Foglio 5

Exercise 2 Let $X \overset{def}{=} V_p(x_1^2 + x_2^2 - x_0^2) \subseteq \mathbb{P}^2_{\mathbb{C}}$ be the projective closure of the (complex?) unit circle. Let

$$U \overset{def}{=} X \backslash V_p(x_1, x_0 - x_2)$$

and consider the morphism $\phi : U \to \mathbb{P}^1$ given by $\phi(x_0 : x_1 : x_2) = (x_0 - x_2 : x_1)$. Prove the following:

1. the rational map $f =< U, \phi >$ is in fact a moprhism
2. the morphism f is an extension to the north pole of the stereographic projection
3. find the image of the north pole
4. the morphism is an isomorphism, find its inverse.

Proof. 1. Let $\psi : V \to \mathbb{P}^1$ be defined by $[x_0 : x_1 : x_2] \overset{\psi}{\mapsto} [x_1 : x_0 + x_2]$ where $V \overset{def}{=} X \backslash V_p(x_1, x_0 + x_2)$. We claim that $f =< \phi, U >=< \psi, V >$. Consider the open dense set $U \cap V \ni x$, then

$$\left[1 : \frac{x_1}{x_0 - x_2}\right] = \left[1 : \frac{x_0 + x_2}{x_1}\right] \iff x_1^2 = x_0^2 - x_2^2$$

which is always true by definition of X. Finally, since $Dom f = X$ and ϕ, ψ agree on the overlap we may define

$$f(x = [x_0, x_1, x_2]) = \begin{cases} [x_0 - x_2, x_1] & if\ x \in U \\ [x_1, x_0 + x_2] & if\ x \in V \end{cases}$$

which is a morphism since can be locally written as homogeneous polimials of degree 1.

2. Let $(x, y) \in S^1 \setminus \{(0, 1)\}$, let us denote by π_N the stereographic projection. To begin with, let us compute $\pi_N\big((x, y)\big)$. Consider the parametric equation for the line r passing through $N = (0, 1)$ and (x, y):

$$r : \quad t(0, 1) + (1 - t)(x, y), \quad t \in \mathbb{R}.$$

Finding the projection of (x, y) is the same as finding the intersection between the $x-$axis and r. To do it, we just need to find t such that the second coordinate of the expression above equals zero, then substitute it into the first coordinate.
After performing some easy computations, we get that $t = 1 + \dfrac{1}{y - 1}$, hence $\pi_N(x, y) = (1 - t)x = \dfrac{x}{1 - y}$.

Let us now consider a point $p = (x_0 : x_1 : x_2) \in \big(X \setminus V_p(x_1, x_0 - x_2)\big) \cap U_0$. We have that the point corresponding to p in $\mathbb{A}^2$ is $\tilde{p} = (\frac{x_1}{x_0}, \frac{x_2}{x_0})$, hence, by the previous computation, we get that

$$\pi_N(\tilde{p}) = \frac{\frac{x_1}{x_0}}{1 - \frac{x_2}{x_0}} = \frac{x_1}{x_0 - x_2}.$$

To conclude, we have that the point in $\mathbb{P}^1$ corresponding to $\pi_N(\tilde{p})$ is $(\frac{x_1}{x_0 - x_2} : 1) = (x_1 : x_0 - x_2)$, which is by definition the image of p through ϕ. Hence, we have that ϕ is an extension of the stereographic projection.

3. The image of the north pole, $[1 : 0 : 1]$ is $g(1 : 0 : 1) = (1 : 0)$.

4. We write explicitly the inverse. Consider $U_0 \subseteq \mathbb{P}^1$ and note that again from the above equation we have that

$$x_0 + x_2 = \frac{x_1^2}{x_0 - x_2}$$

Let $(y_0 : y_1)$ be the coordinates on the codomain and note that since $\phi(x_0 : x_1 : x_2) = (x_1 : x_0 + x_2) = (y_0 : y_1)$ the above equation becomes

$$x_0 - x_2 = \frac{y_0^2}{y_1} \tag{1}$$

Since

$$x_0 = \frac{(x_0 - x_2) + (x_0 + x_2)}{2} = \frac{y_1}{2} + \frac{y_0^2}{2y_1}$$

and

$$x_2 = \frac{(x_0 - x_2) - (x_0 + x_2)}{2} = \frac{y_1}{2} - \frac{y_0^2}{2y_1}$$

we get that the function $\psi_0(y_0 : y_1) \overset{def}{=} (y_1^2 + y_0^2 : 2y_0y_1 : y_1 - y_0^2)$ (where we already multiplied by the common denominator $2y_1$) is in fact an inverse in U_1 of ϕ. Since it is defined on U_0 too it is a global inverse of the given moprphism.

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