Dynamics of EW & Strong Interactions

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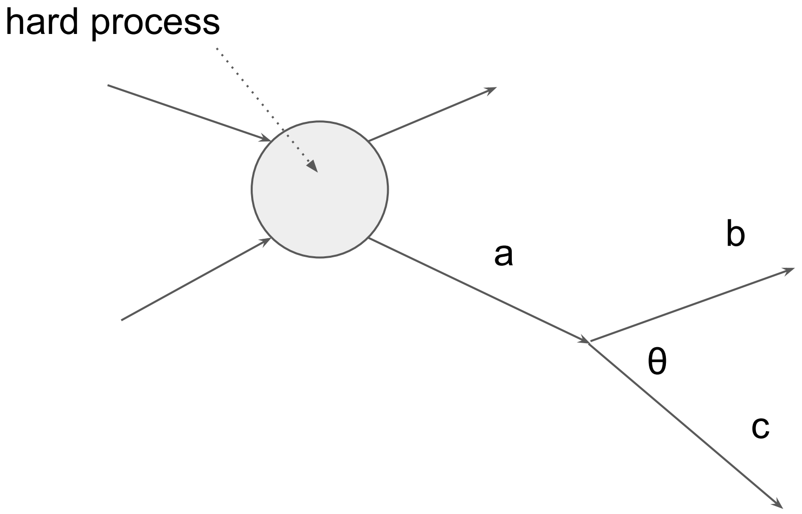
Lecture 2 - Trieste, 13/12/2022

IR divergences and jets

IR divergences - generalization to all (FSR) partons

- What if we add mass terms?
- What about other splittings like g → gg, g → q$\bar{q}$?



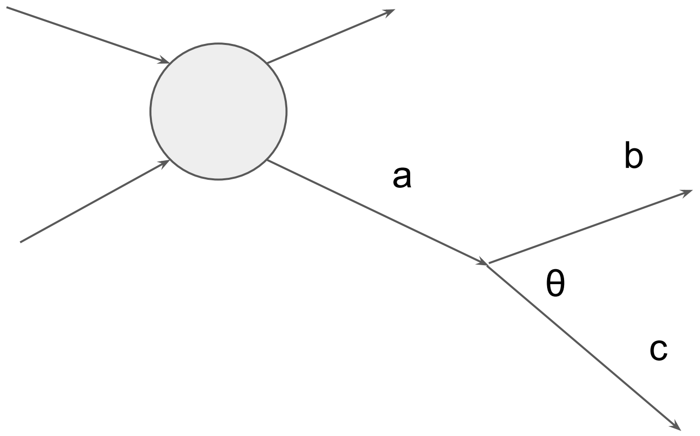
- Propagator factor: $\dfrac{1}{p_a^2 - m_a^2}$
  - and we can write:

$$\begin{aligned} p_a^2 &= (p_b + p_c)^2 \\ &= m_b^2 + m_c^2 + 2(E_b E_c - |\vec{p}_b||\vec{p}_c| cos\theta) \\ &= m_b^2 + m_c^2 + 2E_b E_c(1 - v_b v_c cos\theta) \end{aligned}$$

IR divergences - generalization to all (FSR) partons

- Final formula is then:

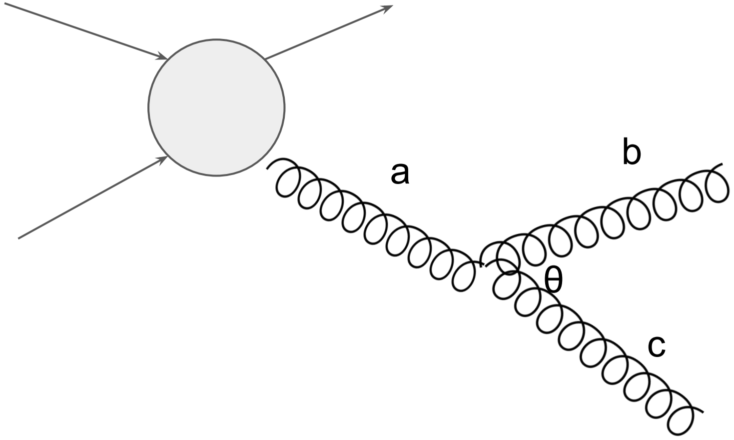
$$\frac{1}{p_a^2 - m_a^2} = \frac{1}{m_b^2 + m_c^2 + 2E_bE_c(1 - v_bv_c cos\theta) - m_a^2}$$



IR divergences - generalization to all (FSR) partons

- Final formula is then:

$$\frac{1}{p_a^2 - m_a^2} = \frac{1}{\cancel{m_b^2} + \cancel{m_c^2} + 2E_bE_c(1 - \underset{1}{\cancel{v_b}}\underset{1}{\cancel{v_c}}cos\theta) - \cancel{m_a^2}}$$



- g → gg (a=g, b=g, c=g):
  - $m_a = m_b = m_c = 0$
  - $v_a = v_b = v_c = 1$

$$= \frac{1}{2E_bE_c(1 - cos\theta)}$$

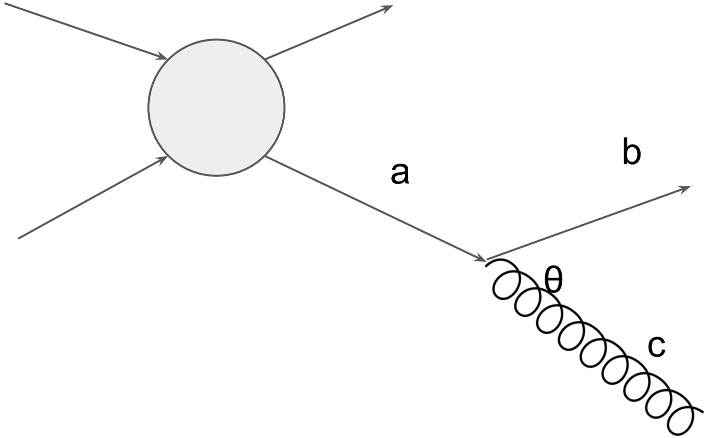
soft divergence

collinear divergence

IR divergences - generalization to all (FSR) partons

- Final formula is then:

$$\frac{1}{p_a^2 - m_a^2} = \frac{1}{\cancel{m_b^2} + \cancel{m_c^2} + 2E_bE_c(1 - v_b \underset{1}{\cancel{v_c}} cos\theta) - \cancel{m_a^2}}$$



- q → qg (a=q, b=q, c=g):
  - $m_a = m_b = m_q$
  - $m_c = 0$
  - $v_c = 1$

$$= \frac{1}{2E_bE_c(1 - v_b cos\theta)}$$

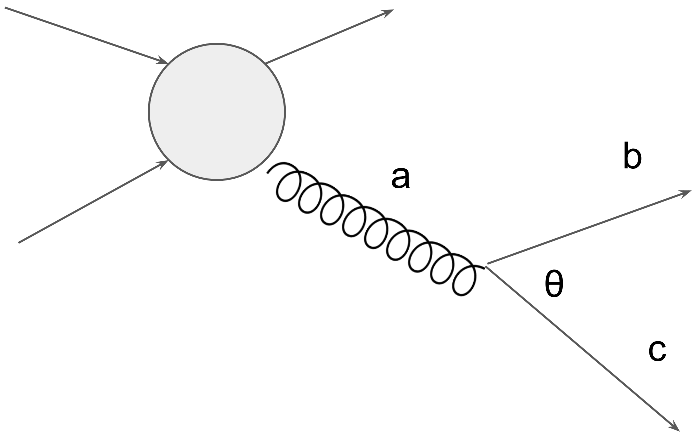
soft divergence

collinear divergence *only* if $m_q \to 0$

IR divergences - generalization to all (FSR) partons

- Final formula is then:

$$\frac{1}{p_a^2 - m_a^2} = \frac{1}{m_b^2 + m_c^2 + 2E_bE_c(1 - v_bv_c cos\theta) - \cancel{m_a^2}}$$



- g → q$\overline{\text{q}}$ (a=g, b=q, c=q):
  - $m_b = m_c = m_q$
  - $m_a = 0$

$$= \frac{1}{2m_q^2 + 2E_bE_c(1 - v_bv_c cos\theta)}$$

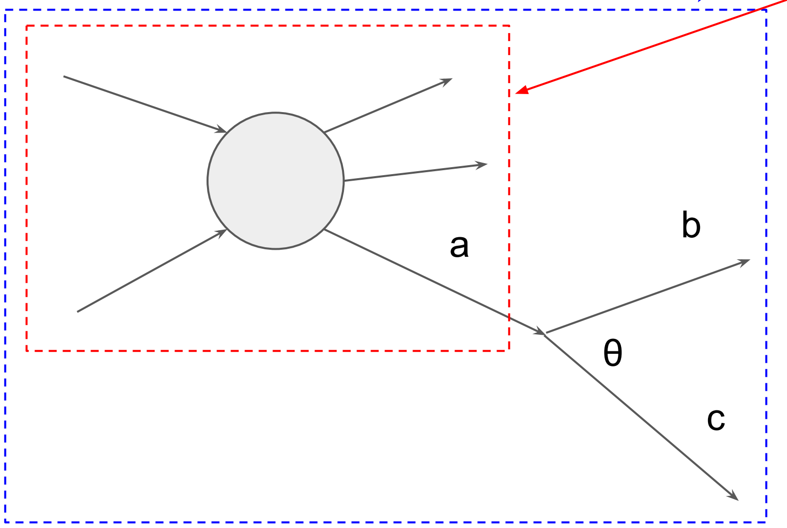
soft divergence
only if $m_q \rightarrow 0$

collinear divergence
only if $m_q \rightarrow 0$

Branching probabilities & splitting functions

$$z = \frac{E_b}{E_a} = 1 - \frac{E_c}{E_a}$$

- General formulas for getting $d\sigma_{n+1}$ given $d\sigma_n$
  - (neglecting m_q)



already includes integral on ϕ

$$d\sigma_{n+1} = d\sigma_n \underbrace{\frac{dt}{t} dz \frac{\alpha_s}{2\pi} \hat{P}_{ba}(z)}_{dS}$$

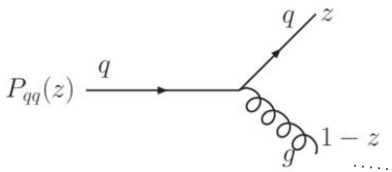
$t = -Q^2$ momentum transfer = virtuality of a (if b is on-shell)

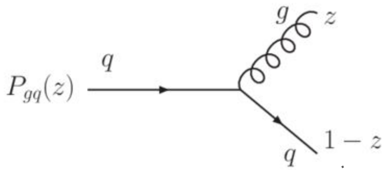
"Unregulated" branching probability for a → bc

Branching probabilities & splitting functions

- General formulas for getting $d\sigma_{n+1}$ given $d\sigma_n$

$$z = \frac{E_b}{E_a} = 1 - \frac{E_c}{E_a}$$

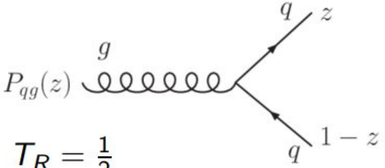


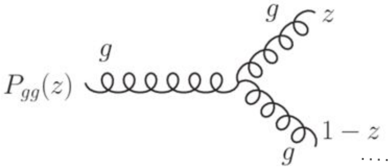


$$d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz \frac{\alpha_s}{2\pi} \boxed{\hat{P}_{ba}(z)}$$

$$\hat{P}_{qq}(z) = \hat{P}_{q\to qg} = \hat{P}_{\bar{q}\to \bar{q}g} = C_F \left[\frac{1+z^2}{1-z}\right]$$

$$\hat{P}_{gq}(z) = \hat{P}_{q\to gq} = \hat{P}_{\bar{q}\to g\bar{q}} = C_F \left[\frac{1+(1-z)^2}{z}\right]$$





$$\hat{P}_{gg}(z) = \hat{P}_{g\to gg} = C_A \left[\frac{z}{(1-z)} + \frac{1-z}{z} + z(1-z)\right]$$

$$\hat{P}_{qg}(z) = \hat{P}_{g\to q\bar{q}} = T_R \left[z^2 + (1-z)^2\right]$$

$T_R = \frac{1}{2}$

$C_A = N_c = 3$

$C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}$

Evolution of the shower

- Splitting function rules can be combined to allow successive emissions in several steps:

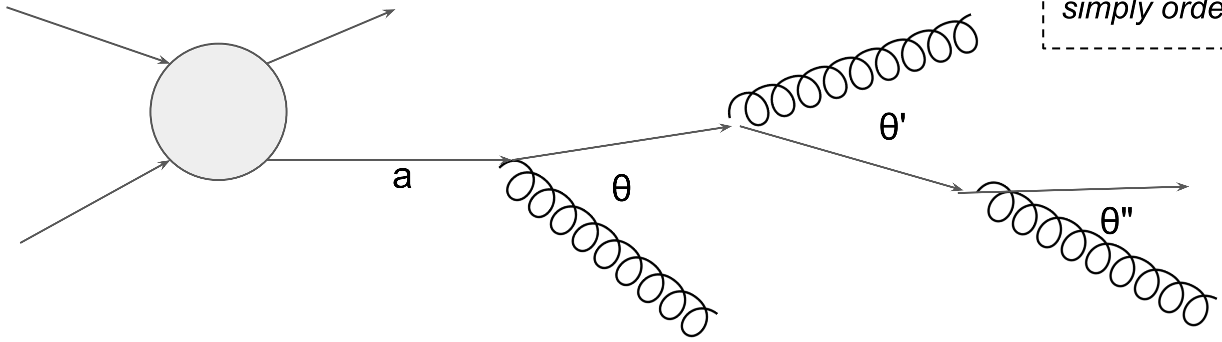
$$d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz \frac{\alpha_s}{2\pi} \hat{P}_{qq}(z)$$

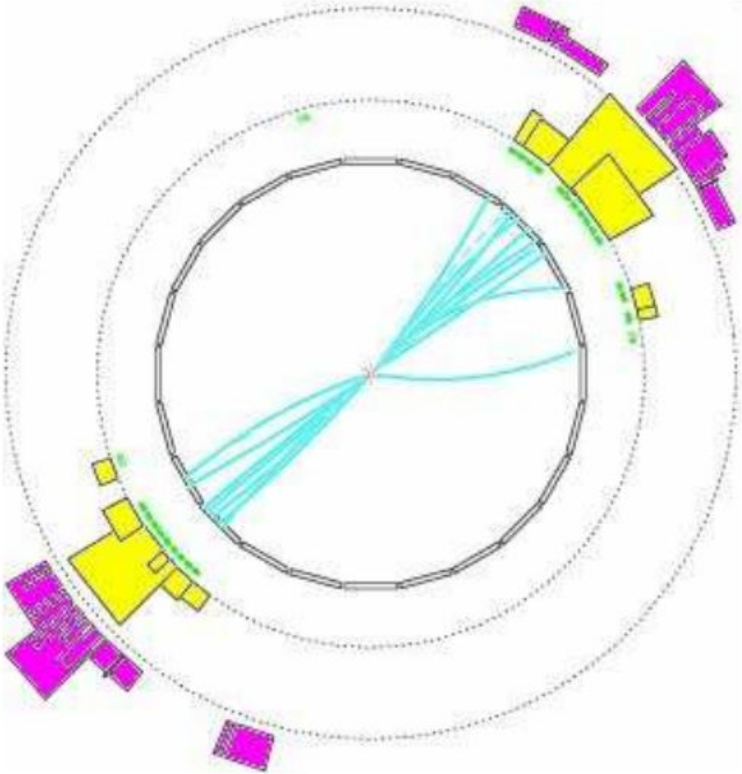
$$d\sigma_{n+2} = d\sigma_{n+1} \frac{dt'}{t'} dz' \frac{\alpha_s}{2\pi} \hat{P}_{qq}(z')$$

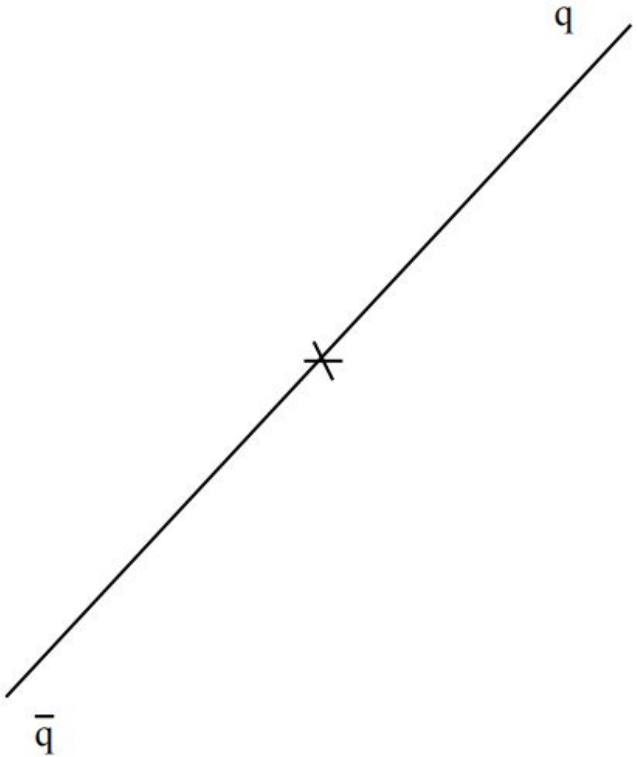
$$\ldots$$

In the collinear limit the cross section factorizes. The splitting can be iterated.

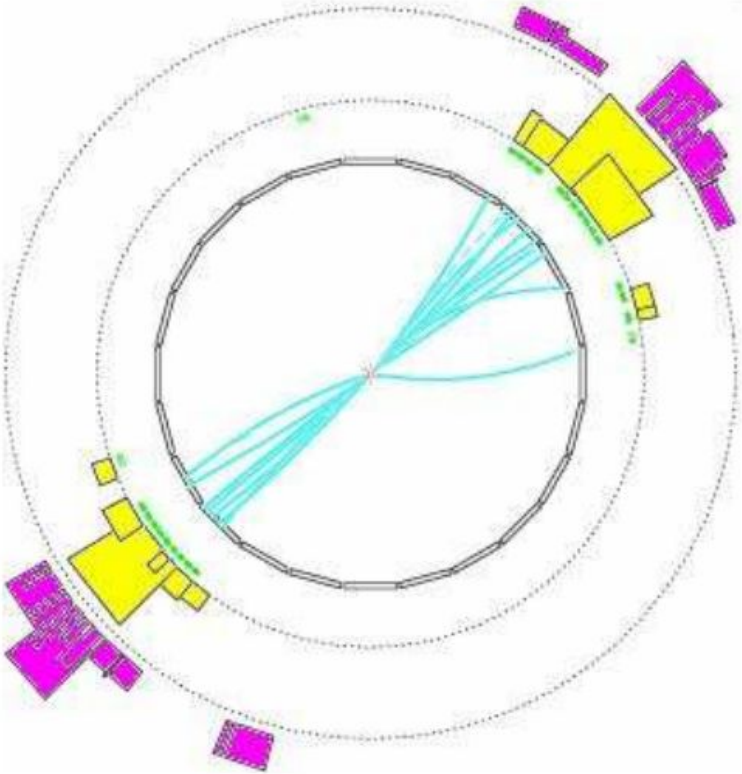
Note: *picture reliable in case of strongly ordered showers $Q^2_1 \gg Q^2_2 \gg Q^2_3$ but meaningful approximate answer also for simply ordered showers $Q^2_1 > Q^2_2 > Q^2_3$*

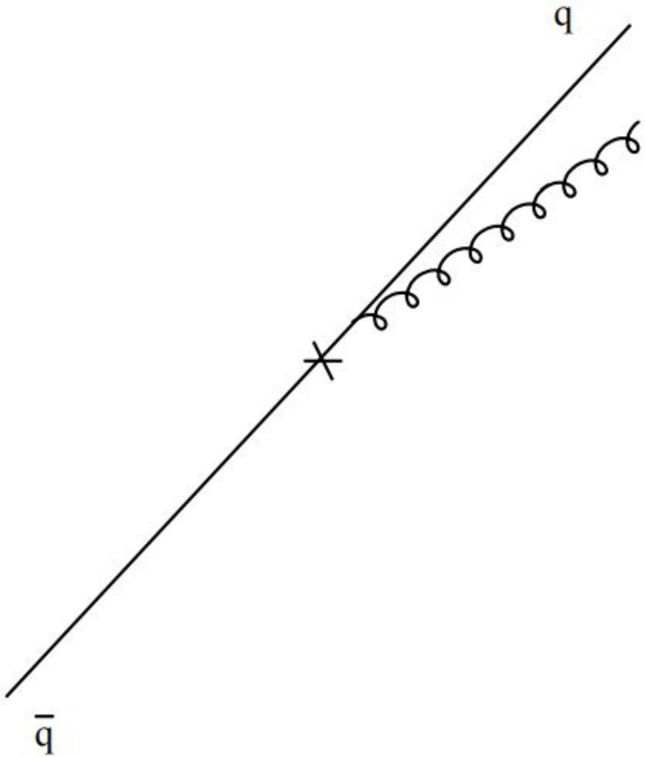




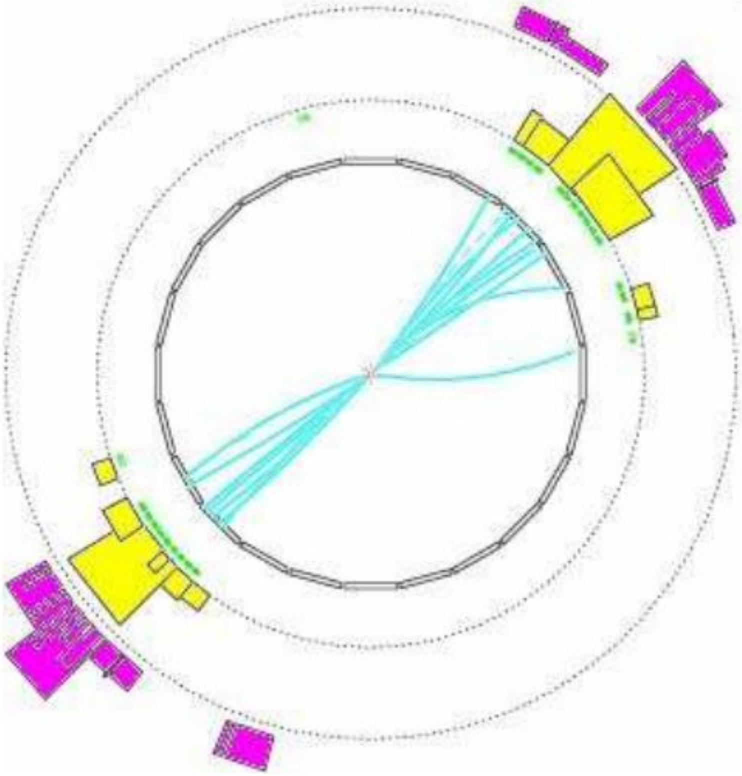


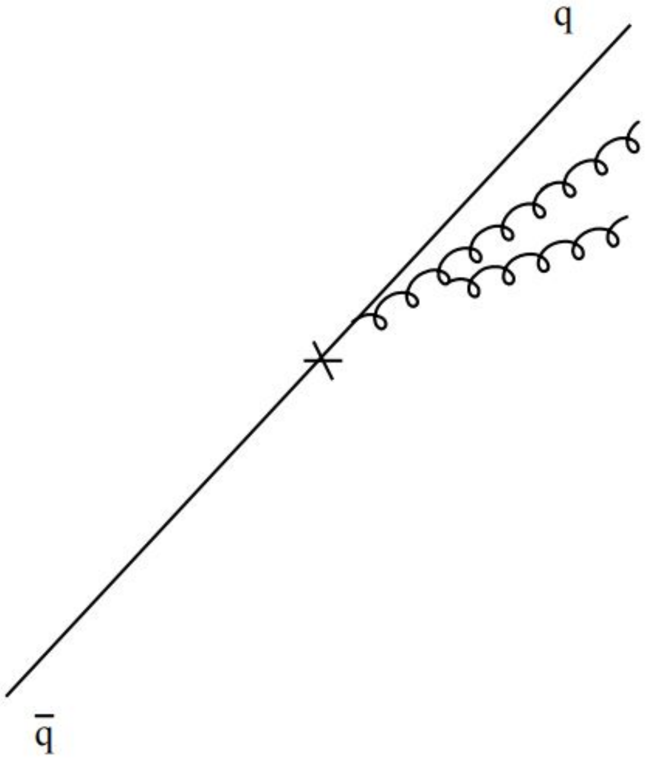
Start of with $q\bar{q}$



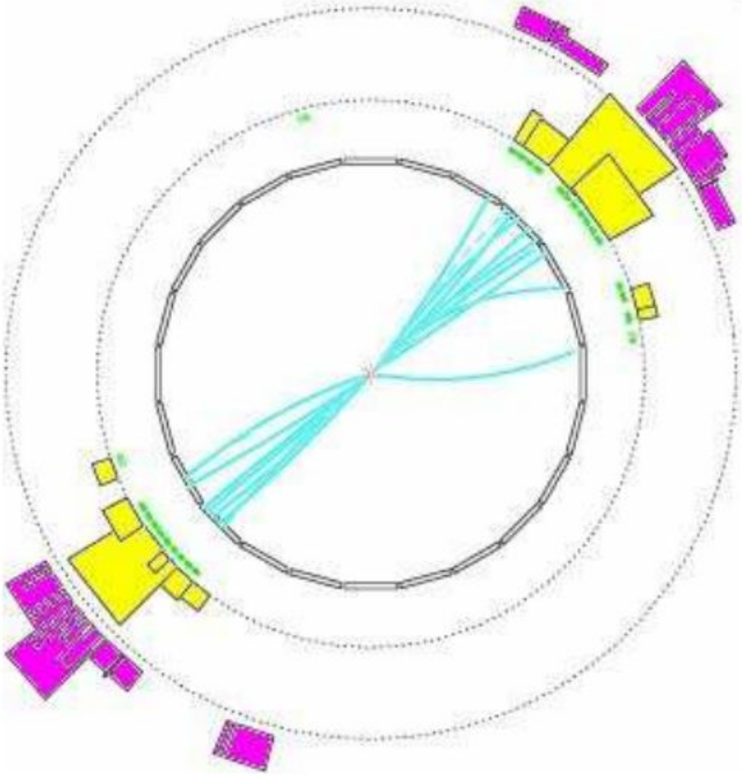


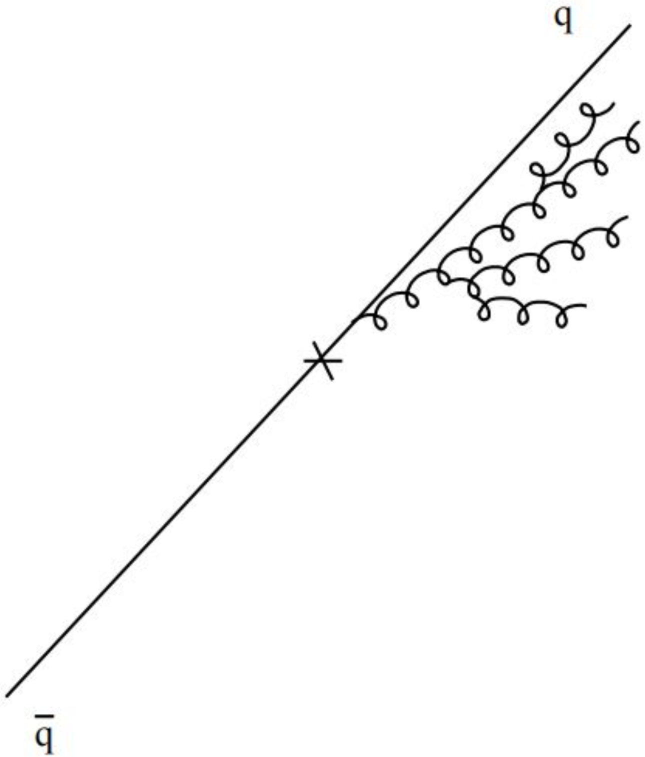
A gluon gets emitted at small angles



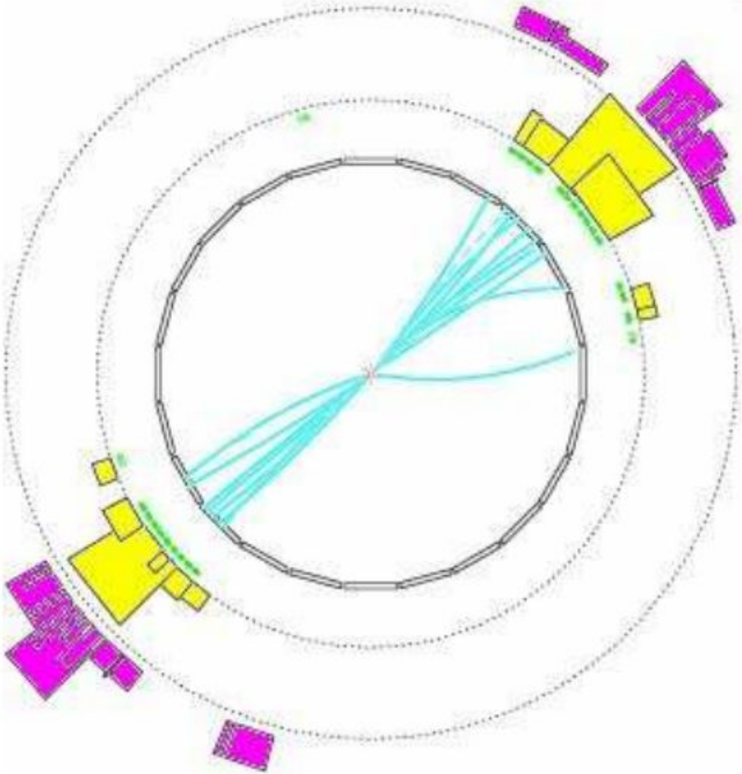


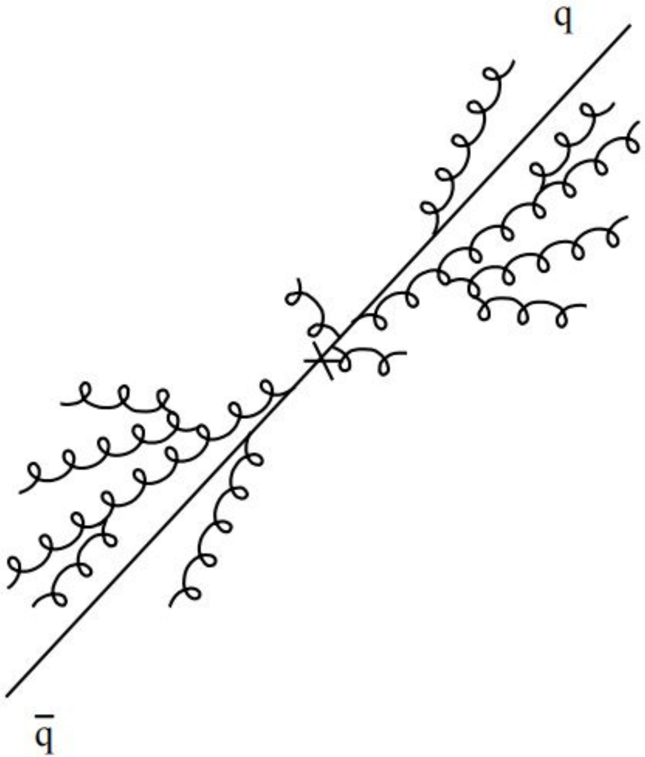
It radiates a further gluon



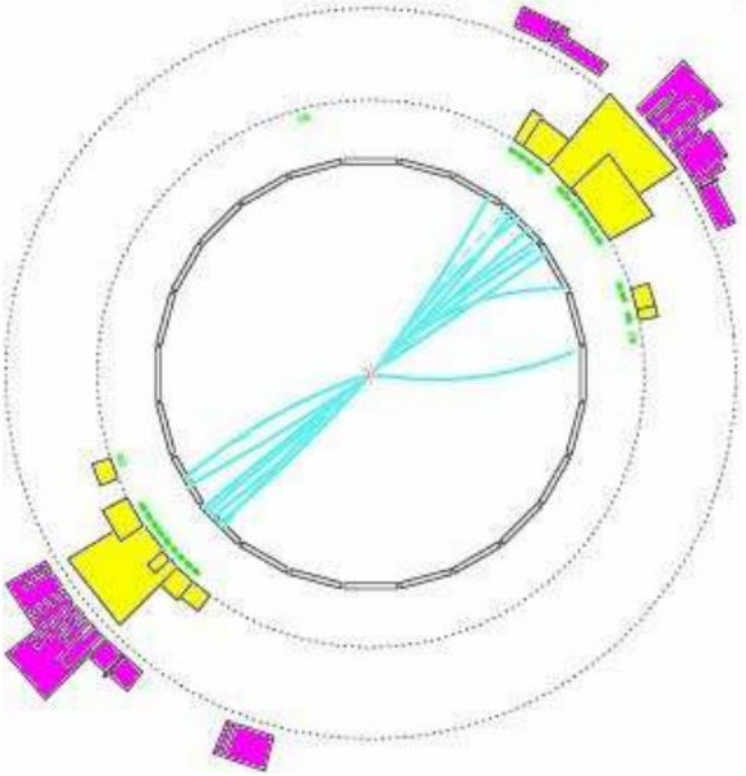


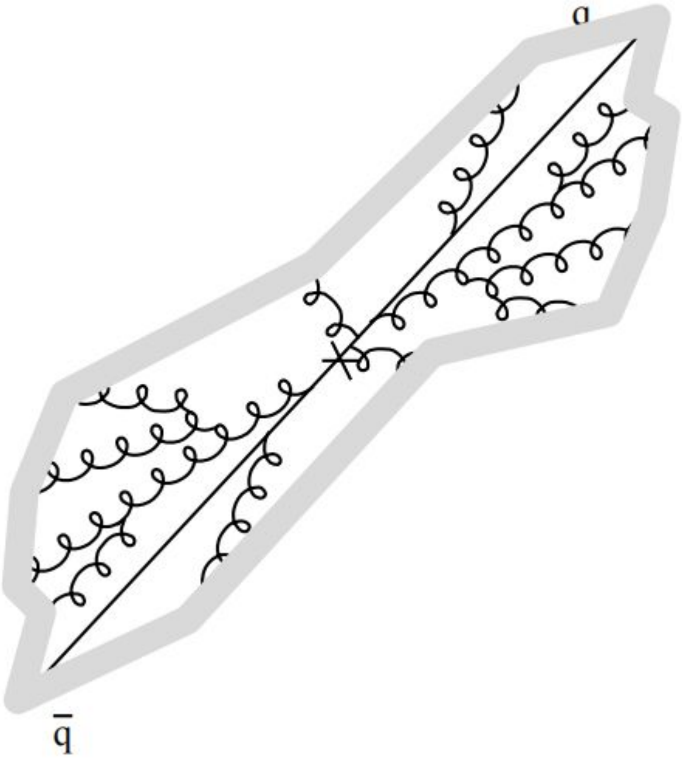
And so forth



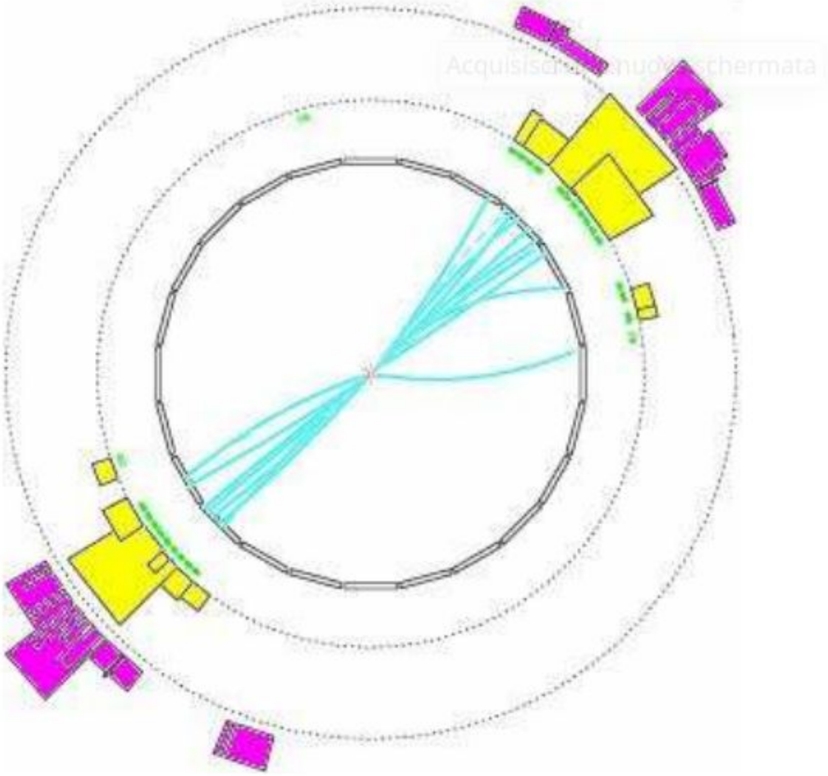


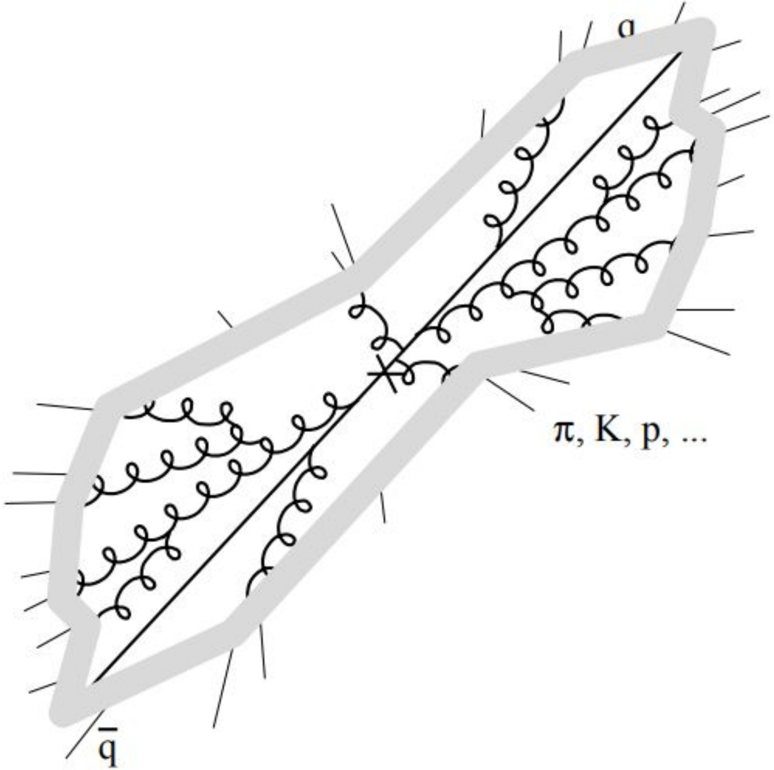
Meanwhile the same happened on other side of event





And then a non-perturbative transition occurs





Giving a pattern of hadrons that "remembers" the gluon branching

Hadrons mostly produced at small angle wrt $q\bar{q}$ directions or with low energy

Cut-off

step function:
0 if $E\theta < Q_0$
1 otherwise

- Consider $\sigma(q\bar{q}g)$ expression obtained before
  - can add a cut-off into integral to avoid divergence:

$$\frac{2\alpha_s C_F}{\pi} \int^{Q} \frac{dE}{E} \int^{\pi/2} \frac{d\theta}{\theta} \Theta(E\theta > Q_0)$$

- Interpreted as cross-section of $q\bar{q}$ production plus gluon with $E\theta > Q_0$
  - natural choice $Q_0 = \Lambda_{QCD}$ (below this value language of quarks and gluons becomes meaningless)
- In this way can perform integration and get expression for mean number of gluons emitted:

$$\langle N_g \rangle \simeq \frac{\alpha_s C_F}{\pi} \ln^2 \frac{Q}{Q_0} + \mathcal{O}(\alpha_s \ln Q)$$

Cut-off and mean gluon multiplicity

- Setting $Q_0 = \Lambda_{QCD}$ and $\alpha_S(Q) = \frac{1}{2b \cdot \ln \frac{Q}{\Lambda_{QCD}}}$

$$\langle N_g \rangle \simeq \frac{\alpha_s C_F}{\pi} \ln^2 \frac{Q}{\Lambda_{QCD}} \to \frac{C_F}{2b\pi} \ln \frac{Q}{\Lambda_{QCD}}$$

  - putting in some numbers:

    $Q = 100$ GeV, $\Lambda_{QCD} \approxeq 0.2$ GeV, $C_F = 4/3$, $b \approxeq 0.6$

$$\langle N_g \rangle \simeq 2.2$$

Cut-off and mean gluon multiplicity

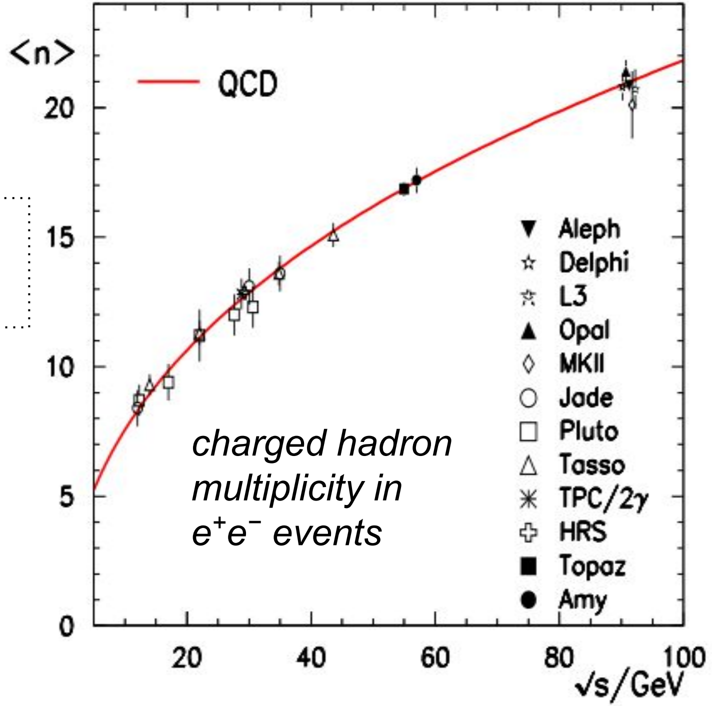
- It turns out you can calculate the gluon multiplicity analytically, by summing all orders (n) of perturbation theory:

$$\langle N_g \rangle \sim \sum_n \frac{1}{(n!)^2} \left(\frac{C_A}{\pi b} \ln \frac{Q}{\Lambda} \right)^n$$

$$\sim \exp \sqrt{\frac{4 C_A}{\pi b} \ln \frac{Q}{\Lambda}}$$

example of "resummation"

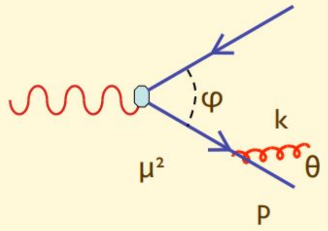
  - compare with hadron multiplicity (assuming hadron pattern remembers gluon one)
  - $Q = \sqrt{s}$



charged hadron multiplicity in e^+e^- events

Angular ordering

- Essentially tells that:
  - $\theta > \theta' > \theta'' \ldots$
- Intuitive explanation



Lifetime of the virtual intermediate state:

$\tau < \gamma/\mu = E/\mu^2 = 1/(k_0\theta^2) = 1/(k_\perp\theta)$

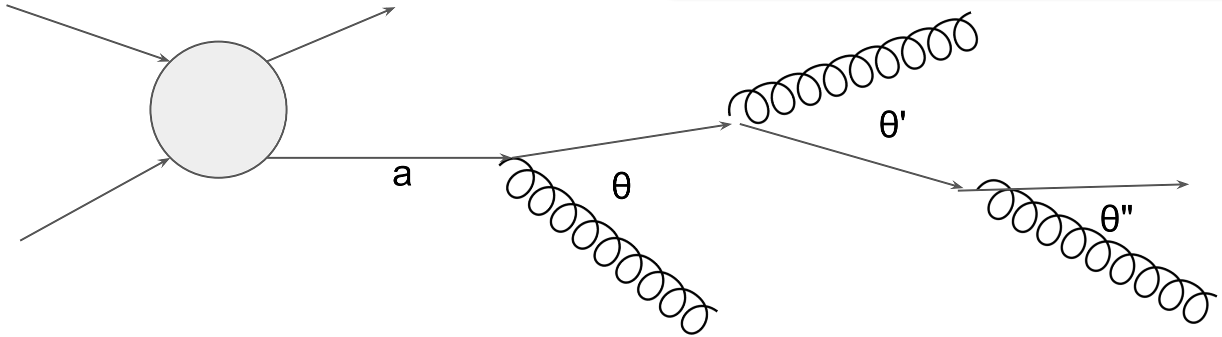
$\mu^2 = (p+k)^2 = 2E\,k_0\,(1-\cos\theta)$
$\sim E\,k_0\,\theta^2 \sim E\,k_\perp\,\theta$

Distance between q and qbar after τ:

$d = \varphi\tau = (\varphi/\theta)\ 1/k_\perp$

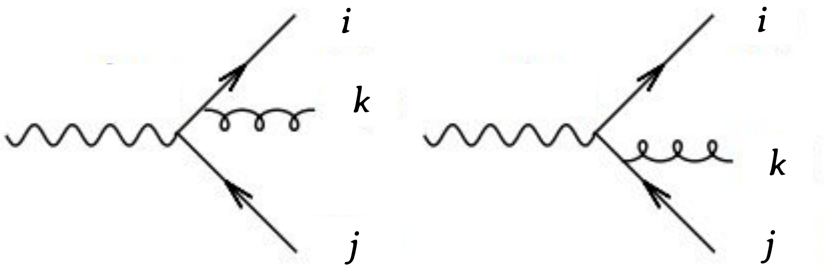
If the transverse wavelength of the emitted gluon is longer than the separation between q and qbar, the gluon emission is suppressed, because the q qbar system will appear as colour neutral (=> dipole-like emission, suppressed)

Therefore $d > 1/k_\perp$, which implies $\theta < \varphi$



Angular ordering - (part of) the formal proof

- Consider:



$$dS \simeq E dE \frac{d\phi}{2\pi} d\cos\theta \cdot \frac{2\alpha_S C_F}{\pi} \frac{2p_i \cdot p_j}{(2p_i \cdot p_k)(2p_j \cdot p_k)}$$

$$= \frac{dE}{E} \frac{d\phi}{2\pi} d\cos\theta \cdot \frac{\alpha_S C_F}{\pi} \frac{(1-\cos\theta_{ij})}{(1-\cos\theta_{ik})(1-\cos\theta_{jk})}$$

$$\begin{aligned} p_i \cdot p_j &= E_i E_j - \vec{p}_i \cdot \vec{p}_j \\ &= E_i E_j (1-\cos\theta_{ij}) \end{aligned}$$

$$W_{ij}^i + W_{ij}^j$$

with:

(see backup)

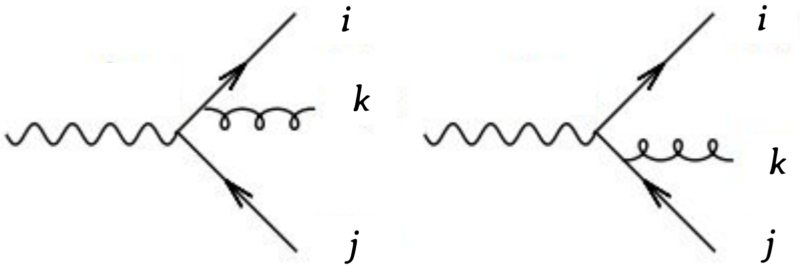
each containing only collider divergence along i or j

$$W_{ij}^i = \frac{1}{2}\left[\frac{(\cos\theta_{ik} - \cos\theta_{ij})}{(1-\cos\theta_{ik})(1-\cos\theta_{jk})} + \frac{1}{1-\cos\theta_{ik}}\right]$$

$$W_{ij}^j = \frac{1}{2}\left[\frac{(\cos\theta_{jk} - \cos\theta_{ij})}{(1-\cos\theta_{ik})(1-\cos\theta_{jk})} + \frac{1}{1-\cos\theta_{jk}}\right]$$

(see backup)

Angular ordering

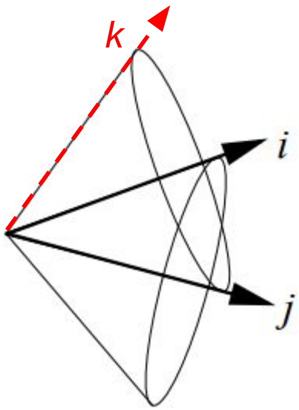


- Can consider that each W_{ij}^{i} or W_{ij}^{j} describes emission from leg *i* or *j*

$$W_{ij}^{i} = \frac{1}{2}\left[\frac{(\cos\theta_{ik} - \cos\theta_{ij})}{(1-\cos\theta_{ik})(1-\cos\theta_{jk})} + \frac{1}{1-\cos\theta_{ik}}\right]$$

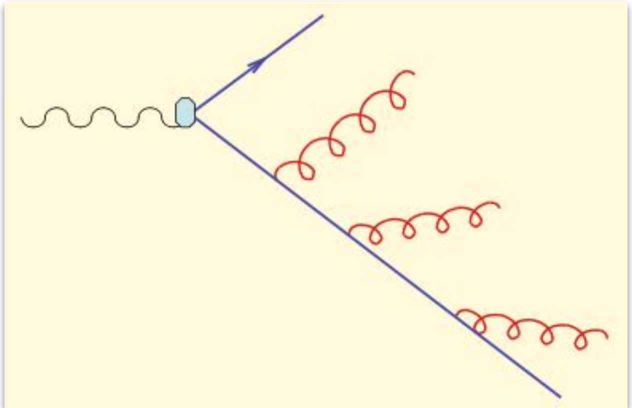
$$W_{ij}^{j} = \frac{1}{2}\left[\frac{(\cos\theta_{jk} - \cos\theta_{ij})}{(1-\cos\theta_{ik})(1-\cos\theta_{jk})} + \frac{1}{1-\cos\theta_{jk}}\right]$$

- Consider W_{ij}^{i}:
  - can proof that integrating over ϕ_k gives:

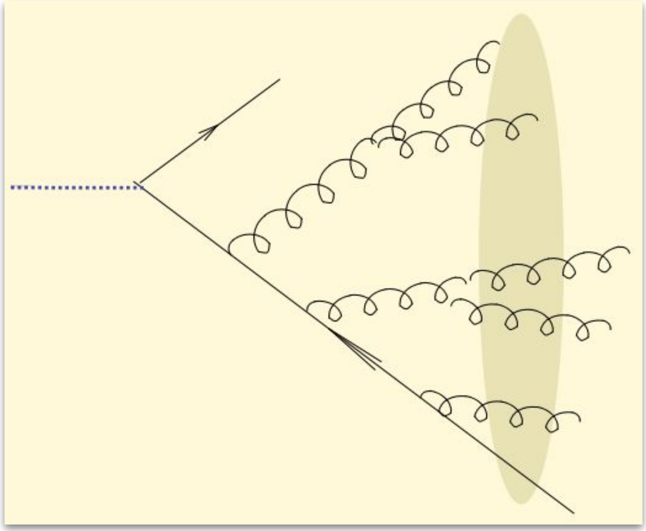


$$\int_0^{2\pi} \frac{d\phi_{ik}}{2\pi} W_{ij}^{i} = \begin{cases} \dfrac{1}{1-\cos\theta_{ik}} & \text{if } \theta_{ik} < \theta_{ij} \\ 0 \text{ otherwise} \end{cases}$$

Further branchings will obey angular ordering relative to the new angles



- Construction can be iterated to next emission:
  - angles keep getting smaller and smaller
  - jet structure



- Total colour charge of the system
  = quark colour charge

IR safety and factorization

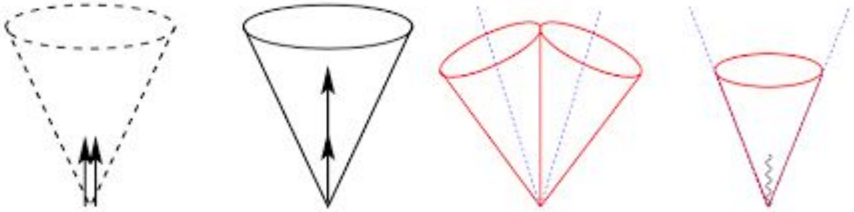
- Complications:
  1. didn't consider yet α_S running: at low enough energy $Q^2 < \Lambda_{QCD}{}^2$ (soft emission) perturbation theory not valid anymore ⇒ even stronger divergence, non perturbative effects (hadronization)
  2. total cross-section not affected by soft gluon emission, ok
     - but what if one wants to calculate explicitly $\sigma(e^+e^- \to q\bar{q}g)$?

- Distinguish between:
  1. infrared-safe quantities: insensitive to soft or collinear branching (e.g. $\sigma(e^+e^- \to q\bar{q}+X)$)
     - determined by hard / short-distance physics
     - long-distance effects → corrections suppressed by inverse powers of energy scale
  2. factorisable quantities: infrared divergent part absorbed into overall non-perturbative factor(s)

IR-safe jet algorithms

- Things like σ(qq+g) are not IR-safe
  - but can define "jets" instead of partons, in a IR-safe way

- Jet algorithms should:
  - allow to use common language between theory and experiments
  - be applicable to partons, particles and signals in particle detectors (energy deposits in calorimeters, charged particle tracks...)
  - be IR-safe:



IR-safe jet algorithms

- Idea: follow in "reverse order" the development of a jet via sequential parton branchings:
  - angular ordering, soft / collinear branching more and more likely

$$[dk_j]\,|M_{g\to g_i g_j}|^2 = \frac{2\alpha_s C_A}{\pi}\frac{dE_i}{E_i E_j}\frac{d\theta_{ij}}{\theta_{ij}} \quad E_j \ll E_i, \theta_{ij} \ll 1$$

  - to invert branching process, take pair "closest" in metric defined by the divergence structure

Example: k_T/Durham algorithm

1. Calculate (or update) distances between all (pseudo-)particles i and j, (related to the relative k_T between the particles)

$$y_{ij} = 2min(E_i^2, E_j^2)(1 - \cos\theta_{ij})$$

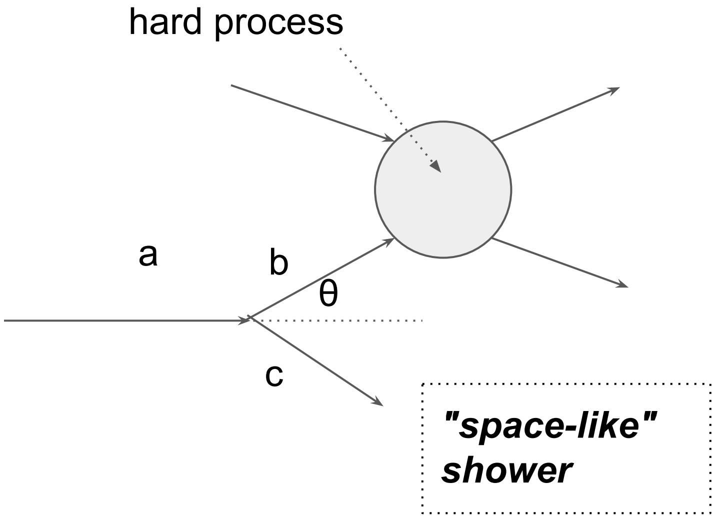
2. Find smallest of y_{ij}

- ⋆ If $y > y_{cut}$ stop clustering
- ⋆ else recombine i and j and repeat from step 1

Initial State Radiation

Initial-state radiation

- Consider now hadron collisions (or DIS)
- Incoming parton can radiation in the same way as outgoing parton



- Propagator factor

  (massless partons): $\dfrac{1}{p_b^2}$

  - and we can write:

$$p_b^2 = (p_a - p_c)^2 = -2E_a E_c (1 - \cos\theta) \leq 0$$

soft divergence collinear divergence

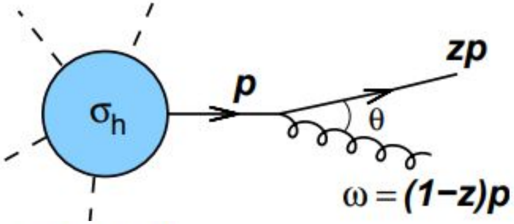
Initial vs. final state radiation

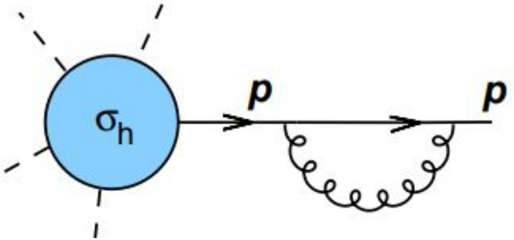
*: $$dS = \frac{2\alpha_s C_F}{\pi}\frac{dE}{E}\frac{d\theta}{\sin\theta}\frac{d\phi}{2\pi}$$ → 1 integrating over ϕ

~dθ/θ = ½ dθ²/θ²

- Final-state radiation:

Real emission





Virtual correction

*

$$\sigma_{h+g} \simeq \sigma_h \frac{\alpha_s C_F}{\pi}\frac{dE}{E}\frac{d\theta^2}{\theta^2}$$

$$E = (1-z)p$$
$$k_t = E\sin\theta \simeq E\theta$$

$$\sigma_{h+g} \simeq \sigma_h \frac{\alpha_s C_F}{\pi}\frac{dz}{1-z}\frac{dk_t^2}{k_t^2}$$

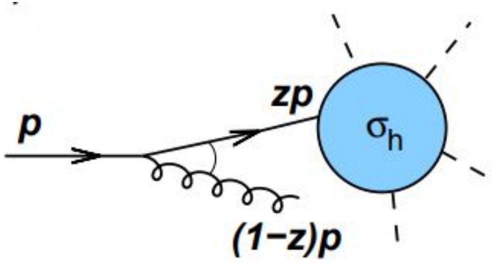
$$\sigma_{h+V} \simeq -\sigma_h \frac{\alpha_s C_F}{\pi}\frac{dz}{1-z}\frac{dk_t^2}{k_t^2}$$

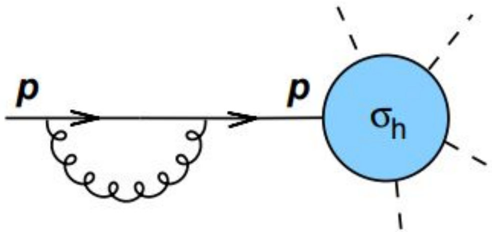
perfect cancellation in both soft and collinear limits

Initial vs. final state radiation

- Initial-state radiation:

Real emission





Virtual correction

incoming parton with reduced momentum

$$\sigma_{g+h}(p) \simeq \sigma_h(zp)\frac{\alpha_s C_F}{\pi}\frac{dz}{1-z}\frac{dk_t^2}{k_t^2}$$

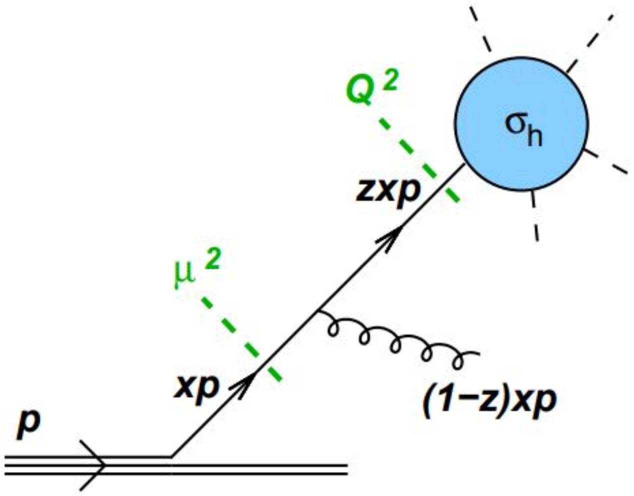
$$\sigma_{V+h}(p) \simeq -\sigma_h(p)\frac{\alpha_s C_F}{\pi}\frac{dz}{1-z}\frac{dk_t^2}{k_t^2}$$

Initial vs. final state radiation

$$\sigma_{g+h} + \sigma_{V+h} \simeq \frac{\alpha_s C_F}{\pi} \underbrace{\int_0^{Q^2} \frac{dk_t^2}{k_t^2}}_{\text{infinite}} \underbrace{\int \frac{dz}{1-z}[\sigma_h(zp) - \sigma_h(p)]}_{\text{finite}}$$

- soft limit ($z \to 1$), $\sigma_h(zp) - \sigma_h(p) \to 0 \Rightarrow$ soft divergence cancels
- for $1 - z \neq 0$, $\sigma_h(zp) - \sigma_h(p) \neq 0$ $\Rightarrow$ z integral is non-zero but finite
  - k_t integral is infinite
- This is a collinear ($k_t \to 0$) divergence.
  - i.e. cross section with incoming parton is not collinear safe...

Collinear cutoff and PDFs



- For kt → 0 QCD becomes non-perturbative
- Cut out divergent region and add non-perturbative quark distribution is proton (PDF)

$$\sigma_0 = \int dx\, \sigma_h(xp)\, \boxed{q(x,\mu^2)}$$

proton momentum fraction carried by quark

factorisation scale

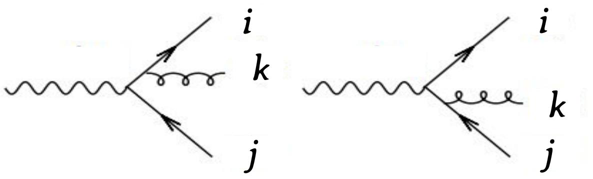
$$\sigma_1 \simeq \frac{\alpha_s C_F}{\pi} \underbrace{\int_{\mu^2}^{Q^2} \frac{dk_t^2}{k_t^2}}_{\text{finite (large)}} \underbrace{\int \frac{dx\,dz}{1-z} [\sigma_h(zxp) - \sigma_h(xp)]\, q(x,\mu^2)}_{\text{finite}}$$

Backup

$W_{ij}^{i/j}$ function properties

$$W_{ij}^{i} = \frac{1}{2}\left[\frac{(\cos\theta_{ik} - \cos\theta_{ij})}{(1-\cos\theta_{ik})(1-\cos\theta_{jk})} + \frac{1}{1-\cos\theta_{ik}}\right]$$

$$W_{ij}^{j} = \frac{1}{2}\left[\frac{(\cos\theta_{jk} - \cos\theta_{ij})}{(1-\cos\theta_{ik})(1-\cos\theta_{jk})} + \frac{1}{1-\cos\theta_{jk}}\right]$$



$$W_{ij}^{i} + W_{ij}^{j} = \frac{1}{2}\left[\frac{(\cancel{\cos\theta_{ik}} - \cos\theta_{ij}) + (\cancel{\cos\theta_{jk}} - \cos\theta_{ij}) + (1 - \cancel{\cos\theta_{jk}}) + (1 - \cancel{\cos\theta_{ik}})}{(1-\cos\theta_{ik})(1-\cos\theta_{jk})}\right]$$

$$= \frac{1-\cos\theta_{ij}}{(1-\cos\theta_{ik})(1-\cos\theta_{jk})}$$

$$\lim_{\theta_{ik}\to 0} W_{ij}^{i} = \frac{1}{2}\left[\frac{1-\cos\theta_{ij}}{\underbrace{(1-1)}_{\to 0}(1-\cos\theta_{ij})} + \frac{1}{\underbrace{1-1}_{\to 0}}\right] = \infty$$

$$\lim_{\theta_{ik}\to 0} W_{ij}^{j} = \frac{1}{2}\left[\cancel{\frac{\cos\theta_{ij} - \cos\theta_{ij}}{(1-1)(1-\cos\theta_{ij})}} + \frac{1}{1-\cos\theta_{ij}}\right] \neq \infty$$

$$\lim_{\theta_{jk}\to 0} W_{ij}^{i} = \frac{1}{2}\left[\cancel{\frac{\cos\theta_{ij} - \cos\theta_{ij}}{(1-\cos\theta_{ij})(1-1)}} + \frac{1}{1-\cos\theta_{ij}}\right] \neq \infty$$

$$\lim_{\theta_{jk}\to 0} W_{ij}^{j} = \frac{1}{2}\left[\frac{1-\cos\theta_{ij}}{(1-\cos\theta_{ij})\underbrace{(1-1)}_{\to 0}} + \frac{1}{\underbrace{1-1}_{\to 0}}\right] = \infty$$

References

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  - Perturbative QCD and the parton model - Keith Ellis
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