Proof. 1. Fix $y \in \mathbb{R}^n$, $0 < s < t$ and choose $z \in \mathbb{R}^n$ so that

$$u(y,s) = sL\left(\frac{y-z}{s}\right) + g(z). \tag{23}$$

Now since L is convex and $\frac{x-z}{t} = \left(1 - \frac{s}{t}\right)\frac{x-y}{t-s} + \frac{s}{t}\frac{y-z}{s}$, we have

$$L\left(\frac{x-z}{t}\right) \leq \left(1 - \frac{s}{t}\right) L\left(\frac{x-y}{t-s}\right) + \frac{s}{t} L\left(\frac{y-z}{s}\right).$$

Thus

$$\begin{aligned} u(x,t) \leq tL\left(\frac{x-z}{t}\right) + g(z) &\leq (t-s)L\left(\frac{x-y}{t-s}\right) + sL\left(\frac{y-z}{s}\right) + g(z) \\ &= (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s), \end{aligned}$$

by (23). This inequality is true for each $y \in \mathbb{R}^n$. Therefore, since $y \mapsto u(y,s)$ is continuous (according to the first part of the proof Lemma 2 below), we have

$$u(x,t) \leq \min_{y \in \mathbb{R}^n} \left\{ (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}. \tag{24}$$

2. Now choose w such that

$$u(x,t) = tL\left(\frac{x-w}{t}\right) + g(w), \tag{25}$$

and set $y := \frac{s}{t}x + \left(1 - \frac{s}{t}\right) w$. Then $\frac{x-y}{t-s} = \frac{x-w}{t} = \frac{y-w}{s}$. Consequently

$$\begin{aligned} &(t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s) \\ &\qquad \leq (t-s)L\left(\frac{x-w}{t}\right) + sL\left(\frac{y-w}{s}\right) + g(w) \\ &\qquad = tL\left(\frac{x-w}{t}\right) + g(w) = u(x,t), \end{aligned}$$

by (25). Hence

$$\min_{y \in \mathbb{R}^n} \left\{ (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\} \leq u(x,t). \tag{26}$$

□

LEMMA 2 (Lipschitz continuity). *The function u is Lipschitz continuous in $\mathbb{R}^n \times [0,\infty)$, and*

$$u = g \quad \text{on } \mathbb{R}^n \times \{t=0\}.$$

Proof. 1. Fix $t>0$, $x,\hat{x} \in \mathbb{R}^n$. Choose $y \in \mathbb{R}^n$ such that

$$tL\left(\frac{x-y}{t}\right) + g(y) = u(x,t). \tag{27}$$

Then

$$\begin{aligned} u(\hat{x},t) - u(x,t) &= \min_z \left\{ tL\left(\frac{\hat{x}-z}{t}\right) + g(z)\right\} - tL\left(\frac{x-y}{t}\right) - g(y) \\ &\leq g(\hat{x}-x+y) - g(y) \leq \mathrm{Lip}(g)|\hat{x}-x|. \end{aligned}$$

Hence

$$u(\hat{x},t) - u(x,t) \leq \mathrm{Lip}(g)|\hat{x}-x|;$$

and, interchanging the roles of $\hat{x}$ and x, we find

$$|u(x,t) - u(\hat{x},t)| \leq \mathrm{Lip}(g)|x-\hat{x}|. \tag{28}$$

2. Now select $x \in \mathbb{R}^n$, $t>0$. Choosing $y=x$ in (21), we discover

$$u(x,t) \leq tL(0) + g(x). \tag{29}$$

Furthermore,

$$\begin{aligned} u(x,t) &= \min_{y\in\mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y)\right\} \\ &\geq g(x) + \min_{y\in\mathbb{R}^n} \left\{ -\mathrm{Lip}(g)|x-y| + tL\left(\frac{x-y}{t}\right)\right\} \\ &= g(x) - t\max_{z\in\mathbb{R}^n}\{\mathrm{Lip}(g)|z| - L(z)\} \quad \left(z = \frac{x-y}{t}\right) \\ &= g(x) - t \max_{w\in B(0,\mathrm{Lip}(g))} \max_{z\in\mathbb{R}^n}\{w\cdot z - L(z)\} \\ &= g(x) - t \max_{B(0,\mathrm{Lip}(g))} H. \end{aligned}$$

This inequality and (29) imply

$$|u(x,t) - g(x)| \leq Ct$$

for

$$C := \max(|L(0)|, \max_{B(0,\mathrm{Lip}(g))} |H|). \tag{30}$$

3. Finally select $x \in \mathbb{R}^n$, $0 < \hat{t} < t$. Then $\mathrm{Lip}(u(\cdot, t)) \leq \mathrm{Lip}(g)$ by (28) above. Consequently Lemma 1 and calculations like those employed in step 2 above imply

$$|u(x,t) - u(x,\hat{t})| \leq C|t - \hat{t}|$$

for the constant C defined by (30). □

Now Rademacher's Theorem (which we will prove later, in §5.8.3) asserts that a Lipschitz function is differentiable almost everywhere. Consequently in view of Lemma 2 our function u defined by the Hopf–Lax formula (21) is differentiable for a.e. $(x,t) \in \mathbb{R}^n \times (0,\infty)$. The next theorem asserts u in fact solves the Hamilton–Jacobi PDE wherever u is differentiable.

THEOREM 5 (Solving the Hamilton–Jacobi equation). *Suppose $x \in \mathbb{R}^n$, $t > 0$, and u defined by the Hopf–Lax formula* (21) *is differentiable at a point $(x,t) \in \mathbb{R}^n \times (0,\infty)$. Then*

$$u_t(x,t) + H(Du(x,t)) = 0.$$

Proof. 1. Fix $v \in \mathbb{R}^n$, $h > 0$. Owing to Lemma 1,

$$\begin{aligned} u(x+hv, t+h) &= \min_{y \in \mathbb{R}^n} \left\{ hL\left(\frac{x+hv-y}{h}\right) + u(y,t) \right\} \\ &\leq hL(v) + u(x,t). \end{aligned}$$

Hence

$$\frac{u(x+hv,t+h) - u(x,t)}{h} \leq L(v).$$

Let $h \to 0^+$, to compute

$$v \cdot Du(x,t) + u_t(x,t) \leq L(v).$$

This inequality is valid for all $v \in \mathbb{R}^n$, and so

$$u_t(x,t) + H(Du(x,t)) = u_t(x,t) + \max_{v \in \mathbb{R}^n}\{v \cdot Du(x,t) - L(v)\} \leq 0. \tag{31}$$

The first equality holds since $H = L^*$.

2. Now choose z such that $u(x,t) = tL\left(\frac{x-z}{t}\right) + g(z)$. Fix $h > 0$ and set $s = t - h$, $y = \frac{s}{t}x + \left(1 - \frac{s}{t}\right)z$. Then $\frac{x-z}{t} = \frac{y-z}{s}$, and thus

$$\begin{aligned} u(x,t) - u(y,s) &\geq tL\left(\frac{x-z}{t}\right) + g(z) - \left[sL\left(\frac{y-z}{s}\right) + g(z)\right] \\ &= (t-s)L\left(\frac{x-z}{t}\right). \end{aligned}$$

That is,

$$\frac{u(x,t) - u((1-\frac{h}{t})\,x + \frac{h}{t}z, t-h)}{h} \geq L\left(\frac{x-z}{t}\right).$$

Let $h \to 0^+$, to see that

$$\frac{x-z}{t} \cdot Du(x,t) + u_t(x,t) \geq L\left(\frac{x-z}{t}\right).$$

Consequently

$$\begin{aligned} u_t(x,t) + H(Du(x,t)) &= u_t(x,t) + \max_{v\in\mathbb{R}^n}\{v \cdot Du(x,t) - L(v)\} \\ &\geq u_t(x,t) + \frac{x-z}{t} \cdot Du(x,t) - L\left(\frac{x-z}{t}\right) \\ &\geq 0. \end{aligned}$$

This inequality and (31) complete the proof. □

We summarize:

THEOREM 6 (Hopf–Lax formula as solution). *The function u defined by the Hopf–Lax formula* (21) *is Lipschitz continuous, is differentiable a.e. in* $\mathbb{R}^n \times (0,\infty)$, *and solves the initial-value problem*

$$\text{(32)} \qquad \begin{cases} u_t + H(Du) = 0 & \textit{a.e. in } \mathbb{R}^n \times (0,\infty) \\ u = g & \textit{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

3.3.3. Weak solutions, uniqueness.

a. Semiconcavity. In view of Theorem 6 above it may seem reasonable to define a weak solution of the initial-value problem (18) to be a Lipschitz function which agrees with g on $\mathbb{R}^n \times \{t=0\}$ and solves the PDE a.e. on $\mathbb{R}^n \times (0,\infty)$. However this turns out to be an inadequate definition, *as such weak solutions would not in general be unique.*

Example. Consider the initial-value problem

$$\text{(33)} \qquad \begin{cases} u_t + |u_x|^2 = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R} \times \{t=0\}. \end{cases}$$

One obvious solution is

$$u_1(x,t) \equiv 0.$$

However the function

$$u_2(x,t) := \begin{cases} 0 & \text{if } |x| \geq t \\ x-t & \text{if } 0 \leq x \leq t \\ -x-t & \text{if } -t \leq x \leq 0 \end{cases}$$

is Lipschitz continuous and also solves the PDE a.e. (everywhere, in fact, except on the lines $x = 0, \pm t$). It is easy to see that actually there are infinitely many Lipschitz functions satisfying (33). □

This example shows we must presumably require more of a weak solution than merely that it satisfy the PDE a.e. We will look to the Hopf–Lax formula (21) for a further clue as to what is needed to ensure uniqueness. The following lemma demonstrates that u inherits a kind of "one-sided" second-derivative estimate from the initial function g.

LEMMA 3 (Semiconcavity). *Suppose there exists a constant C such that*

$$g(x+z) - 2g(x) + g(x-z) \leq C|z|^2 \tag{34}$$

for all $x, z \in \mathbb{R}^n$. Define u by the Hopf–Lax formula (21). *Then*

$$u(x+z,t) - 2u(x,t) + u(x-z,t) \leq C|z|^2$$

for all $x, z \in \mathbb{R}^n$, $t > 0$.

We say g is *semiconcave* provided (34) holds. It is easy to check that (34) is valid if g is C^2 and $\sup_{\mathbb{R}^n} |D^2 g| < \infty$. Note that g is semiconcave if and only if the mapping $x \mapsto g(x) - \frac{C}{2}|x|^2$ is concave for some constant C.

Proof. Choose $y \in \mathbb{R}^n$ so that $u(x,t) = tL\left(\frac{x-y}{t}\right) + g(y)$. Then, putting $y+z$ and $y-z$ in the Hopf–Lax formulas for $u(x+z,t)$ and $u(x-z,t)$, we find

$$\begin{aligned}
u(x+z,t) &- 2u(x,t) + u(x-z,t) \\
&\leq \left[tL\left(\frac{x-y}{t}\right) + g(y+z)\right] - 2\left[tL\left(\frac{x-y}{t}\right) + g(y)\right] \\
&\quad + \left[tL\left(\frac{x-y}{t}\right) + g(y-z)\right] \\
&= g(y+z) - 2g(y) + g(y-z) \\
&\leq C|z|^2, \quad \text{by (34).}
\end{aligned}$$

□

As a semiconcavity condition for u will turn out to be important, we pause to identify some other circumstances under which it is valid. We will no longer assume g to be semiconcave but will suppose the Hamiltonian H to be uniformly convex.

DEFINITION. *A* C^2 *convex function* $H : \mathbb{R}^n \to \mathbb{R}$ *is called* uniformly convex *(with constant* $\theta > 0$*) if*

$$\sum_{i,j=1}^{n} H_{p_i p_j}(p)\xi_i \xi_j \geq \theta|\xi|^2 \quad \text{for all } p, \xi \in \mathbb{R}^n. \tag{35}$$

We now prove that even if g is not semiconcave, the uniform convexity of H forces u to become semiconcave for times $t > 0$: this is a kind of mild regularizing effect for the Hopf–Lax solution of the initial-value problem (18).

LEMMA 4 (Semiconcavity again). *Suppose that* H *is uniformly convex (with constant* θ*) and* u *is defined by the Hopf–Lax formula* (21). *Then*

$$u(x+z,t) - 2u(x,t) + u(x-z,t) \leq \frac{1}{\theta t}|z|^2$$

for all $x, z \in \mathbb{R}^n$, $t > 0$.

Proof. 1. We note first using Taylor's formula that (35) implies

$$H\left(\frac{p_1+p_2}{2}\right) \leq \frac{1}{2}H(p_1) + \frac{1}{2}H(p_2) - \frac{\theta}{8}|p_1 - p_2|^2. \tag{36}$$

Next we claim that for the Lagrangian L we have the estimate

$$\frac{1}{2}L(v_1) + \frac{1}{2}L(v_2) \leq L\left(\frac{v_1+v_2}{2}\right) + \frac{1}{8\theta}|v_1 - v_2|^2 \tag{37}$$

for all $v_1, v_2 \in \mathbb{R}^n$. Verification is left as an exercise.

2. Now choose y so that $u(x,t) = tL\left(\frac{x-y}{t}\right) + g(y)$. Then using the same value of y in the Hopf–Lax formulas for $u(x+z,t)$ and $u(x-z,t)$, we calculate

$$\begin{aligned}
u(x+z,t) &- 2u(x,t) + u(x-z,t) \\
&\leq \left[tL\left(\frac{x+z-y}{t}\right) + g(y)\right] - 2\left[tL\left(\frac{x-y}{t}\right) + g(y)\right] \\
&\quad + \left[tL\left(\frac{x-z-y}{t}\right) + g(y)\right] \\
&= 2t\left[\frac{1}{2}L\left(\frac{x+z-y}{t}\right) + \frac{1}{2}L\left(\frac{x-z-y}{t}\right) - L\left(\frac{x-y}{t}\right)\right] \\
&\leq 2t\frac{1}{8\theta}\left|\frac{2z}{t}\right|^2 \leq \frac{1}{\theta t}|z|^2,
\end{aligned}$$

the next-to-last inequality following from (37). □

b. Weak solutions, uniqueness. In this section we show that semiconcavity conditions of the sorts discovered for the Hopf–Lax solution u in Lemmas 3 and 4 can be utilized as uniqueness criteria.

DEFINITION. *We say that a Lipschitz continuous function* $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ *is a* weak solution *of the initial-value problem:*

$$
\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases} \tag{38}
$$

provided

(a) $u(x,0) = g(x) \quad (x \in \mathbb{R}^n)$,

(b) $u_t(x,t) + H(Du(x,t)) = 0$ *for a.e.* $(x,t) \in \mathbb{R}^n \times (0,\infty)$,

and

(c) $u(x+z,t) - 2u(x,t) + u(x-z,t) \leq C\left(1+\frac{1}{t}\right)|z|^2$

for some constant $C \geq 0$ *and all* $x, z \in \mathbb{R}^n$, $t > 0$.

Next we prove that a weak solution of (38) is unique, the key point being that this uniqueness assertion follows from the *inequality* condition (c).

THEOREM 7 (Uniqueness of weak solutions). *Assume* H *is* C^2 *and satisfies* (19) *and* g *satisfies* (20). *Then there exists at most one weak solution of the initial-value problem* (38).

Proof*. 1. Suppose that u and $\tilde{u}$ are two weak solutions of (38) and write $w := u - \tilde{u}$.

Observe now that at any point (y,s) where both u and $\tilde{u}$ are differentiable and solve our PDE, we have

$$
\begin{aligned}
w_t(y,s) &= u_t(y,s) - \tilde{u}_t(y,s) \\
&= -H(Du(y,s)) + H(D\tilde{u}(y,s)) \\
&= -\int_0^1 \frac{d}{dr} H(rDu(y,s) + (1-r)D\tilde{u}(y,s))\,dr \\
&= -\int_0^1 DH(rDu(y,s) + (1-r)D\tilde{u}(y,s))\,dr \cdot (Du(y,s) - D\tilde{u}(y,s)) \\
&=: -\mathbf{b}(y,s) \cdot Dw(y,s).
\end{aligned}
$$

Consequently

$$
w_t + \mathbf{b} \cdot Dw = 0 \quad \text{a.e.} \tag{39}
$$

*Omit on first reading.

2. Write $v := \phi(w) \geq 0$, where $\phi : \mathbb{R} \to [0,\infty)$ is a smooth function to be selected later. We multiply (39) by $\phi'(w)$ to discover

$$v_t + \mathbf{b} \cdot Dv = 0 \quad \text{a.e.} \tag{40}$$

3. Now choose $\varepsilon > 0$ and define $u^\varepsilon := \eta_\varepsilon * u$, $\tilde{u}^\varepsilon := \eta_\varepsilon * \tilde{u}$, where η_ε is the standard mollifier in the x and t variables. Then according to §C.4

$$|Du^\varepsilon| \leq \mathrm{Lip}(u), \ |D\tilde{u}^\varepsilon| \leq \mathrm{Lip}(\tilde{u}), \tag{41}$$

and

$$Du^\varepsilon \to Du, \ D\tilde{u}^\varepsilon \to D\tilde{u} \quad \text{a.e., as } \epsilon \to 0. \tag{42}$$

Furthermore inequality (c) in the definition of weak solution implies

$$D^2u^\varepsilon, D^2\tilde{u}^\varepsilon \leq C\left(1 + \frac{1}{s}\right) I \tag{43}$$

for an appropriate constant C and all $\varepsilon > 0$, $y \in \mathbb{R}^n$, $s > 2\varepsilon$. Verification is left as an exercise.

4. Write

$$\mathbf{b}_\varepsilon(y,s) := \int_0^1 DH(rDu^\varepsilon(y,s) + (1-r)D\tilde{u}^\varepsilon(y,s))\, dr. \tag{44}$$

Then (40) becomes

$$v_t + \mathbf{b}_\varepsilon \cdot Dv = (\mathbf{b}_\varepsilon - \mathbf{b}) \cdot Dv \quad \text{a.e.;}$$

hence

$$v_t + \mathrm{div}(v\mathbf{b}_\varepsilon) = (\mathrm{div}\, \mathbf{b}_\varepsilon)v + (\mathbf{b}_\varepsilon - \mathbf{b}) \cdot Dv \quad \text{a.e.} \tag{45}$$

5. Now

$$\begin{aligned} \mathrm{div}\, \mathbf{b}_\varepsilon &= \int_0^1 \sum_{k,l=1}^n H_{p_k p_l}(rDu^\varepsilon + (1-r)D\tilde{u}^\varepsilon)(ru^\varepsilon_{x_l x_k} + (1-r)\tilde{u}^\varepsilon_{x_l x_k})\, dr \\ &\leq C\left(1 + \frac{1}{s}\right) \end{aligned} \tag{46}$$

for some constant C, in view of (41), (43). Here we note that H convex implies $D^2H \geq 0$.

6. Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$, and set

$$R := \max\{|DH(p)| \mid |p| \leq \max(\mathrm{Lip}(u), \mathrm{Lip}(\tilde{u}))\}. \tag{47}$$

Define also the cone

$$C := \{(x,t) \mid 0 \leq t \leq t_0, |x - x_0| \leq R(t_0 - t)\}.$$

Next write

$$e(t) = \int_{B(x_0,R(t_0-t))} v(x,t)\,dx$$

and compute for a.e. $t > 0$:

$$\begin{aligned}
\dot{e}(t) &= \int_{B(x_0,R(t_0-t))} v_t\,dx - R\int_{\partial B(x_0,R(t_0-t))} v\,dS \\
&= \int_{B(x_0,R(t_0-t))} -\operatorname{div}(v\mathbf{b}_\varepsilon) + (\operatorname{div}\mathbf{b}_\varepsilon)v + (\mathbf{b}_\varepsilon - \mathbf{b})\cdot Dv\,dx \\
&\quad - R\int_{\partial B(x_0,R(t_0-t))} v\,dS \quad \text{by (45)} \\
&= -\int_{\partial B(x_0,R(t_0-t))} v(\mathbf{b}_\varepsilon \cdot \nu + R)\,dS \\
&\quad + \int_{B(x_0,R(t_0-t))} (\operatorname{div}\mathbf{b}_\varepsilon)v + (\mathbf{b}_\varepsilon - \mathbf{b})\cdot Dv\,dx \\
&\leq \int_{B(x_0,R(t_0-t))} (\operatorname{div}\mathbf{b}_\varepsilon)v + (\mathbf{b}_\varepsilon - \mathbf{b})\cdot Dv\,dx \quad \text{by (41), (44)} \\
&\leq C\left(1 + \frac{1}{t}\right)e(t) + \int_{B(x_0,R(t_0-t))} (\mathbf{b}_\varepsilon - \mathbf{b})\cdot Dv\,dx
\end{aligned}$$

by (46). The last term on the right-hand side goes to zero as $\varepsilon \to 0$, for a.e. $t > 0$, according to (41), (42) and the Dominated Convergence Theorem. Thus

$$\dot{e}(t) \leq C\left(1 + \frac{1}{t}\right)e(t) \quad \text{for a.e. } 0 < t < t_0. \tag{48}$$

7. Fix $0 < \varepsilon < r < t_0$ and choose the function $\phi(z)$ to equal zero if

$$|z| \leq \varepsilon[\mathrm{Lip}(u) + \mathrm{Lip}(\tilde{u})]$$

and to be positive otherwise. Since $u \equiv \tilde{u}$ on $\mathbb{R}^n \times \{t = 0\}$,

$$v = \phi(w) = \phi(u - \tilde{u}) = 0 \quad \text{at } \{t = \varepsilon\}.$$

Thus $e(\varepsilon)=0$. Consequently Gronwall's inequality (§B.2) and (48) imply

$$e(r) \leq e(\varepsilon)e^{\int_\varepsilon^r C\left(1+\frac{1}{s}\right)ds} = 0.$$

Hence

$$|u-\tilde{u}| \leq \varepsilon[\mathrm{Lip}(u)+\mathrm{Lip}(\tilde{u})] \quad \text{on } B(x_0, R(t_0-r)).$$

This inequality is valid for all $\varepsilon > 0$, and so $u \equiv \tilde{u}$ in $B(x_0, R(t_0-r))$. Therefore, in particular, $u(x_0,t_0)=\tilde{u}(x_0,t_0)$. □

In light of Lemmas 3, 4 and Theorem 7, we have

THEOREM 8 (Hopf–Lax formula as weak solution). *Suppose H is C^2 and satisfies* (19) *and g satisfies* (20). *If either g is semiconcave or H is uniformly convex, then*

$$u(x,t) = \min_{y\in\mathbb{R}^n}\left\{tL\left(\frac{x-y}{t}\right)+g(y)\right\}$$

is the unique weak solution of the initial-value problem (38) *for the Hamilton–Jacobi equation.*

Examples. (i) Consider the initial-value problem:

$$\left\{\begin{aligned} u_t + \tfrac{1}{2}|Du|^2 &= 0 && \text{in } \mathbb{R}^n \times (0,\infty) \\ u &= |x| && \text{on } \mathbb{R}^n \times \{t=0\}. \end{aligned}\right. \tag{49}$$

Here $H(p)=\frac{1}{2}|p|^2$ and so $L(v)=\frac{1}{2}|v|^2$. The Hopf–Lax formula for the unique, weak solution of (49) is

$$u(x,t) = \min_{y\in\mathbb{R}^n}\left\{\frac{|x-y|^2}{2t}+|y|\right\}. \tag{50}$$

Assume $|x|>t$. Then

$$D_y\left(\frac{|x-y|^2}{2t}+|y|\right) = \frac{y-x}{t}+\frac{y}{|y|} \quad (y\neq 0);$$

and this expression equals zero if $x = y+\frac{y}{|y|}t$, $y=(|x|-t)\frac{x}{|x|}\neq 0$. Thus $u(x,t)=|x|-\frac{t}{2}$ if $|x|>t$. If $|x|\leq t$, the minimum in (50) is attained at $y=0$. Consequently

$$u(x,t) = \begin{cases} |x| - t/2 & \text{if } |x| \geq t \\ \frac{|x|^2}{2t} & \text{if } |x| \leq t. \end{cases}$$

Observe that the solution becomes semiconcave at times $t > 0$, even though the initial function $g(x) = |x|$ is not semiconcave. This accords with Lemma 4.

(ii) We next examine the problem with reversed initial conditions:

$$
\begin{cases} u_t + \frac{1}{2}|Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \qquad\qquad u = -|x| & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{51}
$$

Then

$$
u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ \frac{|x-y|^2}{2t} - |y| \right\}.
$$

Now

$$
D_y \left(\frac{|x-y|^2}{2t} - |y| \right) = \frac{y-x}{t} - \frac{y}{|y|} \quad (y \neq 0),
$$

and this equals zero if $x = y - \frac{y}{|y|}t$, $y = (|x| + t)\frac{x}{|x|}$. Thus

$$
u(x,t) = -|x| - \frac{t}{2} \quad (x \in \mathbb{R}^n,\ t \geq 0).
$$

The initial function $g(x) = -|x|$ is semiconcave, and the solution remains so for times $t > 0$. □

In Chapter 10 we will again study Hamilton–Jacobi PDE and discover another and better notion of weak solution, applicable even if H is not convex.

3.4. INTRODUCTION TO CONSERVATION LAWS

In this section we investigate the initial-value problem for scalar conservation laws in one space dimension:

$$
\begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ \qquad\quad u = g & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1}
$$

Here $F : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are given and $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ is the unknown, $u = u(x,t)$. As noted in §3.2, the method of characteristics demonstrates that there does not in general exist a smooth solution of (1), existing for all times $t > 0$. By analogy with the developments in §3.3.3, we therefore look for some sort of weak or generalized solution.