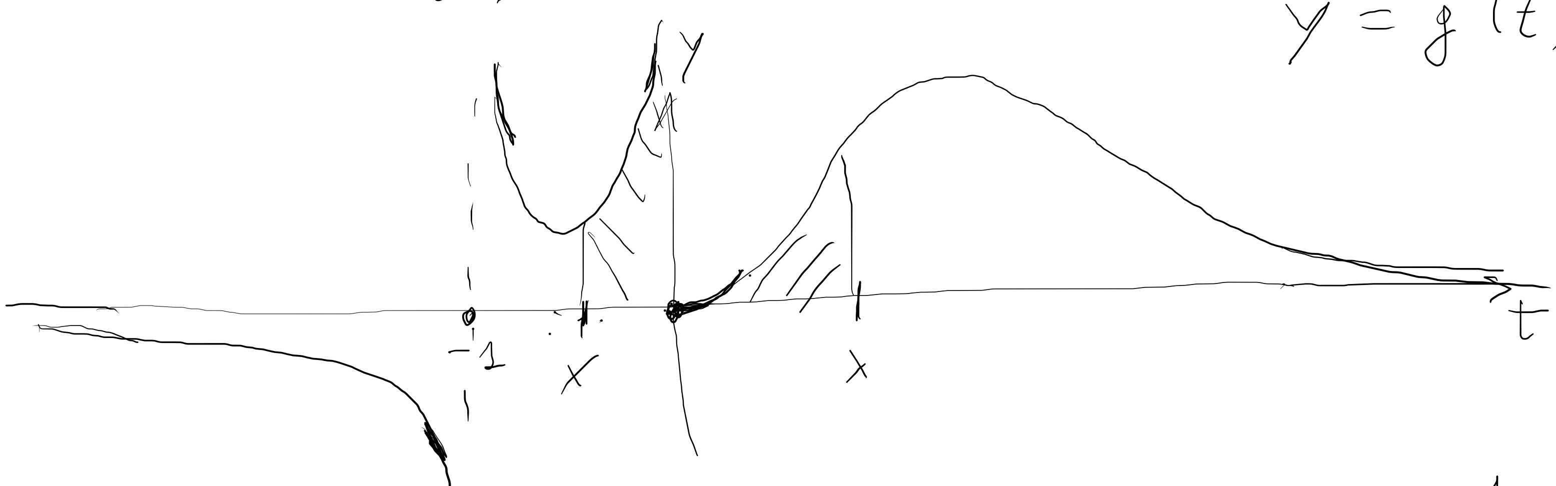
1:0 [illegible]

10/1/22

$$f(x)=\begin{cases}\displaystyle\int_0^x \underbrace{e^{-\frac{1}{t}}\,\frac{1}{1+t}}_{g(t)}\,dt & x\geq 0\\[2ex] \displaystyle\int_0^x \frac{t}{t^3-1}\,dt & x\leq 0\end{cases}$$

$g(t)$ $\qquad \lim\limits_{t\to\infty} e^{-\frac{1}{t}}\frac{1}{1+t}=0$ $\qquad$ [illegible] $g(0)=0$

$y=g(t)$



$\lim\limits_{t\to 0^+} g(t)=0$ $\qquad \lim\limits_{t\to 0^+} e^{-\frac{1}{t}}\frac{1}{1+t}=\lim\limits_{t\to 0^+} e^{-\frac{1}{t}}=e^{-\infty}=0$

$\lim\limits_{t\to 0^-} g(t)=+\infty$ $\qquad \lim\limits_{t\to 0^-} e^{-\frac{1}{t}}\frac{1}{1+t}=\lim\limits_{t\to 0^-} e^{-\frac{1}{t}}=e^{+\infty}=+\infty$

$$f(x) = \begin{cases} \int_0^x e^{-\frac{1}{t}} \frac{1}{1+t} & x \geq 0 \\ \int_0^x \frac{t}{t^3-1}\, dt & x \leq 0 \end{cases}$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \int_0^x e^{-\frac{1}{t}} \frac{1}{1+t}\, dt = +\infty$$

$$\lim_{t \to +\infty} \frac{e^{-\frac{1}{t}} \frac{1}{1+t}}{\frac{1}{t}} = \lim_{t \to +\infty} \frac{t}{1+t} = 1$$

perché $\frac{1}{t} \notin L[1, +\infty)$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \int_0^x \frac{t}{t^3-1}\,dt = \int_0^{-\infty} \frac{t}{t^3-1}\,dt \in \mathbb{R}$$

$$= -\int_{-\infty}^{0} \frac{t}{t^3-1}\,dt \in \mathbb{R}_-$$

$$\frac{t}{(t-1)(t^2+t+1)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+t+1} \qquad \frac{t_\pm}{(t-t_+)}$$

$$\frac{A}{t-1} + \frac{\beta}{t-t_+} + \frac{\gamma}{t-t_-} = \frac{A}{t-1} + \frac{\beta(t-t_-)+\gamma(t-t_+)}{t^2+t+1}$$

$$A = \frac{t}{t^2+t+1}\Big|_{t=1} = \frac{1}{3} \implies B = -\frac{1}{3}$$

$$\frac{t^2}{(t-1)(t^2+t+1)} = A\frac{t}{t-1} + \frac{Bt^2}{t^2+t+1} + \frac{Ct}{t^2+t+1}$$

$$\downarrow$$

$$0 = A + B \implies A = -B$$

$$\frac{t}{(t-1)(t^2+t+1)} = \frac{\frac{1}{3}}{t-1} + \frac{-\frac{1}{3}t+C}{t^2+t+1}$$

$$= \frac{\frac{1}{3}(t^2+t+1) + (-\frac{1}{3}t+C)(t-1)}{(t-1)(t^2+t+1)}$$

$$\frac{t}{(t-1)(t^2+t+1)} = \frac{t(\frac{1}{3}+\frac{1}{3}+C) + (\frac{1}{3}-C)}{(t-1)(t^2+t+1)}$$

$$t = t\left(\frac{1}{3}+\frac{1}{3}+C\right) + \frac{1}{3} - C$$

$$\begin{cases} \frac{2}{3}+C = 1 \\ \frac{1}{3}-C = 0 \end{cases} \qquad \boxed{C = \frac{1}{3}}$$

$$\int_0^x \frac{t}{t^3-1}\,dt =$$

$$\frac{1}{3}\lg|x-1| - \frac{1}{6}\lg(x^2+x+1) + \frac{1}{\sqrt3}\operatorname{arctan}\left(\frac{2}{\sqrt3}(x+1)\right) - \frac{2}{\sqrt3}\operatorname{arctan}\left(\frac{1}{\sqrt3}\right)$$

$$\downarrow\ ? \qquad\qquad \downarrow\ -\frac{\pi}{2}\frac{1}{\sqrt3}$$

$$\int_0^x \frac{t}{(t-1)(t^2+t+1)} = \frac{1}{3}\int_0^x \frac{1}{t-1} + \int_0^x \frac{-\frac{1}{3}t+\frac{1}{3}}{t^2+t+1}$$

$$= \frac{1}{3}\lg|t-1|\Big]_0^x - \frac{1}{3}\int_0^x \frac{t-1}{t^2+t+1}\,dt$$

$$\frac{\lg|x-1|}{3}$$

$$\int_0^x \frac{t-1}{t^2+t+1}\,dt = \frac{1}{2}\int_0^x \frac{2t+1}{t^2+t+t}\,dt - \frac{3}{2}\int_0^x \frac{1}{t^2+t+1}\,dt$$

$$= \frac{1}{2}\lg(t^2+t+1)\Big]_0^x - \frac{3}{2}\int_0^x \frac{1}{t^2+t+1}\,dt$$

$$= \frac{1}{2}\lg(x^2+x+1)$$

$$\int \frac{1}{y^2+1}\,dy$$

$$\int_0^x \frac{1}{t^2+t+1} = \int_0^x \frac{dt}{t^2+2\,\frac{1}{2}t+1} = \int_0^x \frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}}$$

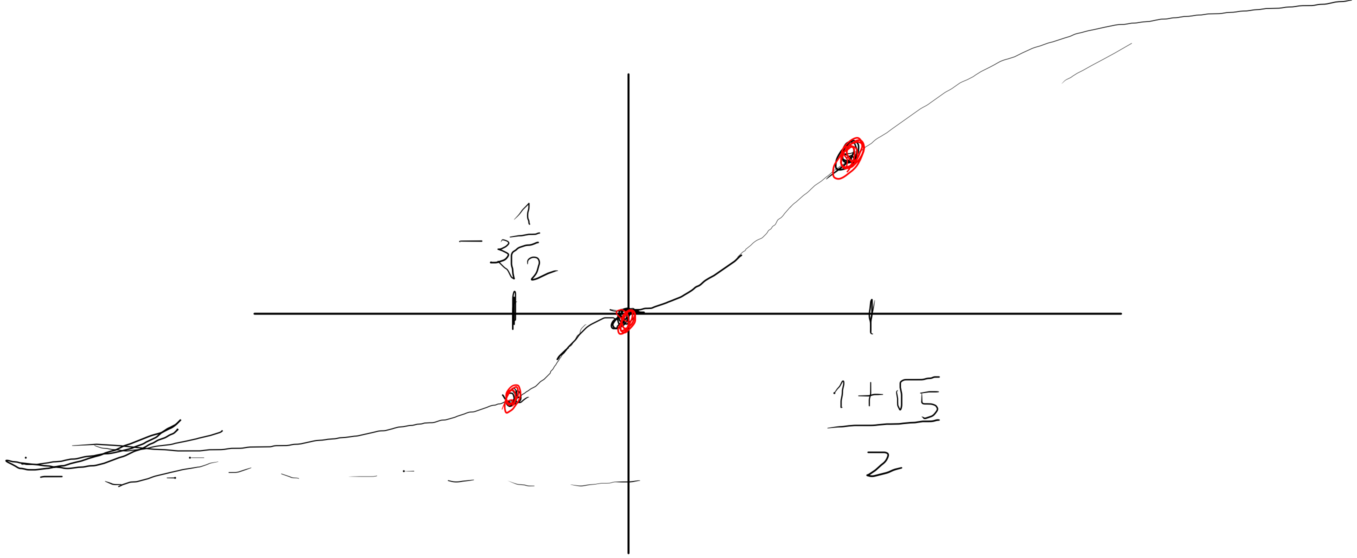
$$\frac{\sqrt3}{2}y = \left(t+\frac{1}{2}\right) \qquad dt = \frac{\sqrt3}{2}\,dy$$

$$= \int_{\frac{1}{\sqrt3}}^{\frac{2}{\sqrt3}(x+1)} \frac{\frac{\sqrt3}{2}\,dy}{\left(\frac{\sqrt3}{2}\right)^2 y^2 + \left(\frac{\sqrt3}{2}\right)^2} = \frac{1}{\frac{\sqrt3}{2}}\int_{\frac{1}{\sqrt3}}^{\frac{2}{\sqrt3}(x+1)} \frac{dy}{y^2+1}$$

$$= \frac{2}{\sqrt3}\left[\operatorname{arctan}\left(\frac{2}{\sqrt3}(x+1)\right) - \operatorname{arctan}\frac{1}{\sqrt3}\right]$$

$$\frac{1}{3}\lg|x-1| - \frac{1}{6}\lg(x^2+x+1) =$$

$$= \lg\frac{|x-1|^{\frac{1}{3}}}{(x^2+x+1)^{\frac{1}{6}}} = \lg\left(\frac{\cancel{|x|^{\frac{1}{3}}}}{\cancel{|x|^{\frac{1}{3}}}}(1+o(1))\right) \xrightarrow{x\to-\infty} 0$$



$$f'(x) = \begin{cases} e^{-\frac{1}{x}} \dfrac{1}{1+x} & x > 0 \\ \dfrac{x}{x^3-1} & x < 0 \end{cases}$$

$$f'_d(0) = \lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0^+} f'(x) = 0$$

$$f'_s(0) = \lim_{x \to 0^-} \frac{f(x)}{x} = \lim_{x \to 0^-} \frac{x}{x^3-1} = 0 \qquad f'(0) = 0$$

$$f'(x) > 0 \quad \forall\, x \neq 0$$

So che per $x >> 1$ $f(x)$ è concava

$$f(x) = \int_0^x e^{-\frac{1}{t}} \frac{1}{1+t}\, dt =$$

dimostrare che $\lim_{x \to +\infty} \left(f(x) - \log x \right) \in \mathbb{R}$

$x < 0$

$$f'(x) = \frac{x}{x^3-1}$$

$x < 0$

$$f''(x) = \left(\frac{x}{x^3-1}\right)' = \frac{x^3-1-x\,3x^2}{(x^3-1)^2} =$$

$$= \frac{-1-2x^3}{(x^3-1)^2} = 0$$

$$-1-2x^3 = 0 \qquad 2x^3 = -1 \qquad x^3 = -\frac{1}{2}$$

$$x^3 = -\frac{1}{\sqrt[3]{2}}$$

$x > 0$ $\qquad f'(x) = e^{-\frac{1}{x}} \frac{1}{1+x}$

$$f''(x) = \left(e^{-\frac{1}{x}} \frac{1}{1+x} \right)' = \frac{1}{x^2} e^{-\frac{1}{x}} \frac{1}{1+x} - \frac{1}{(1+x)^2} e^{-\frac{1}{x}}$$

$$= e^{-\frac{1}{x}} \frac{1}{1+x} \left(\frac{1}{x^2} - \frac{1}{1+x} \right)$$

$$= e^{-\frac{1}{x}} \frac{1+x-x^2}{x^2(1+x)^2}$$

$$x^2 - x - 1 = 0 \qquad x_\pm = \frac{1}{2} \pm \frac{\sqrt{1+4}}{2} =$$

$$= \frac{1 \pm \sqrt{5}}{2} \longrightarrow x_+ = \frac{1+\sqrt{5}}{2}$$

$$f(x) = \int_0^x e^{-\frac{1}{t}} \frac{1}{1+t}\, dt - \lg x$$

$$= \int_0^1 e^{-\frac{1}{t}} \frac{1}{1+t}\, dt + \int_1^x e^{-\frac{1}{t}} \frac{1}{1+t}\, dt - \lg x$$

$$e^{-\frac{1}{t}} = 1 - \frac{1}{t} + \frac{1}{2t^2} + o\left(\frac{1}{t^2}\right)$$

$$\frac{1}{1+t} = \frac{1}{t} \frac{1}{1+t^{-1}} = \frac{1}{t}\left(1 - \frac{1}{t} + \frac{1}{t^2} + o\left(\frac{1}{t^2}\right)\right)$$

$$e^{-\frac{1}{t}} \frac{1}{1+t} = \frac{1}{t}\left(1 - \frac{1}{t} + \frac{1}{2t^2} + o\left(\frac{1}{t^2}\right)\right)\left(1 - \frac{1}{t} + \frac{1}{t^2} + o\left(\frac{1}{t^2}\right)\right)$$

$$= \frac{1}{t}\left(1 - \frac{2}{t} + o\left(\frac{1}{t}\right)\right)$$

$$f(x) - \lg x = f(1) + \int_1^x \frac{1}{t}\left(1 - \frac{2}{t} + o\left(\frac{1}{t}\right)\right) dt - \lg x$$

$$= f(1) + \int_1^x \left(\cancel{\frac{1}{t}} - \frac{2}{t^2} + o\left(\frac{1}{t^2}\right)\right) dt - \cancel{\lg x}$$

$$\xrightarrow{x \to +\infty} f(1) + C$$