AG 3 - exercises of the third assignment

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Definition 1. *Let X be a qp variety, let $Aut(X)$ be the set of automorphism of X (i.e the isomorphism from X to X), then clearly $Aut(X)$ is a group with respect to the composition operation.*

In particular one can consider $Aut(\mathbb{P}^1_{\mathbb{K}})$.

For any

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{K})$$

Let $\varphi_A : \mathbb{P}^1_{\mathbb{K}} \to \mathbb{P}^1_{\mathbb{K}}$ be the map defined in this way

$$\varphi_A([x_0, x_1]) := [ax_0 + bx_1, cx_0 + dx_1]$$

so basically $\varphi_A([x]) := [A \cdot x]$.

Let $\mathbb{PGL}(2, \mathbb{K}) = \mathbb{P}(GL(2, \mathbb{K})) = \{\varphi_A \ : \ A \in GL(2, \mathbb{K})\}$

Then it's clear that $\mathbb{PGL}(2, \mathbb{K})$ is a subgroup of $Aut(\mathbb{P}^1_{\mathbb{K}})$

($\varphi_A \circ \varphi_B = \varphi_{A \cdot B}$ therefore $\varphi_{A^{-1}} = (\varphi_A)^{-1}$)

Exercise 3. *If $\mathbb{K} = \overline{\mathbb{K}}$ then $\mathbb{PGL}(2, \mathbb{K}) = Aut(\mathbb{P}^1_{\mathbb{K}})$*

Proof. $\subset$ has already been proven, therefore I only prove $\supset$

Let $\varphi \in Aut(\mathbb{P}^1_{\mathbb{K}})$, I want to show that $\varphi \in \mathbb{PGL}(2, \mathbb{K})$

Claim : there exists $A \in GL(2, \mathbb{K})$ such that $\varphi_A(\varphi([0, 1])) = [0, 1]$.

This is just an easy linear algebra exercise, in this particular case let $[P_0, P_1] = \varphi([0, 1])$, then

$$\text{if } P_0 \neq 0 \ \ A = \begin{pmatrix} P_1 & -P_0 \\ -1/P_0 & 0 \end{pmatrix} \qquad \text{if } P_1 \neq 0 \ \ A = \begin{pmatrix} P_1 & -P_0 \\ 0 & 1/P_1 \end{pmatrix}$$

Since $\mathbb{PGL}(2, \mathbb{K})$ is closed under composition and

$$\varphi = \varphi_{A^{-1}} \circ (\varphi_A \circ \varphi)$$

I have that

$$\varphi \in \mathbb{PGL}(2, \mathbb{K}) \iff \varphi_A \circ \varphi \in \mathbb{PGL}(2, \mathbb{K})$$

therefore, wlog, I can assume $\varphi([0, 1]) = [0, 1]$

Now since φ is bijective it is in particular injective, therefore for any $P \in U_0$ $\varphi(P) \in U_0$, therefore I can restrict the domain and the image of φ on U_0, obtaining $\varphi_{|U_0}$ which will be an automorphism of U_0. then since $j_0 : \mathbb{A}^1_{\mathbb{K}} \to U_0$ and $\rho_0 : U_0 \to \mathbb{A}^1_{\mathbb{K}}$ are both isomorphism I have that

$$\varphi' := \rho_0 \circ \varphi_{|U_0} \circ j_0 \; : \; \mathbb{A}^1_{\mathbb{K}} \to \mathbb{A}^1_{\mathbb{K}}$$

Is an automorphism of $\mathbb{A}^1_{\mathbb{K}}$, thus using the theorem about morphism of affine varietis (which needs the closureness of $\mathbb{K}$ as hypothesis!) I have that $\varphi \in \mathbb{K}[x]$, in particular since φ has to be invertible and the inverse need to be a morphism, it's easy to see that $\varphi(x) = a + bx$ with $b \neq 0$

Now let

$$A := \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}$$

I claim that $\varphi = \varphi_A$

if $P = [0, 1]$ then it's clear that $\varphi(P) = [0, 1] = \varphi_A(P)$

if $P \in U_0$ then $P = [1, P_1]$ for some $P_1 \in \mathbb{K}$ and

$$\varphi(P) = j_0 \circ \varphi' \circ \rho_0(P) = j_0 \circ \varphi'(P_1) = j_0(a + bP_1) = [1, a + bP_1] = \varphi_A(P)$$

Thus $\varphi = \varphi_A \in \mathbb{PGL}(2, \mathbb{K})$ as wanted to show □

Exercise 4. *Let $\varphi \; : \; X \to Y$ be a morphism between affine varietis $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$, and consider the pull-back $\varphi^* \; : \; \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Prove that $ker\varphi^*)I(\varphi(X)) \subset \mathcal{O}_Y(Y)$, where $I(\phi(X))$ denotes the ideal of the image in of φ in the ring $\mathcal{O}_Y(Y)$*

Proof.

$$\begin{aligned} ker\phi^* &= \{f \in \mathcal{O}_Y(Y) \; : \; \varphi^*(f) = 0\} \\ &= \{f \in \mathcal{O}_Y(Y) \; : \; \varphi^*(f)(x) = 0 \; \forall x \in X\} \\ &= \{f \in \mathcal{O}_Y(Y) \; : \; f(\varphi(x)) = 0 \; \forall x \in X\} \\ &= \{f \in \mathcal{O}_Y(Y) \; : \; f(y) = 0 \; \forall y \in Y\} \\ &= I(\phi(X)) \end{aligned}$$

□

Exercise 5. *Prove that the projection*

$$\pi \; : \; \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n \,, \quad \pi(a_0, \ldots, a_n) = [a_0 : \cdots : a_n]$$

is a morphism of qp varietis

Proof. I prove the theorem applying the definition.

First I prove that π is continuos. Let $Z \subset \mathbb{P}^n$ be closed, then $Z = V_H(f_1, \ldots, f_r)$ with $f_i \in \mathbb{K}[x_0, \ldots, x_n]$ all homogeneous. Then it's clear that

$$\begin{aligned} \pi^{-1}(Z) &= \{x \in \mathbb{A}^{n+1} \setminus \{0\} \; : \; \pi(x) \in Z\} \\ &= \{x \in \mathbb{A}^{n+1} \setminus \{0\} \; : \; \pi(x) \in V_H(f_1, \ldots, f_r)\} \\ &= \{x \in \mathbb{A}^{n+1} \setminus \{0\} \; : \; f_1(x) = \cdots = f_r(x) = 0\} \\ &= \mathbb{A}^{n+1} \setminus \{0\} \cap V(f_1, \ldots, f_r) \end{aligned}$$

Therefore it is clear that $\pi^{-1}(Z)$ is closed in $\mathbb{A}^{n+1} \setminus \{0\}$

Now I prove that for any regular function $\varphi \in \mathcal{O}_{\mathbb{P}^n}(U)$

$\varphi \circ \pi \in \mathcal{O}_{\mathbb{A}^{n+1}\setminus\{0\}}(\pi^{-1}(U))$

Let $P \in \pi^{-1}(U)$, then $\pi(P) = [P] \in U$ so there exists $f, g \in \mathbb{K}[x_0, x_1, \ldots, x_n]$ homogeneous of the same degree such that $g(P) \neq 0$ and $\varphi = f/g$

Observe that $\varphi([x]) = f(x)/g(x)$ (at least when $x \in U \setminus V(g)$), therefore it's clear that $\phi(\pi(x)) = \phi([x]) = f(x)/g(x)$

Since P is arbitrary, by definition I have proved

$\varphi \circ \pi \in \mathcal{O}_{\mathbb{A}^{n+1}\setminus\{0\}}(\pi^{-1}(U))$ □