Jan 11

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11) Find a Banach space X and a closed bounded $S \subseteq X$ and a continuous function

$$f : S \longrightarrow \mathbb{R}$$

$$\sup_{x \in S} f = +\infty$$

Answer

Recall that if W is a closed

vector subspace of X $\exists$ a sequence

~~off~~ $\{x_n\}$ in X. with $|x_n| = 1$

s.t. $\operatorname{dist}(x_n, W) \longrightarrow 1$

If we ~~fix~~ $0 < r < 1$ then

$\forall$ W like above $\exists$ $x \in X$

$|x| = 1$ st $\operatorname{dist}(x, W) > r$

~~Ex~~ if $\dim X = +\infty$

we can then conclude that $\exists \{x_n\}$ in

X with $|x_n| = 1$ st

$|x_n - x_m| > r \quad \forall\ n \neq m.$

$$\Rightarrow \overline{D_X\left(x_n, \tfrac{r}{3}\right)} \cap \overline{D_X\left(x_m, \tfrac{r}{3}\right)} = \phi$$

$\left\{ \overline{D_X\left(x_n, \tfrac{r}{3}\right)} \right\}$ $\qquad x_n \in \partial D_X(0, 1)$

$$g : \mathbb{R} \longrightarrow \mathbb{R} \qquad g \in C_c^0(\mathbb{R}, [0,1])$$

$$\operatorname{supp} g \subset \left[-\frac{r}{4}, \frac{r}{4}\right] \qquad g(0) = 1$$

$$g(\|x\|) \in C^0(X, [0,1])$$

$$n\, g(\|x - x_n\|_{\cancel{X}})$$

$$f(x) = \sum_{n=1}^{\infty} n\, g(\|x - x_n\|_X)$$

$$x \notin \bigcup \overline{D_X\left(x_n, \frac{r}{3}\right)} \longleftarrow$$ it is a ~~clo~~ closed set

$$\Rightarrow \quad \|x - x_n\|_X \geq \frac{r}{3} > \frac{r}{4} \quad \forall n$$

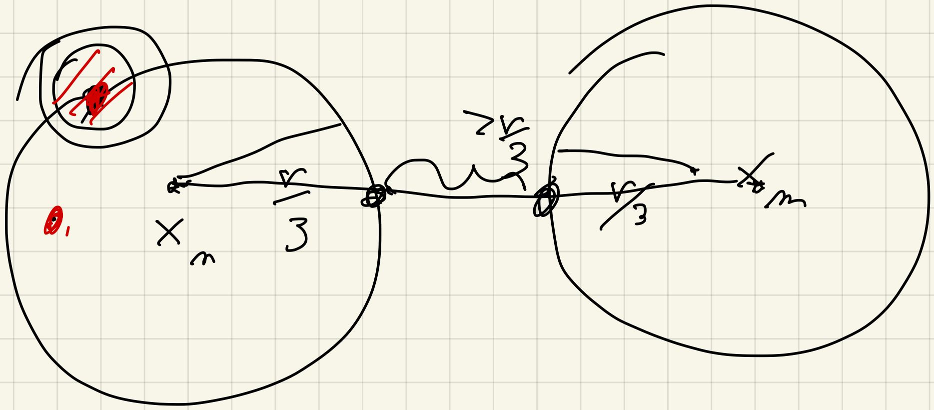
$$\Rightarrow \quad g(\|x - x_n\|_X) = 0 \quad \forall n$$

If $x \in \overline{D_{\cancel{X}}\left(x_{n_0}, \frac{r}{3}\right)} \Rightarrow$

$$x \notin \overline{D_X\left(x_m, \frac{r}{3}\right)} \quad \forall\, m \neq n_0$$

$$\Rightarrow \quad f(x) = n_0\, g(\|x - x_{n_0}\|)$$

$D_n($



$$r < \mathrm{dist}(x_n, x_m)$$

$$f|_{\text{red disk}} = n\, g(\|x - x_n\|) \quad \text{is continuous}$$

$$\bigcup_{n=1}^{\infty} \overline{D_X(x_n, r/3)} \subset \overline{D_X(0, 1 + \frac{r}{3})} = S$$

$$x \in \overline{D_X(x_n, r/3)}$$

$$\|x\| = \|x - x_n + x_n\| \leq \underbrace{\|x - x_n\|}_{\leq r/3} + \underbrace{\|x_n\|}_{1}$$

$$f : S \longrightarrow \mathbb{R}$$

$$f(x_n) = n$$

$H^s(\mathbb{T}^d)$

$$\hat{f}(n) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-ix\cdot n} f(x)\, dx$$

$n \in \mathbb{Z}^d$

$$\langle \xi \rangle = \left(1 + |\xi|^2\right)^{\frac{1}{2}}$$

$$\langle n \rangle^s \hat{f}(n) \in \ell^2(\mathbb{Z}^d)$$

$$\sum_{n\in\mathbb{Z}^d} \langle n \rangle^{2s} |\hat{f}(n)|^2 = \|f\|^2_{H^s}$$

$H^s(\mathbb{T})$ is the completion of the vector space of trigonometric polynomials using the norm $\|\cdot\|_{H^s}$

Show ~~th~~ that for $s \in \mathbb{N}$ $\|\cdot\|_{H^s}$ is equivalent to the norm

$f \in TP$

$$|||f|||_{TP} := \sqrt{\sum_{|\alpha|\le s} |\partial_x^\alpha f|^2_{L^2(\mathbb{T}^d)}}$$

X $\qquad \|\ \|_1 = \sum \|\partial_x^\alpha f\|_{L^2}$

$\|\cdot\|_2$ they are

equivalent if $\exists\, C \geq 1$ s.t.

$\forall\, x \in X$

$$\frac{1}{C}\|x\|_2 \leq \|x\|_1 \leq C\|x\|_2$$

If $s > 0$ then

$$\begin{tikzcd}
H^s(\mathbb{T}^d) \arrow[r, hook] & L^2(\mathbb{T}^2) \\
\ell^{2,s}(\mathbb{Z}^d) \arrow[u, "\text{map isomorphism}"] \arrow[r, "s>0"'] & \ell^2(\mathbb{Z}^d) \arrow[u, leftrightarrow, "\text{isomorphism}"']
\end{tikzcd}$$

is compact

$$\ell^{2,s}(\mathbb{Z}^d) = \left\{ (x_n)_{n\in\mathbb{Z}^d} : \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} |x_n|^2 < \infty \right\}$$

~~def~~ $\quad x_n \xrightarrow{n\to+\infty} x$ in X, Y B-spaces

$T : X \longrightarrow Y$ compact

$\Longrightarrow \quad Tx_n \longrightarrow Tx$ strongly in Y.

If false $\exists \ \varepsilon_0 > 0$ and a subsequence

x_{n_k} s.t. $\quad \|Tx_{n_k} - Tx\|_Y \geq \varepsilon_0 \quad \forall k \in \mathbb{N}$.

On the other hand

$$T : (X, \sigma(X, X')) \longrightarrow (Y, \sigma(Y, Y'))$$

$\Longrightarrow \quad Tx_n \longrightarrow Tx$

~~$\{Tx_{n_k}\}$ is a k~~

x_{n_k} is a bounded sequence

$\Longrightarrow \quad Tx_{n_k}$ has ~~a~~ convergent subsequence $\exists \ y \in Y$

s.t. $\quad Tx_{n_k} \longrightarrow y$ strongly in Y.

$\Longrightarrow \quad Tx_{n_k} \longrightarrow y$

$Tx_{n_k} \longrightarrow Tx \qquad Tx = y$

$$\|Tx_{n_k} - Tx\| \geq \varepsilon_0 > 0$$

$$\Big\Downarrow k \to +\infty$$

$$0 = \|y - Tx\| \geq \varepsilon_0$$

8 6. $I = [0,1]$ $u_n, v_n \in L^2(I) \cap C^0(I)$

u_n they are all increasing functions

$$\|u_n\|_{L^2} \leq C < +\infty$$

$$\|v_n\|_{L^2} \leq C < +\infty$$

Then $f_n(x,y) \doteq u_n(x)\, v_n(y) \in L^2(I \times I)$

is a <u>relatively compact sequence in</u> $\underline{L^1(I \times I)}$

u_n is relatively compact in $L^1(I)$

$L^2(I)$

$\chi \in C^\infty_c([0,+\infty))$

$n^{\frac{1}{2}} \chi(nx)$

$f \in L^p(\mathbb{R}^d)$ $n^{\frac{d}{p}} f(nx)$

$$\sup n^{\frac{1}{2}} \chi(n x) \subseteq [0, \tfrac{1}{n}]$$

$$n^{\frac{1}{2}} \chi(n x) \longrightarrow 0 \quad \text{in } L^2([0,1])$$

$$\| n^{\frac{1}{2}} \chi(n \cdot) \|_{L^2} = \| \chi \|_{L^2}$$

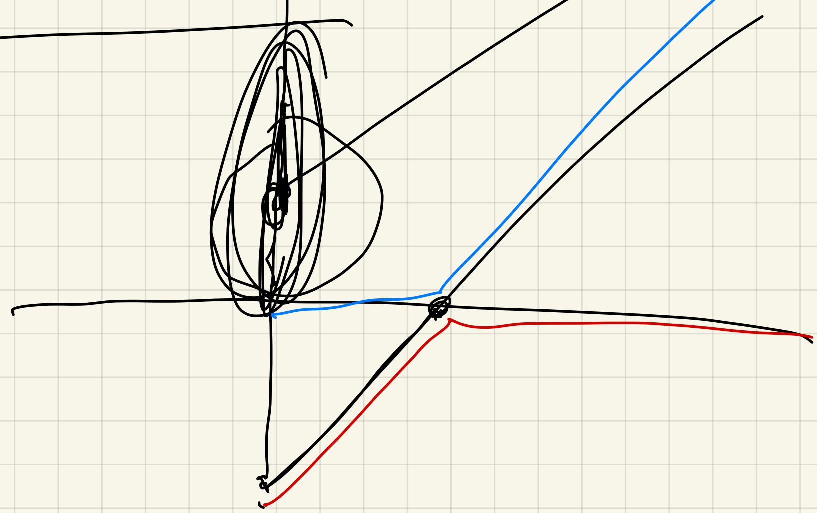
$$\| n^{\frac{1}{2}} \chi(n \cdot) \|_{L^1} = n^{-\frac{1}{2}} \| \chi \|_{L^1}$$

$$\| n^{\frac{1}{2}} \chi(n \cdot) \|_{L^p} = n^{\frac{1}{2}} \| \chi(n \cdot) \|_{L^p} =$$

$$= n^{\frac{1}{2}} \, n^{-\frac{1}{p}} \| \chi \|_{L^p}$$

$$= n^{\frac{1}{2}} \, n^{-1} \| \chi \|_{L^1} \xrightarrow{n \to +\infty} 0$$

$$u_n(0) = 0 \quad \forall n$$

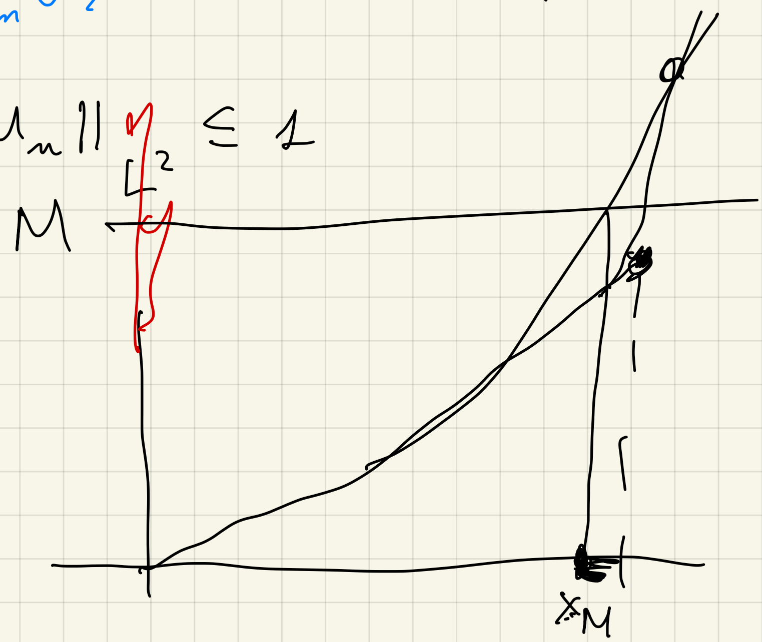


$$\lim_{n \to +\infty} u_n(0) = +\infty$$

$$\|u_n\|_{L^2} \geq \|u_n(0)\|_{L^2} = u_n(0) \longrightarrow +\infty$$

$$u_n(0) = 0 \qquad \forall n$$

$$\|u_n\|_{L^2} \le 1$$



$M > 0$

$$1 - x_M^{(n)} \quad \left|\{\cancel{x} : u_n^2(x) \geq M^2\}\right| \leq \frac{\|u_n\|_{L^2}^2}{M^2} \leq \frac{1}{M^2}$$

$$|\{x : |f(x)| > M\}| \leq \frac{1}{M}\|f\|_{L^1}$$

$\forall n$

$$\{x : u_n(x) \geq M\} = [x_M^{(n)}, 1]$$

$$1 - x_M^{(n)} \leq \frac{1}{M^2} \qquad x_M^{(n)} \geq 1 - \frac{1}{M^2}$$

in $\left[0, 1-\frac{1}{M^2}\right]$

$$0 \le u_n(x) \le M \qquad \forall n.$$

$$u_n\big|_{\left[0, 1-\frac{1}{M^2}\right]} \in L^2\left(\left[0, 1-\frac{1}{M^2}\right]\right)$$

is relatively compact sequence.

$[0, 2\pi]$

$u_n(0) = 0$ $\qquad C^0([0, 2\pi])$

$u_n(2\pi) \le M$ $\qquad$ increasing

$$\Rightarrow \quad |\hat{u}_n(k)| \le \frac{C_M}{\langle k \rangle} \qquad \forall k \cancel{\ne} 0$$

$$\hat{u}_n(k) \quad \ell$$

$$\Rightarrow u_n \in H^s(\mathbb{T}) \quad s < 1$$

$$\sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} |\hat{u}_n(k)|^2 \le \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} \frac{C_M^2}{\langle k \rangle^2} < \infty$$

convergent for $\sum_{k \in \mathbb{Z}} \langle k \rangle^{2s-2}$

$$2s - 2 < -1$$

$$2s < 1$$

$$0 < s < \frac{1}{2}$$

$$H^s(\mathbb{T}) \hookrightarrow L^2(\mathbb{T}) \qquad L^1([0,1])$$

$$u_n = \chi_{[0, 1-\frac{1}{M^2}]} u_n + \chi_{[1-\frac{1}{M^2}, 1]} u_n$$

$\{u_n\}$ can be covered by finitely balls of radius $2\varepsilon > 0$ in $L^1(I)$

$\forall \varepsilon > 0$

$$\| \chi_{[1-\frac{1}{M^2}, 1]} u_n \|_{L^1} =$$

$$\leq M \quad \| \chi_{[1-\frac{1}{M^2}, 1]} \|_{L^2}$$

$$= \cancel{M} \quad \frac{1}{M^2} = \frac{1}{M} < \varepsilon$$