag{26}$$

This is an expression having the form

$$G(Dv(y), v(y), y) = 0 \quad \text{in } V.$$

In addition $v = h$ on $\Delta$, where $\Delta := \mathbf{\Phi}(\Gamma)$ and $h(y) := g(\mathbf{\Psi}(y))$.

In summary, our problem (1), (2) transforms to read

$$\begin{cases} G(Dv, v, y) = 0 & \text{in } V \\ v = h & \text{on } \Delta, \end{cases} \tag{27}$$

for $G$, $h$ as above. The point is that if we change variables to straighten out the boundary near $x^0$, the boundary-value problem (1), (2) converts into a problem having the same form.

**b. Compatibility conditions on boundary data.** In view of the foregoing computations, if we are given a point $x^0 \in \Gamma$, *we may as well assume from the outset that* $\Gamma$ *is flat near* $x^0$, lying in the plane $\{x_n = 0\}$.

We intend now to utilize the characteristic ODE to construct a solution (1), (2), at least near $x^0$, and for this we must discover appropriate initial conditions

$$\mathbf{p}(0) = p^0, \; z(0) = z^0, \; \mathbf{x}(0) = x^0. \tag{28}$$

Now clearly if the curve $\mathbf{x}(\cdot)$ passes through $x^0$, we should insist that

$$z^0 = g(x^0). \tag{29}$$

What should we require concerning $\mathbf{p}(0) = p^0$? Since (2) implies $u(x_1, \dots x_{n-1}, 0) = g(x_1, \dots, x_{n-1})$ near $x^0$, we may differentiate to find

$$u_{x_i}(x^0) = g_{x_i}(x^0) \quad (i = 1, \dots, n-1).$$

As we also want the PDE (1) to hold, we should therefore insist $p^0 = (p_1^0, \dots, p_n^0)$ satisfies these relations:

$$\begin{cases} p_i^0 = g_{x_i}(x^0) & (i = 1, \dots, n-1) \\ F(p^0, z^0, x^0) = 0. \end{cases} \tag{30}$$

These identities provide $n$ equations for the $n$ quantities $p^0 = (p_1^0, \dots, p_n^0)$.

We call (29) and (30) the *compatibility conditions.* A triple $(p^0, z^0, x^0) \in \mathbb{R}^{2n+1}$ verifying (29), (30) is *admissible.* Note $z^0$ is uniquely determined by the boundary condition and our choice of the point $x^0$, but a vector $p^0$ satisfying (30) may not exist or may not be unique.

**c. Noncharacteristic boundary data.** So now assume as above that $x^0 \in \Gamma$, that $\Gamma$ near $x^0$ lies in the plane $\{x_n = 0\}$, and that the triple $(p^0, z^0, x^0)$ is admissible. We are planning to construct a solution $u$ of (1), (2) in $U$ near $x^0$ by integrating the characteristic ODE (11). So far we have ascertained $\mathbf{x}(0) = x^0$, $z(0) = z^0$, $\mathbf{p}(0) = p^0$ are appropriate boundary conditions for the characteristic ODE, with $\mathbf{x}(\cdot)$ intersecting $\Gamma$ at $x^0$. But we will need in fact to solve these ODE for *nearby* initial points as well and must consequently now ask if we can somehow appropriately perturb $(p^0, z^0, x^0)$, keeping the compatibility conditions.

In other words, given a point $y = (y_1, \ldots, y_{n-1}, 0) \in \Gamma$, with $y$ close to $x^0$, we intend to solve the characteristic ODE

$$(31) \quad \begin{cases} \text{(a) } \dot{\mathbf{p}}(s) = -D_x F(\mathbf{p}(s), z(s), \mathbf{x}(s)) - D_z F(\mathbf{p}(s), z(s), \mathbf{x}(s))\mathbf{p}(s) \\ \text{(b) } \dot{z}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) \\ \text{(c) } \dot{\mathbf{x}}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)), \end{cases}$$

with the initial conditions

$$(32) \quad \mathbf{p}(0) = \mathbf{q}(y), \;\; z(0) = g(y), \;\; \mathbf{x}(0) = y.$$

Our task then is to find a function $\mathbf{q}(\cdot) = (q^1(\cdot), \ldots, q^n(\cdot))$, so that

$$(33) \quad \mathbf{q}(x^0) = p^0$$

and $(\mathbf{q}(y), g(y), y)$ is admissible; that is, the compatibility conditions

$$(34) \quad \begin{cases} q^i(y) = g_{x_i}(y) \qquad (i = 1, \ldots, n-1) \\ F(\mathbf{q}(y), g(y), y) = 0 \end{cases}$$

hold for all $y \in \Gamma$ close to $x^0$.

**LEMMA 1** (Noncharacteristic boundary conditions)**.** *There exists a unique solution* $\mathbf{q}(\cdot)$ *of* (33), (34) *for all* $y \in \Gamma$ *sufficiently close to* $x^0$, *provided*

$$(35) \quad F_{p_n}(p^0, z^0, x^0) \neq 0.$$

We say the admissible triple $(p^0, z^0, x^0)$ is *noncharacteristic* if (35) holds. We henceforth assume this condition.

**Proof.** Our problem is to find $q^n(y)$ so that

$$F(\mathbf{q}(y), g(y), y) = 0,$$

where $q^i(y) = g_{x_i}(y)$ for $i = 1, \ldots, n-1$. Since $F(p^0, z^0, x^0) = 0$, the Implicit Function Theorem (§C.7) implies we can indeed locally and uniquely solve for $q^n(y)$, provided that the noncharacteristic condition (35) is valid. □

**General noncharacteristic condition.** If $\Gamma$ is not flat near $x^0$, the condition that $\Gamma$ be noncharacteristic reads

$$D_pF(p^0, z^0, x^0) \cdot \boldsymbol{\nu}(x^0) \neq 0, \tag{36}$$

$\boldsymbol{\nu}(x^0)$ denoting the outward unit normal to $\partial U$ at $x^0$. See Problem 7.

### 3.2.4. Local solution.

Remember that our aim is to use the characteristic ODE to build a solution $u$ of (1), (2), at least near $\Gamma$. So as before we select a point $x^0 \in \Gamma$ and, as shown in §3.2.3, may as well assume that near $x^0$ the surface $\Gamma$ is flat, lying in the plane $\{x_n = 0\}$. Suppose further that $(p^0, z^0, x^0)$ is an admissible triple of boundary data, which is noncharacteristic. According to Lemma 1 there is a function $\mathbf{q}(\cdot)$ so that $p^0 = \mathbf{q}(x^0)$ and the triple $(\mathbf{q}(y), g(y), y)$ is admissible, for all $y$ sufficiently close to $x^0$.

Given any such point $y = (y_1, \dots, y_{n-1}, 0)$, we solve the characteristic ODE (31), subject to initial conditions (32).

**NOTATION.** Let us write

$$\begin{cases} \mathbf{p}(s) = \mathbf{p}(y, s) = \mathbf{p}(y_1, \dots, y_{n-1}, s) \\ z(s) = z(y, s) = z(y_1, \dots, y_{n-1}, s) \\ \mathbf{x}(s) = \mathbf{x}(y, s) = \mathbf{x}(y_1, \dots, y_{n-1}, s) \end{cases}$$

to display the dependence of the solution of (31), (32) on $s$ and $y$. Also, we will henceforth when convenient regard $x^0$ as lying in $\mathbb{R}^{n-1}$. □

**LEMMA 2** (Local invertibility). *Assume we have the noncharacteristic condition $F_{p_n}(p^0, z^0, x^0) \neq 0$. Then there exist an open interval $I \subseteq \mathbb{R}$ containing 0, a neighborhood $W$ of $x^0$ in $\Gamma \subset \mathbb{R}^{n-1}$, and a neighborhood $V$ of $x^0$ in $\mathbb{R}^n$, such that for each $x \in V$ there exist unique $s \in I$, $y \in W$ such that*

$$x = \mathbf{x}(y, s).$$

*The mappings $x \mapsto s, y$ are $C^2$.*

**Proof.** We have $\mathbf{x}(x^0, 0) = x^0$. Consequently the Inverse Function Theorem (§C.6) gives the result, provided $\det D\mathbf{x}(x^0, 0) \neq 0$. Now

$$\mathbf{x}(y, 0) = (y, 0) \quad (y \in \Gamma);$$

and so if $i = 1, \dots, n-1$,

$$x^j_{y_i}(x^0, 0) = \begin{cases} \delta_{ij} & (j = 1, \dots, n-1) \\ 0 & (j = n). \end{cases}$$