EXERCISE 1 OF 4^{TH} ASSIGNMENT

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FIND TWO PROJECTIVE VARIETIES X AND Y THAT ARE ISOMORPHIC BUT SUCH THAT $S(X) \not\cong S(Y)$

Solution: Let $X = \mathbb{P}^1$, and $Y := V_p(x_0x_2 - x_1^2) \subseteq \mathbb{P}^2$

We can consider the Veronese embedding $\nu_{1,2} : (t_0 : t_1) \in \mathbb{P}^1 \to (t_0^2 : t_0t_1 : t_1^2) \in \mathbb{P}^2$

$\subseteq$) OBVIOUS

$$\operatorname{Im} \nu_{1,2} = \{ (y_0 : y_1 : y_2) \in \mathbb{P}^2 \mid y_0 y_2 = y_1^2 \} = V_p(x_0x_2 - x_1^2) = Y \subseteq \mathbb{P}^2$$

$\supseteq$) take $(y_0 : y_1 : y_2)$ if $y_0 = 0 \Rightarrow y_1 = 0 \Rightarrow (y_0 : y_1 : y_2) = (0:0:1) \in \operatorname{Im} \nu_{1,2}$

if $y_0 \neq 0 \Rightarrow (y_0 : y_1 : y_2) = (y_0^2 : y_0y_1 : y_0y_2) = (y_0^2 : y_0y_1 : y_1^2) \in \operatorname{Im} \nu_{1,2}$

$y_0 y_2 = y_1^2$

So, denoting with $f : X \to Y$ the function $\nu_{1,2}$ with codomain restricted to the image

We know from the theory that f is an isomorphism of projective varieties (*)

We, now, want to prove that $S(X)$ and $S(Y)$ are not isomorphic.

We know that $S(X) = S(\mathbb{P}^1) = \dfrac{K[x_0,x_1]}{\langle 0 \rangle} \cong K[x_0,x_1]$, this is a Unique Factorization Domain (UFD)

While $S(Y) = \dfrac{K[x_0,x_1,x_2]}{\langle x_0x_2 - x_1^2 \rangle}$ is NOT a UFD since the classes $\bar{x}_0, \bar{x}_1, \bar{x}_2 \in S(Y)$ are such that

$\overline{x_0x_2 - x_1^2} = \bar{0} \Rightarrow \bar{x}_0 \, \bar{x}_2 = \bar{x}_1^2 = \bar{x}_1 \cdot \bar{x}_1$, so the element $\bar{x}_1^2 \in S(Y)$ can be written in two ways as a product of irreducible elements of $S(Y)$ (thus $S(Y)$ is NOT a UFD)

Thus $S(X)$ and $S(Y)$ cannot be isomorphic because the property of being a Unique factorization domain is invariant under isomorphism

(*): We can also prove that f is an isomorphism directly:

$f : (t_0 : t_1) \in X \to (t_0^2 : t_0t_1 : t_1^2) \in Y$

f is a morphism since it is locally given by homogeneous polynomials of the same degree

we know f is surjective by definition and that f is injective since $\nu_{1,2}$ is injective

Moreover it's easy to check that φ, defined as follows, is well-defined and it's the inverse of f

$$\varphi : Y \to X \quad \text{such that } \varphi(x_0 : x_1 : x_2) = \begin{cases} (x_0 : x_1) & \text{if } x_0 \neq 0 \\ (x_1 : x_2) & \text{if } x_2 \neq 0 \end{cases}$$

$\varphi(f(t_0:t_1)) = \varphi(t_0^2 : t_0t_1 : t_1^2) = \begin{cases} (t_0^2 : t_0t_1) = (t_0 : t_1) & \text{if } t_0 \neq 0 \\ (t_0t_1 : t_1^2) = (t_0 : t_1) & \text{if } t_1 \neq 0 \end{cases}$

$f(\varphi(t_0^2 : t_0t_1 : t_1^2)) = f(t_0 : t_1) = (t_0^2 : t_0t_1 : t_1^2)$

In the end, φ is a morphism since it's locally given by homogeneous polynomials of the same degree