Suppose not. Then there exist $a \in A$ and $\theta > 0$ such that

$$v_t(x,t) + \mathbf{f}(x,a) \cdot Dv(x,t) + r(x,a) \leq -\theta < 0 \tag{27}$$

for all points (x,t) sufficiently close to (x_0, t_0), say

$$|x - x_0| + |t - t_0| < \delta. \tag{28}$$

Since $u - v$ has a local maximum at (x_0, t_0), we may as well also suppose

$$\begin{cases} (u-v)(x,t) \leq (u-v)(x_0,t_0) \\ \text{for all } (x,t) \text{ satisfying (28).} \end{cases} \tag{29}$$

Consider now the constant control $\boldsymbol{\alpha}(s) \equiv a$ $(t_0 \leq s \leq T)$ and the corresponding dynamics

$$\begin{cases} \dot{\mathbf{x}}(s) = \mathbf{f}(\mathbf{x}(s), a) \quad (t_0 < s < T) \\ \mathbf{x}(t_0) = x_0. \end{cases} \tag{30}$$

Choose $0 < h < \delta$ so small that $|\mathbf{x}(s) - x_0| < \delta$ for $t_0 \leq s \leq t_0 + h$. Then

$$v_t(\mathbf{x}(s),s) + \mathbf{f}(\mathbf{x}(s),a) \cdot Dv(\mathbf{x}(s),s) + r(\mathbf{x}(s),a) \leq -\theta \quad (t_0 \leq s \leq t_0 + h), \tag{31}$$

according to (27), (28). But utilizing (29), we find

$$\begin{aligned} &u(\mathbf{x}(t_0+h), t_0+h) - u(x_0,t_0) \leq v(\mathbf{x}(t_0+h), t_0+h) - v(x_0,t_0) \\ &= \int_{t_0}^{t_0+h} \frac{d}{ds} v(\mathbf{x}(s),s)\,ds = \int_{t_0}^{t_0+h} v_t(\mathbf{x}(s),s) + Dv(\mathbf{x}(s),s) \cdot \dot{\mathbf{x}}(s)\,ds \\ &= \int_{t_0}^{t_0+h} v_t(\mathbf{x}(s),s) + \mathbf{f}(\mathbf{x}(s),a) \cdot Dv(\mathbf{x}(s),s)\,ds. \end{aligned} \tag{32}$$

In addition, the optimality condition (7) provides us with the inequality

$$u(x_0,t_0) \leq \int_{t_0}^{t_0+h} r(\mathbf{x}(s),a)\,ds + u(\mathbf{x}(t_0+h), t_0+h). \tag{33}$$

Combining (32) and (33), we discover

$$0 \leq \int_{t_0}^{t_0+h} v_t(\mathbf{x}(s),s) + \mathbf{f}(\mathbf{x}(s),a) \cdot Dv(\mathbf{x}(s),s) + r(\mathbf{x}(s),a)\,ds \leq -\theta h,$$

according to (31). This contradiction establishes (26).

3. Now suppose

$$u - v \text{ has a local minimum at a point } (x_0,t_0) \in \mathbb{R}^n \times (0,T);$$

we must prove

$$v_t(x_0,t_0) + \min_{a\in A}\{\mathbf{f}(x_0,a)\cdot Dv(x_0,t_0) + r(x_0,a)\} \leq 0. \tag{34}$$

Suppose not. Then there exists $\theta > 0$ such that

$$v_t(x,t) + \mathbf{f}(x,a)\cdot Dv(x,t) + r(x,a) \geq \theta > 0 \tag{35}$$

for all $a \in A$ and all (x,t) sufficiently close to (x_0,t_0), say

$$|x - x_0| + |t - t_0| < \delta. \tag{36}$$

Since $u - v$ has a local minimum at (x_0,t_0), we may as well also suppose

$$\begin{cases} (u-v)(x,t) \geq (u-v)(x_0,t_0) \\ \text{for all } (x,t) \text{ satisfying (36).} \end{cases} \tag{37}$$

Choose $0 < h < \delta$ so small that $|\mathbf{x}(s) - x_0| < \delta$ for $t_0 \leq s \leq t_0 + h$, where $\mathbf{x}(\cdot)$ solves

$$\begin{cases} \dot{\mathbf{x}}(s) = \mathbf{f}(\mathbf{x}(s),\boldsymbol{\alpha}(s)) \quad (t_0 < s < T) \\ \mathbf{x}(t_0) = x_0 \end{cases} \tag{38}$$

for some control $\boldsymbol{\alpha}(\cdot) \in \mathcal{A}$. This is possible owing to hypothesis (3).

Then utilizing (37), we find for any control $\boldsymbol{\alpha}(\cdot)$ that

$$\begin{aligned} u(\mathbf{x}(t_0+h),\, t_0+h) &- u(x_0,t_0) \\ &\geq v(\mathbf{x}(t_0+h), t_0+h) - v(x_0,t_0) \\ &= \int_{t_0}^{t_0+h} \frac{d}{ds} v(\mathbf{x}(s),s)\,ds \\ &= \int_{t_0}^{t_0+h} v_t(\mathbf{x}(s),s) + \mathbf{f}(\mathbf{x}(s),\boldsymbol{\alpha}(s))\cdot Dv(\mathbf{x}(s),s)\,ds, \end{aligned} \tag{39}$$

by (38). On the other hand, according to the optimality condition (7) we can select a control $\boldsymbol{\alpha}(\cdot) \in \mathcal{A}$ so that

$$u(x_0,t_0) \geq \int_{t_0}^{t_0+h} r(\mathbf{x}(s),\boldsymbol{\alpha}(s))\,ds + u(\mathbf{x}(t_0+h),t_0+h) - \frac{\theta h}{2}. \tag{40}$$

Combining (39) and (40), we discover

$$\begin{aligned} \frac{\theta h}{2} \geq \int_{t_0}^{t_0+h} & v_t(\mathbf{x}(s),s) + \mathbf{f}(\mathbf{x}(s),\boldsymbol{\alpha}(s))\cdot Dv(\mathbf{x}(s),s) \\ & + r(\mathbf{x}(s),\boldsymbol{\alpha}(s))\,ds \geq \theta h, \end{aligned}$$

according to (35). This contradiction proves (34). □

Design of optimal controls. We have now shown that the value function u, defined by (6), is the unique viscosity solution of the terminal-value problem (22) for the Hamilton–Jacobi–Bellman equation. How does this PDE help us solve the problem of synthesizing an optimal control? In informal terms, the method is this. Given an initial time $0 < t \leq T$ and an initial state $x \in \mathbb{R}^n$, we consider the optimal ODE

$$\begin{cases} \dot{\mathbf{x}}^*(s) = \mathbf{f}(\mathbf{x}^*(s), \boldsymbol{\alpha}^*(s)) & (t < s < T) \\ \mathbf{x}^*(t) = x, \end{cases} \tag{41}$$

where at each time s, $\boldsymbol{\alpha}^*(s) \in A$ is selected so that

$$\begin{aligned} \mathbf{f}(\mathbf{x}^*(s), \boldsymbol{\alpha}^*(s)) \cdot Du(\mathbf{x}^*(s), s) + r(\mathbf{x}^*(s), \boldsymbol{\alpha}^*(s)) \\ = H(Du(\mathbf{x}^*(s), s), \mathbf{x}^*(s)). \end{aligned} \tag{42}$$

In other words, given that the system is at the point $\mathbf{x}^*(s)$ at time s, we adjust the optimal control value $\boldsymbol{\alpha}^*(s)$ so as to attain the minimum in the definition (23) of the Hamiltonian H. We call $\boldsymbol{\alpha}^*(\cdot)$ so defined a *feedback control.*

It is fairly easy to check that this prescription does in fact generate a minimum cost trajectory, at least in regions where u and $\boldsymbol{\alpha}^*(\cdot)$ are smooth (so that (42) makes sense). There are however problems in interpreting (42) at points where the gradient Du does not exist.

10.3.4. Hopf–Lax formula revisited.

Remember that earlier in §3.3 we investigated this initial-value problem for the Hamilton–Jacobi equation:

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0, T] \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases} \tag{43}$$

under the assumptions that

$$p \mapsto H(p) \text{ is convex}, \quad \lim_{|p| \to \infty} \frac{H(p)}{|p|} = +\infty,$$

and

$$g : \mathbb{R}^n \to \mathbb{R} \text{ is Lipschitz continuous.}$$

Notice that we are now taking $0 \leq t \leq T$, to be consistent with §10.2. We introduced as well the Hopf–Lax formula for a solution:

$$u(x, t) = \min_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x - y}{t}\right) + g(y) \right\} \quad (x \in \mathbb{R}^n, t > 0), \tag{44}$$

where L is the Legendre transform of H:

$$L(v) = \sup_{p \in \mathbb{R}^n} \{p \cdot v - H(p)\} \quad (q \in \mathbb{R}^n). \tag{45}$$

In order to tie together the theory set forth here and in §3.3, let us now check that the Hopf–Lax formula gives the correct viscosity solution, as defined in §10.1.1. (The proof is really just a special case of that for Theorem 2.)

THEOREM 3 (Hopf–Lax formula as viscosity solution). *Assume in addition that g is bounded. Then the unique viscosity solution of the initial-value problem* (43) *is given by the formula* (44).

Proof. 1. As shown in §3.3 the function u defined by (44) is Lipschitz continuous and takes on the initial function g at time $t = 0$. It is easy to verify as well that u is also bounded on $\mathbb{R}^n \times (0, T]$, since g is bounded.

2. Now let $v \in C^\infty(\mathbb{R}^n \times (0, \infty))$ and assume $u - v$ has a local maximum at $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$. According to Lemma 1 in §3.3.2,

$$u(x_0, t_0) = \min_{x \in \mathbb{R}^n} \left\{ (t_0 - t) L\left(\frac{x_0 - x}{t_0 - t}\right) + u(x, t) \right\} \tag{46}$$

for each $0 \le t < t_0$. Thus for each $0 \le t < t_0$, $x \in \mathbb{R}^n$

$$u(x_0, t_0) \le (t_0 - t) L\left(\frac{x_0 - x}{t_0 - t}\right) + u(x, t). \tag{47}$$

But since $u - v$ has a local maximum at (x_0, t_0),

$$u(x_0, t_0) - v(x_0, t_0) \ge u(x, t) - v(x, t)$$

for (x, t) close to (x_0, t_0). Combining this estimate with (47), we find

$$v(x_0, t_0) - v(x, t) \le (t_0 - t) L\left(\frac{x_0 - x}{t_0 - t}\right) \tag{48}$$

for $t < t_0$, (x, t) close to (x_0, t_0). Now write $h = t_0 - t$ and set $x = x_0 - hv$, where $v \in \mathbb{R}^n$ is given. Inequality (48) becomes

$$v(x_0, t_0) - v(x_0 - hv, t_0 - h) \le hL(v).$$

Divide by $h > 0$ and send $h \to 0$:

$$v_t(x_0, t_0) + Dv(x_0, t_0) \cdot v - L(v) \le 0.$$

This is true for all $v \in \mathbb{R}^n$ and so

$$v_t(x_0, t_0) + H(Dv(x_0, t_0)) \leq 0, \tag{49}$$

since

$$H(p) = \sup_{v \in \mathbb{R}^n} \{p \cdot v - L(v)\}, \tag{50}$$

by the convex duality of H and L. We have, as desired, established the inequality (49) whenever $u - v$ has a local maximum at (x_0, t_0).

3. Now suppose instead $u - v$ has a local minimum at a point $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$. We must prove

$$v_t(x_0, t_0) + H(Dv(x_0, t_0)) \geq 0. \tag{51}$$

Suppose to the contrary that estimate (51) fails, in which case

$$v_t(x, t) + H(Dv(x, t)) \leq -\theta < 0$$

for some $\theta > 0$ and all points (x, t) close enough to (x_0, t_0). In view of (50)

$$v_t(x, t) + Dv(x, t) \cdot v - L(v) \leq -\theta \tag{52}$$

for all (x, t) near (x_0, t_0) and all $v \in \mathbb{R}^n$.

Now from (46) we see that if $h > 0$ is small enough,

$$u(x_0, t_0) = hL\left(\frac{x_0 - x_1}{h}\right) + u(x_1, t_0 - h) \tag{53}$$

for some point x_1 close to x_0. We then compute

$$\begin{aligned} v(x_0, t_0) - v(x_1, t_0 - h) &= \int_0^1 \frac{d}{ds} v(sx_0 + (1-s)x_1,\ t_0 + (s-1)h)\, ds \\ &= \int_0^1 Dv(sx_0 + (1-s)x_1,\ t_0 + (s-1)h) \cdot (x_0 - x_1) \\ &\qquad + v_t(sx_0 + (1-s)x_1,\ t_0 + (s-1)h)h\, ds \\ &= h \int_0^1 Dv(\cdots) \cdot \left(\frac{x_0 - x_1}{h}\right) + v_t(\cdots)\, ds. \end{aligned}$$

Now if $h > 0$ is sufficiently small, we may apply (52), to find

$$v(x_0, t_0) - v(x_1, t_0 - h) \leq hL\left(\frac{x_0 - x_1}{h}\right) - \theta h.$$

But then (53) forces

$$v(x_0, t_0) - v(x_1, t_0 - h) \leq u(x_0, t_0) - u(x_1, t_0 - h) - \theta h,$$

a contradiction, since $u - v$ has a local minimum at (x_0, t_0). Consequently the desired inequality (51) is indeed valid. $\square$

10.4. PROBLEMS

1. Assume u is a viscosity solution of

$$u_t + H(Du, x) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

Show that $\hat{u} := -u$ is a viscosity solution of

$$\hat{u}_t + \hat{H}(D\hat{u}, x) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

for $\hat{H}(p, x) := -H(-p, x)$.

2. Let $\{u^k\}_{k=1}^{\infty}$ be viscosity solutions of the Hamilton–Jacobi equations

$$u_t^k + H(Du^k, x) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

$(k = 1, \dots)$, and suppose $u^k \to u$ uniformly. Assume as well that H is continuous. Show u is a viscosity solution of

$$u_t + H(Du, x) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

Hence the uniform limits of viscosity solutions are viscosity solutions.

3. Suppose for each $\epsilon > 0$ that u^ϵ is a smooth solution of the parabolic equation

$$u_t^\epsilon + H(Du^\epsilon, x) - \epsilon \sum_{i,j=1}^{n} a^{ij} u^\epsilon_{x_i x_j} = 0$$

in $\mathbb{R}^n \times (0, \infty)$, where the smooth coefficients a^{ij} $(i, j = 1, \dots, n)$ satisfy the uniform ellipticity condition from Chapter 6. Suppose also that H is continuous and that $u^\epsilon \to u$ uniformly as $\epsilon \to 0$.

Prove that u is a viscosity solution of $u_t + H(Du, x) = 0$. (This exercise shows that viscosity solutions do not depend upon the precise structure of the parabolic smoothing.)

4. Let u^i $(i = 1, 2)$ be viscosity solutions of

$$\begin{cases} u_t^i + H(Du^i, x) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u^i = g^i & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Assume H satisfies condition (3) in §10.2. Prove the *contraction* property

$$\sup_{\mathbb{R}^n} |u^1(\cdot, t) - u^2(\cdot, t)| \leq \sup_{\mathbb{R}^n} |g^1 - g^2| \quad (t \geq 0).$$

5. (a) Show that $u(x) := 1 - |x|$ is a viscosity solution of

$$(*) \qquad \begin{cases} |u'| = 1 & \text{in } (-1, 1) \\ u(-1) = u(1) = 0. \end{cases}$$

This means that for each $v \in C^\infty(-1,1)$, if $u - v$ has a maximum (minimum) at a point $x_0 \in (-1,1)$, then $|v'(x_0)| \le 1$ (≥ 1).

(b) Show that $\tilde{u}(x) := |x| - 1$ is *not* a viscosity solution of $(*)$.

(c) Show that $\tilde{u}$ is a viscosity solution of

$$(**) \qquad \begin{cases} -|\tilde{u}'| = -1 & \text{in } (-1,1) \\ \tilde{u}(-1) = \tilde{u}(1) = 0. \end{cases}$$

(Hint: What is the meaning of a viscosity solution of $(**)$?)

(d) Why do problems $(*), (**)$ have different viscosity solutions?

6. Let $U \subset \mathbb{R}^n$ be open, bounded. Set $u(x) := \operatorname{dist}(x, \partial U)$ $(x \in U)$. Prove that u is Lipschitz continuous and that it is a viscosity solution of the eikonal equation

$$|Du| = 1 \quad \text{in } U.$$

This means that for each $v \in C^\infty(U)$, if $u - v$ has a maximum (minimum) at a point $x_0 \in U$, then $|Dv(x_0)| \le 1$ (≥ 1).

7. Suppose an open set $U \subset \mathbb{R}^n$ is subdivided by a smooth hypersurface Γ into the subregions V^+ and V^-. Let ν denote the unit normal to Γ, pointing into V^+. Assume that u is a viscosity solution of

$$H(Du) = 0 \quad \text{in } U$$

and that u is smooth in $\bar{V}^+$ and $\bar{V}^-$. Write u_ν^+ for the limit of $Du \cdot \nu$ along Γ from within V^+, and write u_ν^- for the limit from within V^-. Prove that along Γ we have the inequalities

$$H(\lambda u_\nu^- + (1-\lambda)u_\nu^+) \ge 0 \quad \text{if } \ u_\nu^- \le u_\nu^+$$

and

$$H(\lambda u_\nu^- + (1-\lambda)u_\nu^+) \le 0 \quad \text{if } \ u_\nu^+ \le u_\nu^-,$$

for each $0 \le \lambda \le 1$.

8. A surface described by the graph of $u : \mathbb{R}^2 \to \mathbb{R}$ is illuminated by parallel light rays from the vertical e_3 direction. We assume the surface has constant albedo and in addition is *Lambertian*, meaning that incoming right rays are scattered equally in all directions. Then the intensity $i = i(x)$ of the reflected light above the point $x \in \mathbb{R}^2$ is given by the formula $i = e_3 \cdot \nu$, where ν is the upward pointing unit normal to the surface.