EXERCISE 2, 6th ASSIGNMENT

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Let $X \subseteq \mathbb{P}^m_{\mathbb{C}}$ be a projective variety such that $|X| \geq 2$ and $F \in \mathbb{C}[x_0, \dots, x_m]_d$ a homogeneous polynomial of degree $d \geq 1$

THEN: $X \cap V_p(F) \neq \phi$

Solution:

To solve the exercise we need two results done during the lectures

THEOR 1 $K = \bar{K}$, $X, Y \subseteq \mathbb{P}^m$ proj. variety (i.e. they are Zariski closed and irreducible)

If $\dim X + \dim Y - m \geq 0 \Rightarrow X \cap Y \neq \phi$

THEOR 2: $K = \bar{K}$, $f \in K[x_0, \dots, x_m]$, homogeneous nonconstant $\Rightarrow \dim V_p(f) = m-1$

We are now ready to solve the exercise

Let $F \in \mathbb{C}[x_0, \dots, x_m]_d$ with $d \geq 1$

then $F = F_1 \cdots F_n$ where F_j is irreducible $\forall j = 1, \dots, n$. Since $V_p(F) = \bigcup_{j=1}^{n} V_p(F_j)$, I can prove the exercise in the case that

F is irreducible. (because if $X \cap V_p(F_j) \neq \phi \Rightarrow X \cap V_p(F) \neq \phi$)

So let's assume F is irreducible $\Rightarrow V_p(F)$ is irreducible $\Rightarrow V_p(F)$ is a projective variety

Since $X \subseteq \mathbb{P}^m_{\mathbb{C}}$ is a projective variety such that $|X| \geq 2$ this means that $\dim X \geq 1$ since I can consider $x_0 \in X$

and $\phi \subsetneq \{x_0\} \subsetneq X$

Moreover, since $\mathbb{C}$ is algebraically closed and $d \geq 1$ I know that $\dim V_p(F) \overset{\text{THEOR 2}}{=} m-1$

$\Rightarrow \dim X + \dim V_p(F) - m \geq 1 + m - 1 - m \geq 0 \underset{\text{THEOR 1}}{\Rightarrow} X \cap V_p(F) \neq \phi$