Exercises

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1. Let $\mathbb{K} = \overline{\mathbb{K}}$. Show that $\mathbb{A}^1 \setminus \{pt\}$ is not isomorphic to a projective variety. Similarly, prove that $\mathbb{P}^1 \setminus \{pt\}$ is not isomorphic to a projective variety.

   Proof. Let's prove that under the given conditions of the question $\mathbb{P}^1 \setminus \{pt\}$ is not isomorphic to a projective variety. For simplicity let the point is $[1 : 0]$.

   Consider $\Phi : \mathbb{P}^1 \setminus \{[1 : 0]\} \to \mathbb{A}^1$ define as;

   $$\Phi([x : y]) = \frac{x}{y}.$$

   Clearly it's a morphism and it's inverse, $\Psi : \mathbb{A}^1 \to \mathbb{P}^1$, can be seen as;

   $$\Psi(x) = [x : 1]$$

   This gives that $\mathbb{P}^1$ is isomorphic to an affine variety. And we know that the only varieties which are both affine and projective are points. So $\mathbb{P}^1 \setminus \{pt\}$ can never be projective.

   Now let's prove the other part of the question. Let $pt = 0$

   Consider the second projection $p_2 : \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \to \mathbb{A}^1$.

   Then $V(xy - 1) \subseteq \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1$ is closed but, $p_2(V(xy - 1)) = \mathbb{A}^1 \setminus \{0\}$ is not closed in $\mathbb{A}^1$. Therefor $\mathbb{A}^1 \setminus \{pt\}$ is not complete and hence not projective under the hypothesis of the statement. □

2. Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be two Zariski closed, and let p_Y be the second projection: $p_Y : X \times Y \to Y, p_Y(x, y) = y$. Prove that if $Z \subset X$ is closed and $f : X \to Y$ a morphism, then $f(Z) = p_Y((Z \times Y) \cap \Gamma_f)$.

   Proof. We will show both the inclusion; Let $y \in f(Z), \implies \exists\ x \in Z$ such that $f(x) = y, \implies$

   $$\begin{gathered}(x, y) \in \Gamma_f \wedge (x, y) \in Z \times Y \\ \implies (x, y) \in (Z \times Y) \cap \Gamma_f \\ \implies y \in p_Y((Z \times Y) \cap \Gamma_f)\end{gathered}$$

Similarly, Let $y \in p_Y((Z \times Y) \cap \Gamma_f)$. Then $\exists\ (x, y) \in (Z \times Y) \cap \Gamma_f$, such that;

$$p_Y(x, y) = x.$$

Since $(x, y) \in \Gamma_f \Rightarrow x \in Z, y \in Y$ and $f(x) = y$, $\Rightarrow y \in f(Z)$. $\square$