Third assignment

3) $Aut(\mathbb{P}^1) = PGL_2(\mathbb{K})$

proof: $PGL_2(\mathbb{K}) \subset Aut(\mathbb{P}^1)$: Let $\varphi \in PGL_2(\mathbb{K})$

$\Longrightarrow \varphi([x_0, x_1]) = [ax_0 + bx_1, cx_0 + dx_1]$ where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = A \in GL_2(\mathbb{K})$

Of course φ is bijective since $[A]^{-1} = [A^{-1}] \Rightarrow \varphi^{-1}$ is represented by $[A^{-1}]$.

So φ, φ^{-1} are morphism because are globally linear poly

$\Longrightarrow \varphi \in Aut(\mathbb{P}^1)$.

$PGL_2(\mathbb{K}) \supset Aut(\mathbb{P}^1)$: Let $\varphi \in Aut(\mathbb{P}^1) \Longrightarrow \forall P\ \exists U_P \subsetneq \mathbb{P}^1$:

$\varphi|_{U_P}([x_0, x_1]) = [f_1(x_0, x_1), f_2(x_0, x_1)]$ ($f_{1,2} \in \mathbb{K}[X_0, X_1]_d$ and coprime)

(we can delete this common factor)

and the same for φ^{-1} (also $\varphi^{-1} \in Aut(\mathbb{P}^1)$) $\Longrightarrow \varphi^{-1}|_{\tilde{V}_P}$ is represented

(Call g_1, g_2 the poly. which represent $\varphi^{-1}|_{\tilde{V}_P}$) by poly in $\mathbb{K}[X_0, X_1]_e$ for a $\tilde{V}_P \subsetneq \mathbb{P}^1$ ($\forall P \in \mathbb{P}^1$)

$\Longrightarrow \varphi \circ \varphi^{-1}|_{\tilde{W}_P}([x_0, x_1]) = id|_{\tilde{W}_P}([x_0, x_1])$ where $\tilde{W}_P = \varphi(\tilde{V}_P) \cap \tilde{V}_P \neq \emptyset$

because φ, φ^{-1} are continuous, so $\varphi(\tilde{V}_P) \subsetneq \mathbb{P}^1$.

$[f_1(g_1, g_2), f_2(g_1, g_2)] = [x_0, x_1]$

$\Longrightarrow \exists h \in \mathbb{K}[X_0, X_1] : (*) \begin{cases} f_1(g_1, g_2) = x_0\, h(x_0, x_1) \\ f_2(g_1, g_2) = x_1\, h(x_0, x_1) \end{cases}$

note: $\deg(f \cdot g) = \deg(f)\deg(g) = d \cdot e$ because f, g are homogeneous.

$f_1(X_0, X_1), f_2(X_0, X_1)$ coprime $\Longrightarrow \exists\, \alpha, \beta \in \mathbb{K}[X_0, X_1]$:

$$\alpha(X_0, X_1) f_1(X_0, X_1) + \beta(X_0, X_1) f_2(X_0, X_1) = 1$$

Now apply $ev_{(g_1, g_2)} : \mathbb{K}[X_0, X_1] \longrightarrow \mathbb{K}[X_0, X_1]$ homomorphism

$$f(x_0, x_1) \longmapsto f(g_1(x_0, x_1), g_2(x_0, x_1))$$

$$\Longrightarrow \alpha(g_1, g_2) f_1(g_1, g_2) + \beta(g_1, g_2) f_2(g_1, g_2) = 1$$

$$\overset{(*)}{=} \alpha(g_1, g_2)\, x_0\, h(x_0, x_1) + \beta(x_0, x_1)\, x_1\, h(x_0, x_1)$$

$\Rightarrow h(x_0, x_1)(\alpha(g_1, g_2)x_0 + x_1\beta(x_0, x_1)) = \ell 1$

$\Rightarrow h \mid 1 \Rightarrow \deg(h) = 0$ but

$\deg(f_1(g_1, g_2)) = \deg(f_2(g_1, g_2)) = d \cdot e = \deg(x_0 \cdot h) = 1$

$\Rightarrow e = d = 1 \Rightarrow$ <u>f_i, g_i's are linear.</u>

$\Rightarrow$ The representation on every U_P is given by linear poly (homogeneous).

Suppose to have two different representation in two distinct open's U_P, U_Q : $\begin{cases} \varphi|_{U_P} = [a_P x_0 + b_P x_1, c_P x_0 + d_P x_1] \\ \varphi|_{U_Q} = [a_Q x_0 + b_Q x_1, c_Q x_0 + d_Q x_1] \end{cases}$

$\Rightarrow \varphi|_{U_P \cap U_Q \cap U_0}([1, t]) = [a_P + b_P t, c_P + d_P t] = [a_Q + b_Q t, c_Q + d_Q t]$

$\Rightarrow \forall t \in U_P \cap U_Q \cap U_0 \ \exists \lambda_t : \begin{cases} t(b_P - \lambda_t b_Q) + (a_P - \lambda_t a_Q) = 0 \\ t(d_P - \lambda_t d_Q) + (c_P - \lambda_t c_Q) = 0 \end{cases} \quad (**)$

$\Leftrightarrow \det(A_P - \lambda_t A_Q) = 0 \leftarrow$ but the solutions doesn't depend from $t \Rightarrow \exists \lambda \in K : (**)$ has infinitely many solutions $\forall t \in U_P \cap U_Q \cap U_0$

$\Rightarrow \begin{cases} a_P = \lambda a_Q, & c_P = \lambda c_Q \\ b_P = \lambda b_Q, & d_P = \lambda d_Q \end{cases} \Rightarrow A_P = \lambda A_Q \Rightarrow [A_P] = [A_Q]$

$\Rightarrow$ the representation of φ is unique and since φ is bijective we have $A_P \in GL_2(K) \Rightarrow \varphi \in PGL_2(K)$ ∎

Third assignment

1) $X \subseteq \mathbb{P}^n$ be a qp var., let $U \subseteq X$ be open and let $\{U_i\}_{i \in J}$ be an open cover of U. Prove that a function $h: U \to \mathbb{K}$ is regular in U iff $h_{|U_i}$ is regular $\forall i \in J$

Pf $(\Rightarrow)$ Let $P \in U_i \subseteq U$

$\Rightarrow \exists f, g : \quad h = \frac{f}{g}$ and $g(P) \neq 0$

$h_{|U_i} = \frac{f}{g} \quad g(P) \neq 0$ and $h_{|U_i}$ reg $\forall i \in J$

$(\Leftarrow)$ $P \in U \Rightarrow \exists i \in \{1, \dots, n\} : P \in U_i$

by hypothesis $h_{|U_i} = \frac{f_i}{g_i} \quad g_i(P) \neq 0$

$\Rightarrow \quad h = \frac{f_i}{g_i}$ is well defined because

$x \in U_i \cap U_j \qquad \frac{f_i(x)}{g_i(x)} = h_{|U_i}(x) = h_{|U_j}(x) = \frac{f_j(x)}{g_j}$

$\Leftrightarrow f_i(x) g_j(x) - f_j(x) g_i(x) = 0$

$\Rightarrow \forall x \in U \quad f_i(x) g_j(x) - f_j(x) g_i(x) = 0$

Then $h = \frac{f_i}{g_i}$ is regular in U because

$\forall P \in U \quad \exists j \in \{1, \dots, n\} \quad h = \frac{f_j}{g_j} \quad g_j(P) \neq 0$