FOGLIO 3:

i) Let $X \subseteq \mathbb{P}^n_K$ be a q.p. variety. Let U be open and $\{U_\alpha\}_{\alpha\in J}$ be an open cover of U. Prove that $\varphi : X \to Y$ is a morphism iff $\forall \alpha \in J$, $\varphi|_{U_\alpha}$ is a morphism.

Proof:

Let $\varphi : U \to Y$ be a morphism. Thus $\forall x \in U$, $\exists V_x \in N(x)$ s.t. $\varphi|_{V_x}$ is given by homogeneous poly. of the same degree. Thus $\forall \alpha \in J$, $\forall x \in U_\alpha$, $V_x \cap U_\alpha$ is an open neigh of x in U_α s.t. $\varphi|_{V_x \cap U_\alpha}$ is given by polynomials of the same degree $\Rightarrow \varphi|_{U_\alpha}$ is a morphism $\forall \alpha \in J$.

Now assume $\forall \alpha \in J$, $\varphi|_{U_\alpha}$ is a morphism. Let $x \in U$. By def of cover, $\exists \alpha \in J$ s.t. $x \in U_\alpha \Rightarrow \exists V_x \in N(x)_{U_\alpha}$ s.t. $(\varphi|_{U_\alpha})|_{V_x} := \varphi|_{V_x}$ is given by homogeneous poly. of the same degree. Being U_α open, V_x is open in U $\Rightarrow \varphi$ is locally given by homogeneous poly $\Rightarrow \varphi$ is a morphism.

ii) Show that $Aut(\mathbb{P}^1_K) = PGL(K, 1)$ when $K = \overline{K}$.

Proof:

Let $\varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K$ be an isomorphism.

Obs: φ is locally given by homogeneous polynomials $\Rightarrow$ fixed $[x_0 : x_1]$, $\exists U \subseteq \mathbb{P}^1_K$ open neigh of $[x_0 : x_1]$ s.t. $\varphi|_U = [F_1 : F_1]$. I may assume F_1 and F_2 are coprime. Thus $\forall [x_0 : x_1] \in \mathbb{P}^1_K$ $[F_1[x_0 : x_1] : F_2[x_0 : x_1]] \neq [0:0] \Rightarrow \varphi$ is globally given by F_1 and F_2 homogeneous of the same degree d.

Now φ isomorphism $\Rightarrow \exists [\tilde{x}_0 : \tilde{x}_1], [\tilde{\tilde{x}}_0 : \tilde{\tilde{x}}_1] \in \mathbb{P}^1_K \mid \varphi[\tilde{x}_0 : \tilde{x}_1] = [1:0]$ and $\varphi[\tilde{\tilde{x}}_0 : \tilde{\tilde{x}}_1] = [0:1]$. Thus $\tilde{x}_1 x_0 - \tilde{x}_0 x_1 \mid F_1$ and $\tilde{\tilde{x}}_1 x_0 - \tilde{\tilde{x}}_0 x_1 \mid F_2$.

Being φ injective if $\varphi[a_0 : a_1] = [0:1]$, $\varphi[b_0 : b_1] = [1:0]$, $[a_0 : a_1] = [\tilde{\tilde{x}}_0 : \tilde{\tilde{x}}_1]$ and $[b_0 : b_1] = [\tilde{x}_0 : \tilde{x}_1] \Rightarrow a_1 x_0 - a_0 x_1 \mid F_1 \Rightarrow a_1 = \lambda \tilde{\tilde{x}}_1$, $a_0 = \lambda \tilde{\tilde{x}}_0$ and $b_1 x_0 - b_0 x_1 \mid F_2 \Rightarrow b_1 = \lambda' \tilde{x}_1$, $b_0 = \lambda' \tilde{x}_0$.

Now being $K = \overline{K}$, F_1, F_2 homogeneous of deg d in 2 variables,

$F_1(x_0, x_1) = \prod_{j=1}^{d} (\alpha_{j1} x_0 - \alpha_{j2} x_1)$ and $F_2(x_0, x_1) = \prod_{j=1}^{d} (\beta_{j1} x_0 - \beta_{j2} x_1)$.

By what's seen above, $F_1(x_0, x_1) = \lambda (\tilde{\tilde{x}}_1 x_0 - \tilde{\tilde{x}}_0 x_1)^d$ and $F_2(x_0, x_1) = \mu (\tilde{x}_1 x_0 - \tilde{x}_0 x_1)^d$. Up to projectivities, I may assume $\tilde{\tilde{x}}_0 = 0$, $\tilde{x}_1 = 0 \Rightarrow F_1(x_0, x_1) = \mu x_0^d$, $F_2(x_0, x_1) = \upsilon x_1^d$.

Thus $\varphi[x_0 : x_1] = [u x_0^d : v x_1^d]$. Being φ injective $u, v \neq 0$.
So, up to projectivities I may assume $u = v = 1$. Suppose $d > 1$.

Now let $[a_0 : b_1] \in \varphi^{-1}[1 : 1]$. Chosen ω_d d-th root of unity different from 1, $\varphi[\omega_d a_0 : b_1] := [\omega_d^d a_0^d : b_1^d] = [1 : 1]$, but $[\omega_d a_0 : b_1] \neq [a_0 : b_1]$, contradicting φ injective. ■

iii) Let $\varphi : X \to Y$ be a morphism between affine varieties $X \subseteq \mathbb{A}^n_K$, $Y \subseteq \mathbb{A}^m_K$.
Consider $\varphi^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ the pull-back of φ. Prove that $\mathrm{Ker}\, \varphi^* = I(\varphi(X)) \subseteq \mathcal{O}_Y(Y)$

Proof:

$\mathrm{Ker}\, \varphi^* := \{ f \in \mathcal{O}_Y(Y) \mid f \circ \varphi = 0 \} := \{ f \in \mathcal{O}_Y(Y) \mid \forall x \in X,\ f(\varphi(x)) = 0 \}$
$:= I(\varphi(X))$ □

iv) Prove that $\pi : \mathbb{A}^{m+1}_K \setminus \{0\} \to \mathbb{P}^m_K \mid (a_0, \ldots a_m) \mapsto [a_0 : \ldots a_m]$ is a irreducible and with non-empty intersection. Then $U \cup V$ is irreducible.

Proof:

Suppose $U \cup V$ were to reducible then $\exists\, C_1, C_2$ closed in $U \cup V$ s.t. $U \cup V = C_1 \cup C_2$ and C_1, C_2 proper subsets.

Ok. Suppose $U \not\subseteq C_1$. Then $U \subseteq C_2$. Otherwise, either $U \cap C_1 \neq \emptyset$ and then $(U \cap C_1) \cup (U \cap C_2)$ would make U reducible, or $U \cap C_1 = \emptyset \Rightarrow (U \cup V) \cap U =$
$= U \cap (C_1 \cup C_2) = U \cap C_2 \neq U$. Contradiction $\Rightarrow U \subseteq C_2$.

Now then for the same reason, $V \subseteq C_1$, otherwise, $V \subseteq C_2 \Rightarrow U \cup V \subseteq C_2$ proper subset of $U \cup V$. Contradiction $\Rightarrow U \subseteq C_2$, $V \subseteq C_1$.

Now $\exists\, x \in U, y \in V \mid x \in U \setminus V$, $y \in V \setminus U$, otherwise either $U \cup V = V$ or $U \cup V = U \Rightarrow U \cup V$ irreducible. Contradiction.

I can also assume that either U or V is contained in only one of C_2 or C_1, otherwise $U \cup V \subseteq (C_1 \cap C_2) \cup (C_1 \cap C_2) = C_1 \cap C_2 \subseteq C_1$. Contradiction.

Suppose $V \not\subseteq C_2$. Then we have:

$$V \supseteq (V \cap C_2) \cup (V \cap \complement U) \supseteq (V \cap C_2) \cup (V \cap \complement C_2) = V$$

$\Rightarrow V \cap C_2$ and $V \cap \complement U$ are two closed proper ($y \in V \cap \complement U$) subset of V s.t. their union is $V \Rightarrow V$ reducible. Contradiction. □

Now suppose X were to be reducible $\Rightarrow \exists\, C_1, C_2 \subseteq X$ proper closed subsets s.t. $X = C_1 \cup C_2$.

I ~~know that~~ can suppose that each $U_\alpha \subsetneq X$. Otherwise, X would be irreducible.
$\Rightarrow \forall \alpha \in J, \exists\, x_\alpha \in X \setminus U_\alpha$.

Now $\forall \alpha \in J$, $U_\alpha \subseteq C_1 \vee U_\alpha \subseteq C_2$. In fact if $U_\alpha \not\subseteq C_1 \wedge U_\alpha \not\subseteq C_2$, $U_\alpha \cap C_1$, $U_\alpha \cap C_2$ would be the two close subset of it that would make it reducible. Now for the same reasons seen in the lemma, $\exists\, \alpha_1, \alpha_2 \in J$ s.t. $U_{\alpha_1} \subseteq C_1$, $U_{\alpha_2} \subseteq C_2$ and $U_{\alpha_1} \cap U_{\alpha_2} \neq U_{\alpha_1}, U_{\alpha_2}$.

Otherwise each of them would be contained in either C_1 or C_2.

Now $C_1 \cap (U_{\alpha_1} \cup U_{\alpha_2})$, $C_2 \cap (U_{\alpha_1} \cup U_{\alpha_2})$ are two closed ~~subsets~~ proper subsets of $U_{\alpha_1} \cup U_{\alpha_2}$ whose union is $U_{\alpha_1} \cup U_{\alpha_2}$. Contradicting the lemma. □

$\Rightarrow \mathcal{C} = V(I)$, $\mathcal{D} = V(J)$ are irreducible.

$\mathcal{C} \cap \mathcal{D} = V(x^2+y^2-\frac{1}{4}, x^2+y^2+z^2-1) = \{ \underline{x} \in \mathbb{R}^3 \mid x^2+y^2 = \frac{1}{4} \wedge z = \pm \frac{\sqrt{3}}{2} \} = \{ \underline{x} \in \mathbb{R}^3 \mid x^2+y^2 = \frac{1}{4}, z = \frac{\sqrt{3}}{2} \} \cup \{ \underline{x} \in \mathbb{R}^3 \mid x^2+y^2 = \frac{1}{4}, z = -\frac{\sqrt{3}}{2} \}$.

which is obviously reducible being the union of two disjoint closed subsets: $V(x^2+y^2-\frac{1}{4}, z-\frac{\sqrt{3}}{2})$ and $V(x^2+y^2-\frac{1}{4}, z+\frac{\sqrt{3}}{2})$. //

ix) Let X be a topological space and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ be an open covering s.t. $U_\alpha \cap U_\beta \neq \emptyset$ $\forall \alpha, \beta \in J$ and s.t. U_α is irreducible $\forall \alpha \in J$.

Prove that X is irreducible.

Proof:

Lemma: Let U, V be open subsets of a topological space X. If U and V are

Obs: $I(X) = (x-1, y) \neq \alpha \Rightarrow \alpha$ is not radical. $\square$

v) Let $f_1, \dots f_m \in K[x_1, \dots x_n]$. Let $\varphi : \mathbb{A}^n \to \mathbb{A}^m \mid \underline{x} \mapsto (f_i(\underline{x}))_{i=1}^m$.

Show that
$$\Gamma_\varphi := \{(\underline{x}, y) \mid y = \varphi(\underline{x})\}$$
is closed.

Let $X_i := V(x_{m+i} - f_i(x_1, \dots x_n))$ and $X := \bigcap_{i=1}^m X_i$.

Obs: ~~$x \in$~~ $(\underline{x}, y) \in X \iff \forall\, i = 1, \dots m,\ \ y_i = f_i(x_1, \dots x_n) \iff y = \varphi(\underline{x})$.

$\Rightarrow X = \Gamma_\varphi$. //

vi) Let $f : X \to Y$ be a continuous map between topological spaces.

Prove that if X is irreducible, then $f(X)$ is irreducible.

$g(p) \neq 0 \Rightarrow$ chosen $f = \frac{1}{g(p)} g$, $f(p) = 1$ and $f|_X \equiv 0$. $\square$

iii) $K[x_1, \dots x_n] \subseteq \mathcal{C}^0(\mathbb{A}^n_K, \mathbb{A}^1_K)$.

Proof:

Let $f \in K[x_1, \dots x_n]$ and let $p \in \mathbb{A}^1_K$. $f^{-1}(p) = \{ \underline{x} \in \mathbb{A}^n_K \mid f(\underline{x}) = p \}$

$= V(f - p)$ which is closed. $\square$

iv) Let $\alpha := (x^2 + y^2 - 1, x - 1)$. Determine $X = V(\alpha)$ and $I(X)$.

Is α radical?

$X = \{ (x, y, z) \in \mathbb{A}^{2+n}_K \mid x = 1 \wedge x^2 + y^2 = 1 \} = \{ (x, y, z) \in \mathbb{A}^{2+n}_K \mid x = 1, y = 0 \}$