SECOND ASSIGNEMENT 24/10/2022

3) Consider $X \subseteq \mathbb{A}^n$ a Zariski closed consisting of two points.

Prove that $A(X) \cong \mathbb{K} \oplus \mathbb{K}$

<u>proof</u>: $X = \{p, q\}$, $I(X) = I(\{p\}) \cap I(\{q\})$

$$X = \{p\} \cup \{q\}$$

Note that $I(p)$ and $I(q)$ are maximal and coprime ideals.

$$A(X) = \frac{\mathbb{K}[x_1 .. x_n]}{I(X)} = \frac{\mathbb{K}[x_1, \dots, x_n]}{I(p) \cap I(q)} \underset{\substack{\uparrow \\ \text{chinese} \\ \text{remainder} \\ \text{theorem}}}{\cong} \underbrace{\frac{\mathbb{K}[x_1 .. x_n]}{I(p)}}_{\cong \mathbb{K}} \oplus \underbrace{\frac{\mathbb{K}[x_1 .. x_n]}{I(q)}}_{\cong \mathbb{K}}$$

4) Let $f: X \longrightarrow Y$ be a morphism e let $\Gamma_f = \{(x,y) \in X \times Y \mid y = f(x)\}$ be its graph. Prove that Γ_f is isomorphic to X.

<u>proof</u>: Since $f: X \to Y$ is a morphism $f = (f_1, \dots, f_m)$ for some $f_1, \dots, f_m \in A(X)$ so that

$$f(p) = (f_1(p), \dots, f_m(p)) \quad \forall\, p \in X.$$

Consider the function

$$\begin{aligned} \gamma_f : X &\longrightarrow X \times Y \\ p &\longmapsto (p, f(p)) = \\ &= (p, f_1(p), \dots, f_m(p)) \end{aligned}$$

Then $\gamma_f : X \longrightarrow \Gamma_f \subseteq X \times Y$ is a morphism and bijective.

Moreover its inverse map is the first projection

$$\begin{aligned} \pi_1 : \Gamma_f &\longrightarrow X \\ (p, f_1(p), \dots, f(p)) &\longmapsto p \end{aligned}$$

which is again a morphism. So γ_f is an isomorphism and so $X \cong \Gamma_f$.

FOURTH ASSIGNEMENT 13/11/2022

2) Let X, Y be irred. top. spaces. Assume $X \times Y$ has a topology for which the inclusions

$$j_p : Y \to X \times Y \quad , \quad b \mapsto (p, b)$$

and $$i_q : X \to X \times Y \quad , \quad a \mapsto (a, q)$$

are continous for all $p \in X$ and for all $q \in Y$. Prove that $X \times Y$ is irreducible.

<u>proof</u>: A top. space X is Irreducible iff $\forall U, V$ open in X, $\neq \emptyset$ $U \cap V \neq \emptyset$.

So let us prove that for any $U, V \subseteq X \times Y$ open, $\neq \emptyset$ $U \cap V \neq \emptyset$.
To this end consider $U, V \subseteq X \times Y$ open, $\neq \emptyset$

$U, V \neq \emptyset \Rightarrow$ pick $p \in U$, $q \in V$
$p = (p_1, p_2)$, $q = (q_1, q_2)$

Consider

$$j_{p_1}^{-1}(U) = \{a \in Y \mid (p_1, a) \in U\} \overset{\neq \emptyset}{\supseteq} \{p_2\}$$
$$j_{q_1}^{-1}(V) = \{a \in Y \mid (q_1, a) \in V\} \overset{\neq \emptyset}{\supseteq} \{q_2\}$$

By the continuity of j_{p_1} and j_{q_1}, $j_{p_1}^{-1}(U)$ and $j_{q_1}^{-1}(V)$ are open and $\neq \emptyset$ subset of Y and so by irred. of Y they must have non empty intersection $\Rightarrow \exists\, t \in j_{p_1}^{-1}(U) \cap j_{q_1}^{-1}(V)$
$\Rightarrow (p_1, t) \in U$ and $(q_1, t) \in V$.
Again consider

$$i_t^{-1}(U) = \{b \in X \mid (b, t) \in U\} \overset{\neq \emptyset}{\supseteq} \{p_1\}$$
$$i_t^{-1}(V) = \{b \in X \mid (b, t) \in V\} \underset{\neq \emptyset}{\supseteq} \{q_1\}$$

$i_t^{-1}(U)$, $i_t^{-1}(V)$ are again open, $\neq \emptyset$ in X and so $i_t^{-1}(U) \cap i_t^{-1}(V) \neq \emptyset$
$\Rightarrow \exists\, t_1 \in i_t^{-1}(U) \cap i_t^{-1}(V)$

$\Rightarrow (t_1, t) \in U$ & $(t_1, t) \in V$
$\Rightarrow (t_1, t) \in U \cap V \Rightarrow U \cap V \neq \emptyset$ as claimed. $\square$