Second assignement

(3) Let $X \subseteq \mathbb{A}^n$ Zar closed consisting of two points.
Prove that $A(X) \cong \mathbb{K} \oplus \mathbb{K}$.

proof

$X = \{P, Q\} \subseteq \mathbb{A}^n$ with $P = (a_1, \dots, a_n)$ and $Q = (b_1, \dots, b_n)$

Let $\Phi : \mathbb{K}[x_1, \dots, x_n] \longrightarrow \mathbb{K} \oplus \mathbb{K}$ defined by

$F \longmapsto (F(P), F(Q))$.

$\Phi = (ev_P, ev_Q)$ where ev_P, ev_Q are evaluation maps

$\Rightarrow$ Φ is a $\mathbb{K}$-algebra homom

$$\ker \Phi = \{ F \in \mathbb{K}[x_1, \dots, x_n] \mid (F(P), F(Q)) = (0,0) \} =$$

$$= \{ F \in \mathbb{K}[x_1, \dots, x_n] \mid F|_X \equiv 0 \} = I(X)$$

$P \neq Q \Rightarrow \exists i \in \{1, \dots, n\}$ s.t. $a_i \neq b_i$.

Let $(\lambda, \mu) \in \mathbb{K} \oplus \mathbb{K}$ and cons $F(x_1, \dots, x_n) = \dfrac{\mu - \lambda}{b_i - a_i} x_i + \lambda - \dfrac{\mu - \lambda}{b_i - a_i} a_i$

$$\left.\begin{array}{l} F(P) = F(a_1, \dots, a_n) = \lambda \\ F(Q) = F(b_1, \dots, b_n) = \mu \end{array}\right\} \Rightarrow \Phi(F) = (\lambda, \mu)$$

$$\Rightarrow \operatorname{Im}(\Phi) = \mathbb{K} \oplus \mathbb{K} \Rightarrow \frac{\mathbb{K}[x_1, \dots, x_n]}{I(X)} = \mathbb{K} \oplus \mathbb{K}$$

Sixth assignement

(3) Prove that the blow-up $\hat{\mathbb{A}}^2_{\mathbb{C}}$ of $\mathbb{A}^2_{\mathbb{C}}$ in the origin is not isomorphic to an affine variety.

proof

Suppose $\exists f : \hat{\mathbb{A}}^2_{\mathbb{C}} \longrightarrow X$ isom with X affine var.

$E \cong \mathbb{P}^1_{\mathbb{C}} \Rightarrow f|_E : E \longrightarrow X$ is a morphism from a proj var to an aff. var. $\Rightarrow f(E) = \{\text{point}\}$.

Contradiction because $E \cong \mathbb{P}^1_{\mathbb{C}}$ and f is injective.