Assignment 1

Exercise (9). Let X be a topological space and $\{U_\alpha\}_{\alpha\in A}$ an open covering such that U_α is irreducible for every $\alpha \in A$ and $U_\alpha \cap U_\beta \neq \emptyset$ for every $\alpha, \beta \in A$.

Claim. $X \neq \emptyset$ is irreducible $\iff$ every two non empty open subsets of X have a non empty intersection.

"$\Longrightarrow$" By contradiction: suppose U and V are two arbitrary non empty open subsets of X such that $U \cap V = \emptyset$.
Let's consider the complements: $U^C \cup V^C = X$ (i.e. we have the union of two proper closed sets whose union is X) but this is a contradiction because by definition of irreducible X cannot be written as union of two proper closed subsets.

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Let's consider the complements: $U^C \cap V^C = \emptyset$ (i.e. we have the intersection of two non empty open sets whose intersection is empty) but this is a contradiction of the hypothesis made.

Now we can take any two U and V non empty open subsets of X and show that their intersection is not empty.
Since $\{U_\alpha\}_{\alpha\in A}$ is an open covering, $\exists i, j \in A$ such that $U_i \cap U \neq \emptyset$ and $U_j \cap V \neq \emptyset$.
Since $U_i \cap U$ and $U_i \cap U_j$ are both non empty open subsets of U_i which is irreducible, we have that: $(U_i \cap U) \cap (U_i \cap U_j) = U \cap U_i \cap U_j \neq \emptyset$.
Since $U \cap U_i \cap U_j$ and $U_j \cap V$ are both non empty open subsets of U_j which is irreducible, we have that: $(U \cap U_i \cap U_j) \cap (U_j \cap V) = U \cap V \cap U_i \cap U_j \neq \emptyset$.
Which means that $U \cap V \neq \emptyset$ since by hypothesis $U_i \cap U_j \neq \emptyset$.

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