Furthermore equation (31)(c) implies

$$x^j_s(x^0,0) = F_{p_j}(p^0,z^0,x^0).$$

Thus

$$D\mathbf{x}(x^0,0) = \begin{pmatrix} 1 & & 0 & F_{p_1}(p^0,z^0,x^0) \\ & \ddots & & \vdots \\ 0 & & 1 & \vdots \\ 0 & \cdots & 0 & F_{p_n}(p^0,z^0,x^0) \end{pmatrix}_{n\times n},$$

whence $\det D\mathbf{x}(x^0,0) \neq 0$ follows from the noncharacteristic condition (35). □

In view of Lemma 2 for each $x \in V$, we can locally uniquely solve the equation

(37) $$\begin{cases} x = \mathbf{x}(y,s), \\ \text{for } \; y = \mathbf{y}(x), \;\; s = s(x). \end{cases}$$

Finally, let us define

(38) $$\begin{cases} u(x) := z(\mathbf{y}(x), s(x)) \\ \mathbf{p}(x) := \mathbf{p}(\mathbf{y}(x), s(x)) \end{cases}$$

for $x \in V$ and s, y as in (37).

We come finally to our principal assertion, namely, that we can locally weave together the solutions of the characteristic ODE into a solution of the PDE.

THEOREM 2 (Local Existence Theorem). *The function u defined above is C^2 and solves the* PDE

$$F(Du(x), u(x), x) = 0 \quad (x \in V),$$

with the boundary condition

$$u(x) = g(x) \quad (x \in \Gamma \cap V).$$

Proof. 1. First of all, fix $y \in \Gamma$ close to x^0 and, as above, solve the characteristic ODE (31), (32) for $\mathbf{p}(s) = \mathbf{p}(y,s)$, $z(s) = z(y,s)$, and $\mathbf{x}(s) = \mathbf{x}(y,s)$.

2. We assert that if $y \in \Gamma$ is sufficiently close to x^0, then

(39) $$f(y,s) := F(\mathbf{p}(y,s), z(y,s), \mathbf{x}(y,s)) = 0 \quad (s \in I).$$

To see this, note

$$f(y,0) = F(\mathbf{p}(y,0), z(y,0), \mathbf{x}(y,0)) = F(\mathbf{q}(y), g(y), y) = 0, \tag{40}$$

by the compatibility condition (34). Furthermore

$$\begin{aligned} f_s(y,s) &= \sum_{j=1}^n F_{p_j}\dot{p}^j + F_z\dot{z} + \sum_{j=1}^n F_{x_j}\dot{x}^j \\ &= \sum_{j=1}^n F_{p_j}(-F_{x_j} - F_z p^j) + F_z(\sum_{j=1}^n F_{p_j}p^j) \\ &\qquad + \sum_{j=1}^n F_{x_j}(Fp_j) \quad \text{according to (31)} \\ &= 0. \end{aligned}$$

This calculation and (40) prove (39).

3. In view of Lemma 2 and (37)–(39), we have

$$F(\mathbf{p}(x), u(x), x) = 0 \quad (x \in V).$$

To conclude, we must therefore show

$$\mathbf{p}(x) = Du(x) \quad (x \in V). \tag{41}$$

In order to prove (41), let us first demonstrate for $s \in I$, $y \in W$ that

$$z_s(y,s) = \sum_{j=1}^n p^j(y,s)x_s^j(y,s) \tag{42}$$

and

$$z_{y_i}(y,s) = \sum_{j=1}^n p^j(y,s)x_{y_i}^j(y,s) \quad (i = 1, \dots, n-1). \tag{43}$$

These formulas are obviously consistent with the equality (41) and will later help us prove it. The identity (42) results at once from the characteristic ODE (31)(b),(c). To establish (43), fix $y \in \Gamma$, $i \in \{1, \dots, n-1\}$, and set

$$r^i(s) := z_{y_i}(y,s) - \sum_{j=1}^n p^j(y,s)x_{y_i}^j(y,s). \tag{44}$$

We first note $r^i(0) = g_{x_i}(y) - q^i(y) = 0$ according to the compatibility condition (34). In addition, we can compute

$$\dot{r}^i(s) = z_{y_i s} - \sum_{j=1}^{n} p_s^j x_{y_i}^j + p^j x_{y_i s}^j. \tag{45}$$

To simplify this expression, let us first differentiate the identity (42) with respect to y_i:

$$z_{s y_i} = \sum_{j=1}^{n} p_{y_i}^j x_s^j + p^j x_{s y_i}^j. \tag{46}$$

Substituting (46) into (45), we discover

$$\dot{r}^i(s) = \sum_{j=1}^{n} p_{y_i}^j x_s^j - p_s^j x_{y_i}^j = \sum_{j=1}^{n} p_{y_i}^j F_{p_j} - (-F_{x_j} - F_z p^j) x_{y_i}^j, \tag{47}$$

according to (31)(a). Now differentiate (39) with respect to y_i:

$$\sum_{j=1}^{n} F_{p_j} p_{y_i}^j + F_z z_{y_i} + \sum_{j=1}^{n} F_{x_j} x_{y_i}^j = 0.$$

We employ this identity in (47), thereby obtaining

$$\dot{r}^i(s) = F_z\Big(\sum_{j=1}^{n} p^j x_{y_i}^j - z_{y_i}\Big) = -F_z r^i(s). \tag{48}$$

Hence $r^i(\cdot)$ solves the linear ODE (48), with the initial condition $r^i(0) = 0$. Consequently $r^i(s) = 0$ ($s \in I$, $i = 1, \dots, n-1$), and so identity (43) is verified.

4. We finally employ (42), (43) in proving (41). Indeed, if $j = 1, \dots, n$,

$$\begin{aligned}
u_{x_j} &= z_s s_{x_j} + \sum_{i=1}^{n-1} z_{y_i} y_{x_j}^i \qquad \text{by (38)} \\
&= \sum_{k=1}^{n} p^k x_s^k s_{x_j} + \sum_{i=1}^{n-1} \sum_{k=1}^{n} p^k x_{y_i}^k y_{x_j}^i \qquad \text{by (42), (43)} \\
&= \sum_{k=1}^{n} p^k \Big(x_s^k s_{x_j} + \sum_{i=1}^{n-1} x_{y_i}^k y_{x_j}^i\Big) \\
&= \sum_{k=1}^{n} p^k x_{x_j}^k = \sum_{k=1}^{n} p^k \delta_{jk} = p^j.
\end{aligned}$$

This assertion at last establishes (41) and so finishes up the proof. □

3.2.5. Applications.

We turn now to various special cases, to see how the local existence theory simplifies in these circumstances.

a. F linear. Recall that a linear, homogeneous, first-order PDE has the form

$$F(Du,u,x) = \mathbf{b}(x) \cdot Du(x) + c(x)u(x) = 0 \quad (x \in U). \tag{49}$$

Our noncharacteristic assumption (36) at a point $x^0 \in \Gamma$ as above becomes

$$\mathbf{b}(x^0) \cdot \boldsymbol{\nu}(x^0) \neq 0 \tag{50}$$

and thus does not involve z^0 or p^0 at all. Furthermore if we specify the boundary condition

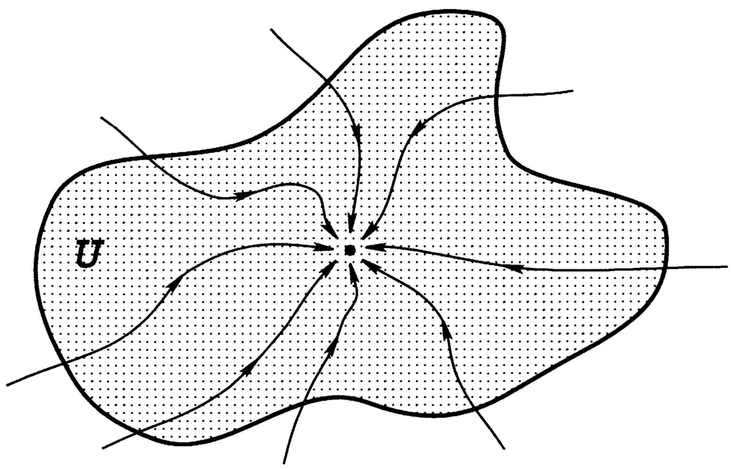
$$u = g \quad \text{on } \Gamma, \tag{51}$$

we can uniquely solve equation (34) for $\mathbf{q}(y)$ if $y \in \Gamma$ is near x^0. Thus we can apply the Local Existence Theorem 2 to construct a unique solution of (49), (51) in some neighborhood V containing x^0. Note carefully that although we have utilized the full characteristic equations (31) in the proof of Theorem 2, once we know the solution exists, we can use the reduced equations (17) (which do not involve $\mathbf{p}(\cdot)$) to compute the solution. Observe also that the projected characteristics $\mathbf{x}(\cdot)$ emanating from distinct points on Γ cannot cross, owing to uniqueness of solutions of the initial-value problem for the ODE (17)(a).

Example 4. Suppose the trajectories of the ODE

$$\dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s)) \tag{52}$$

are as drawn for Case 1. We are thus assuming the vector field $\mathbf{b}$ vanishes within U only at one point, which we will take to be the origin 0, and $\mathbf{b}\cdot\boldsymbol{\nu} < 0$ on $\Gamma := \partial U$.

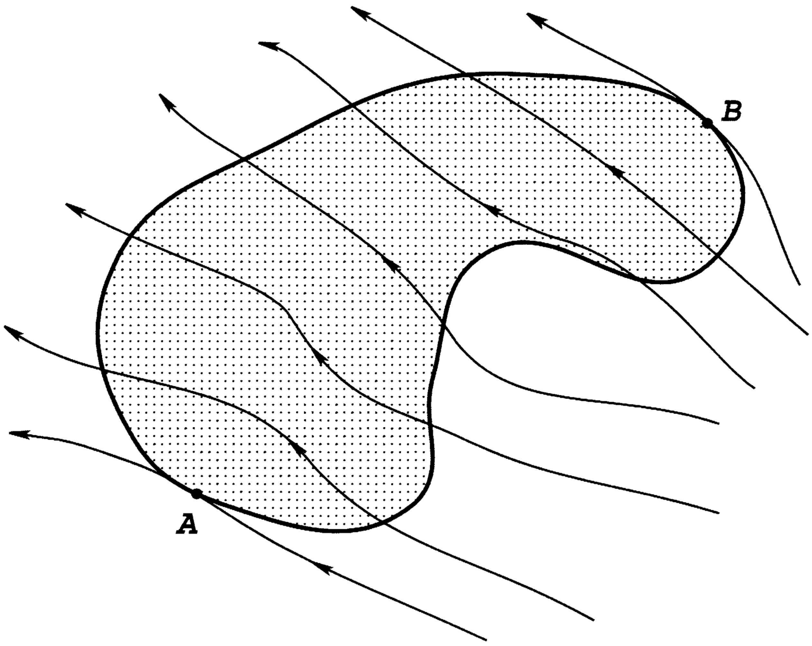


Case 1: flow to an attracting point

Can we solve the linear boundary-value problem

$$\begin{cases} \mathbf{b} \cdot Du = 0 & \text{in } U \\ u = g & \text{on } \Gamma \text{ ?} \end{cases} \tag{53}$$

Invoking Theorem 2, we see that there exists a unique solution u defined *near* Γ and indeed that $u(\mathbf{x}(s)) \equiv u(\mathbf{x}(0)) = g(x^0)$ for each solution of the ODE (52), with the initial condition $\mathbf{x}(0) = x^0 \in \Gamma$. However, this solution cannot be smoothly continued to all of U (unless g is constant): any smooth solution of (53) is constant on trajectories of (52) and thus takes on different values near $x = 0$.



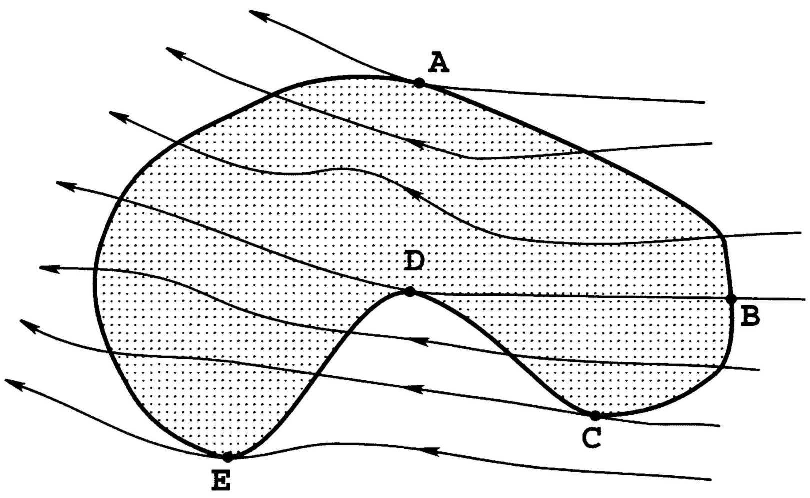
Case 2: flow across a domain

But now suppose the trajectories of the ODE (52) look like the illustration for Case 2. We are assuming that each trajectory of the ODE (except those through the characteristic points A, B) enters U precisely once, somewhere through the set

$$\Gamma := \{x \in \partial U \mid \mathbf{b}(x) \cdot \boldsymbol{\nu}(x) < 0\},$$

and exits U precisely once. In this circumstance we can find a smooth solution of (53) by setting u to be constant along each flow line.

Assume finally the flow looks like Case 3. We can now define u to be constant along trajectories, but then u will be discontinuous (unless $g(B) = g(D)$). Note that the point D is characteristic and that the local existence theory fails near D. □



Case 3: flow with characteristic points

b. F quasilinear. Should F be quasilinear, the PDE (1) is

$$F(Du, u, x) = \mathbf{b}(x, u) \cdot Du + c(x, u) = 0. \tag{54}$$

The noncharacteristic assumption (36) at a point $x^0 \in \Gamma$ reads $\mathbf{b}(x^0, z^0) \cdot \boldsymbol{\nu}(x^0) \neq 0$, where $z^0 = g(x^0)$. As in the preceding example, if we specify the boundary condition

$$u = g \quad \text{on } \Gamma, \tag{55}$$

we can uniquely solve the equations (34) for $\mathbf{q}(y)$ if $y \in \Gamma$ near x^0. Thus Theorem 2 yields the existence of a unique solution of (54), (55) in some neighborhood V of x^0. We can compute this solution in V using the reduced characteristic equations (21), which do not explicitly involve $\mathbf{p}(\cdot)$.

In contrast to the linear case, however, *it is possible that the projected characteristics emanating from distinct points in* Γ *may intersect outside* V; such an occurrence usually signals the failure of our local solution to exist within all of U.

Example 5 (Characteristics for conservation laws). As an instance of a quasilinear first-order PDE, we turn now to the *scalar conservation law*

$$\begin{aligned} G(Du, u_t, u, x, t) &= u_t + \operatorname{div} \mathbf{F}(u) \\ &= u_t + \mathbf{F}'(u) \cdot Du = 0 \end{aligned} \tag{56}$$

in $U = \mathbb{R}^n \times (0, \infty)$, subject to the initial condition

$$u = g \quad \text{on } \Gamma = \mathbb{R}^n \times \{t = 0\}. \tag{57}$$

Here $\mathbf{F} : \mathbb{R} \to \mathbb{R}^n$, $\mathbf{F} = (F^1, \dots, F^n)$, and, as usual, we have set $t = x_{n+1}$. Also, "div" denotes the divergence with respect to the spatial variables $(x_1, \dots, x_n)$, and $Du = D_x u = (u_{x_1}, \dots, u_{x_n})$.

Since the direction $t = x_{n+1}$ plays a special role, we appropriately modify our notation. Writing now $q = (p, p_{n+1})$ and $y = (x, t)$, we have

$$G(q, z, y) = p_{n+1} + \mathbf{F}'(z) \cdot p,$$

and consequently

$$D_q G = (\mathbf{F}'(z), 1),\ D_y G = 0,\ D_z G = \mathbf{F}''(z) \cdot p.$$

Clearly the noncharacteristic condition (35) is satisfied at each point $y^0 = (x^0, 0) \in \Gamma$. Furthermore equation (21)(a) becomes

$$\begin{cases} \dot{x}^i(s) = F^{i\,\prime}(z(s)) \quad (i = 1, \dots, n) \\ \dot{x}^{n+1}(s) = 1. \end{cases} \tag{58}$$

Hence $x^{n+1}(s) = s$, in agreement with our having written $x_{n+1} = t$ above. In other words, we can identify the parameter s with the time t.

Equation (21)(b) reads $\dot{z}(s) = 0$. Consequently

$$z(s) = z^0 = g(x^0); \tag{59}$$

and (58) implies

$$\mathbf{x}(s) = \mathbf{F}'(g(x^0))s + x^0. \tag{60}$$

Thus the projected characteristic $\mathbf{y}(s) = (\mathbf{x}(s), s) = (\mathbf{F}'(g(x^0))s + x^0, s)$ $(s \geq 0)$ is a straight line, along which u is constant.

Crossing characteristics. But suppose now we apply the same reasoning to a different initial point $z^0 \in \Gamma$, where $g(x^0) \neq g(z^0)$. *The projected characteristics may possibly then intersect at some time* $t > 0$. Since Theorem 1 tells us $u \equiv g(x^0)$ on the projected characteristic through x^0 and $u \equiv g(z^0)$ on the projected characteristic through z^0, an apparent contradiction arises. The resolution is that *the initial-value problem* (56), (57) *does not in general have a smooth solution, existing for all times* $t > 0$. □

We will discuss in §3.4 the interesting possibility of extending the local solution (guaranteed to exist for short times by Theorem 2) to all times $t > 0$, as a kind of "weak" or "generalized" solution.

An implicit formula. We can eliminate s from equations (59), (60) to derive an implicit formula for u. Indeed given $x \in \mathbb{R}^n$ and $t > 0$, we see that since $s = t$,

$$u(\mathbf{x}(t), t) = z(t) = g(\mathbf{x}(t) - t\mathbf{F}'(z^0)) = g(\mathbf{x}(t) - t\mathbf{F}'(u(\mathbf{x}(t), t))).$$

Hence

$$u = g(x - t\mathbf{F}'(u)). \tag{61}$$

This implicit formula for u as a function of x and t is a nonlinear analogue of equation (3) in §2.1. It is easy to check that (61) does indeed give a solution, provided

$$1 + tDg(x - t\mathbf{F}'(u)) \cdot \mathbf{F}''(u) \neq 0.$$

In particular if $n = 1$, we require

$$1 + tg'(x - tF'(u))F''(u) \neq 0.$$

Note that if $F'' > 0$, but $g' < 0$, then this will definitely be false at some time $t > 0$. This failure of the implicit formula (61) reflects also the failure of the characteristic method. □

c. F fully nonlinear. The form of the full characteristic equations can be quite complicated for fully nonlinear first-order PDE, but sometimes a remarkable mathematical structure emerges.

Example 6 (Characteristics for the Hamilton–Jacobi equation). We look now at the general Hamilton–Jacobi PDE

$$G(Du, u_t, u, x, t) = u_t + H(Du, x) = 0, \tag{62}$$

where $Du = D_x u = (u_{x_1}, \ldots, u_{x_n})$. Then writing $q = (p, p_{n+1})$, $y = (x, t)$, we have

$$G(q, z, y) = p_{n+1} + H(p, x);$$

and so

$$D_q G = (D_p H(p, x), 1),\ D_y G = (D_x H(p, x), 0),\ D_z G = 0.$$

Thus equation (11)(c) becomes

$$\begin{cases} \dot{x}^i(s) = H_{p_i}(\mathbf{p}(s), \mathbf{x}(s)) \quad (i = 1, \ldots, n) \\ \dot{x}^{n+1}(s) = 1. \end{cases} \tag{63}$$

In particular we can identify the parameter s with the time t. Equation (11)(a) for the case at hand reads

$$\begin{cases} \dot{p}^i(s) = -H_{x_i}(\mathbf{p}(s), \mathbf{x}(s)) \quad (i = 1, \ldots, n) \\ \dot{p}^{n+1}(s) = 0; \end{cases}$$

the equation (11)(b) is

$$\begin{aligned} \dot{z}(s) &= D_p H(\mathbf{p}(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) + p^{n+1}(s) \\ &= D_p H(\mathbf{p}(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) - H(\mathbf{p}(s), \mathbf{x}(s)). \end{aligned}$$

In summary, the characteristic equations for the Hamilton–Jacobi equation are

$$(64) \qquad \begin{cases} \text{(a)} \quad \dot{\mathbf{p}}(s) = -D_x H(\mathbf{p}(s), \mathbf{x}(s)) \\ \text{(b)} \quad \dot{z}(s) = D_p H(\mathbf{p}(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) - H(\mathbf{p}(s), \mathbf{x}(s)) \\ \text{(c)} \quad \dot{\mathbf{x}}(s) = D_p H(\mathbf{p}(s), \mathbf{x}(s)) \end{cases}$$

for $\mathbf{p}(\cdot) = (p^1(\cdot), \ldots, p^n(\cdot))$, $z(\cdot)$, and $\mathbf{x}(\cdot) = (x^1(\cdot), \ldots, x^n(\cdot))$.

The first and third of these equalities,

$$(65) \qquad \begin{cases} \dot{\mathbf{x}} = D_p H(\mathbf{p}, \mathbf{x}) \\ \dot{\mathbf{p}} = -D_x H(\mathbf{p}, \mathbf{x}), \end{cases}$$

are called *Hamilton's equations*. We will discuss these ODE and their relationship to the Hamilton–Jacobi equation in much more detail, just below in §3.3. Observe that the equation for $z(\cdot)$ is trivial, once $\mathbf{x}(\cdot)$ and $\mathbf{p}(\cdot)$ have been found by solving Hamilton's equations. □

As for conservation laws (Example 5), the initial-value problem for the Hamilton–Jacobi equation does not in general have a smooth solution u lasting for all times $t > 0$.

3.3. INTRODUCTION TO HAMILTON–JACOBI EQUATIONS

In this section we study in some detail the initial-value problem for the Hamilton–Jacobi equation:

$$(1) \qquad \begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Here $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is the unknown, $u = u(x, t)$, and $Du = D_x u = (u_{x_1}, \ldots, u_{x_n})$. We are given the *Hamiltonian* $H : \mathbb{R}^n \to \mathbb{R}$ and the initial function $g : \mathbb{R}^n \to \mathbb{R}$.