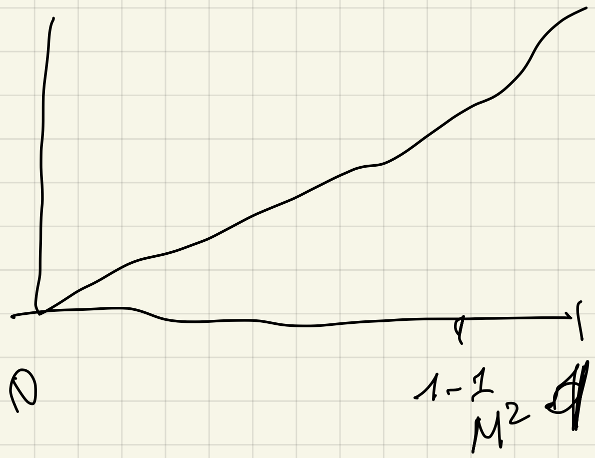


u_n

$|u_n|_2 \le 1$

M

$$\left| \{ u_n(x) : u_n(x) \ge M \} \right| \le 1 - \frac{1}{M^2}$$

$$\boxed{u_n = \chi_{[0, 1-\frac{1}{M^2}]} u_n + \chi_{[1-\frac{1}{M^2}, 1]} u_n}$$

$$\| \chi_{[1-\frac{1}{M^2}, 1]} u_n |_{L^1} \le \| \chi_{[1-\frac{1}{M^2}, 1]} \|_{L^2} \; \overbrace{|u_n|_{L^2}}^{\le 1}$$

$$= \frac{1}{M}$$

4.6
$$\begin{cases} y' = \dfrac{1}{1+ty} & t > 0 \\ y(0) = 1 + \dfrac{1}{n} & n \in \mathbb{N} \end{cases}$$

1) y_n is ~~globobally~~ defined

in $[0, +\infty)$

2) $\{y_n\}$ admits a subsequence uniformly convergent $\forall$ compact subinterval on $[0, +\infty)$

$$\begin{cases} y' = f(t, y) = \dfrac{1}{1+ty} \\ y(0) = y_0 > 0 \end{cases}$$ is globally defined

let $[0, t_0)$ be the maximum forward interval of definition. Let us suppose that $0 < t_0 < +\infty$

$y' > 0 \implies y(t)$ is increasing in t_0.

$\lim\limits_{t \to t_0^-} y(t) = Y_0 \geq y_0$ exists $[0, t_0]$

$y(t_0) = Y_0$

If $Y_0 < +\infty$

it is easy to see that we get a contradiction

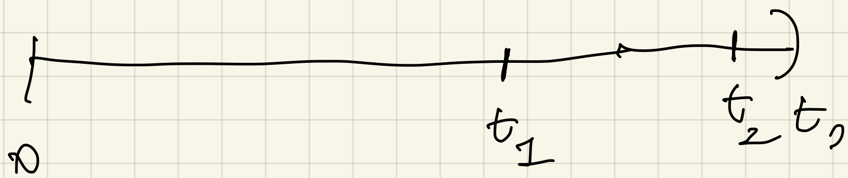
$$\begin{cases} y' = f(t, y) \\ y(t_0) = Y_0 \end{cases}$$ $t \cancel{\geq} t_0$

$[0, t_0] \cup [t_0, t_0+\tau)$

$[0, t_0+\tau)$

$y_0 = +\infty$

$[0, t_0)$



$y(t_1) \geq 1$ $\qquad t_1 < t_2 < t_0$

$$y(t_2)-y(t_1) = \int_{t_1}^{t_2} y'(s)\,ds =$$

$$= \int_{t_1}^{t_2} \frac{1}{1+s\,y(s)}\,ds \le \int_{t_1}^{t_2} \frac{1}{1+t_1}\,ds = \frac{t_2-t_1}{1+t_1}$$

where in $t_1 \le s \le t_2$

$s\,y(s) \geq t_1$

$t = t_2 \qquad t_2 < t < t_0$

$$\underbrace{y(t)-y(t_1)}_{\downarrow\, t\to t_0^- \;\; +\infty} \le \frac{t-t_1}{1+t_1} \xrightarrow{t\to t_0^-} \frac{t_0-t_1}{1+t_1}$$

So All solutions with

positive initial value are globally defined

$$\begin{cases} y_n' = \dfrac{1}{1+ty_n} \\ y_n(0) = 1 + \dfrac{1}{n} \end{cases} \qquad \begin{cases} z' = \dfrac{1}{1+tz} \\ z(0) = 1 \end{cases}$$

$$y_n \longrightarrow z \qquad L^\infty_{loc}([0,+\infty))$$

$[0,T]$

$$y_n(t) - z(t) = \frac{1}{n} + \int_0^t [f(s, y_n(s)) - f(s, z(s))]\, ds$$

$$= \frac{1}{n} + \int_0^t A_n(s)\,(y_n - z)\, ds$$

$$|A_n(s)| \leq \left| \frac{f(s, y_n(s)) - f(s, z(s))}{y_n(s) - z(s)} \right| =$$

$$\leq \sup \left\{ |\partial_y f(t,y)| : \begin{array}{l} 0 \leq t \leq T \\ y \geq 0 \end{array} \right\}$$

$$= T$$

$$\partial_y f(t,y) = \partial_y \frac{1}{1+ty} = -\frac{t}{(1+ty)^2}$$

$$|\partial_t f(t,y)| \leq t \leq T$$

$$y_n(t) - z(t) = \frac{1}{n} + \int_0^t \left[f(s, y_n(s)) - f(s, z(s)) \right] ds$$

$$= \frac{1}{n} + \int_0^t A_n(s) \; (y_n - z) \, ds$$

$$|y_n(t) - z(t)| \leq \frac{1}{n} + T \int_0^t |y_n - z| \, ds$$

$$|y_n(t) - z(t)| \leq e^{tT} \frac{1}{n} \leq e^{T^2} \frac{1}{n}$$

$$y(t) \leq A + \int_0^t B(s) \, y(s) \, dt$$

$$\Downarrow$$

$$y(t) \leq e^{\int_0^t B(s) \, ds} A$$

$$\begin{cases} y' = f(t, y) \\ y(0) = y_0 \end{cases} \qquad [0, t_0)$$

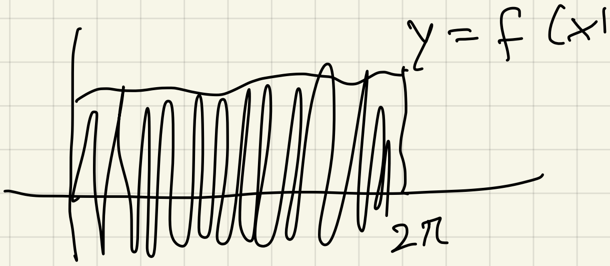
$$\begin{cases} y_n' = f(t, y_n) \\ y_n(0) = y_n \end{cases} \qquad y_n \longrightarrow y_0$$

$$\forall \; 0 < T < t_0$$

$y_n \longrightarrow y$ in $C^0([0,T])$

Ex $\displaystyle\lim_{n\to+\infty} \int_{\mathbb{T}} |\sin(nt)\, f(t)| = \frac{2}{\pi} \int_{\mathbb{T}} |f|\, dt$

$\forall f \in L^1(\mathbb{T})$



$p > q$

$$\boxed{L^p(\mathbb{T}) \hookrightarrow L^q(\mathbb{T})}$$

$\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$

$\infty \geq r \geq 1$

$\|f\|_{L^q} = \|1\, f\|_{L^q} \leq$

$\leq \|1\|_{L^r} \, \|f\|_{L^p}$

$$\begin{array}{ccc} L^2(\mathbb{T}) & \xrightarrow{*} & L^1(\mathbb{T}) \\ \uparrow & & \downarrow \\ \ell^2(\mathbb{Z}) & \dashrightarrow & \ell^\infty(\mathbb{Z}) \end{array}$$

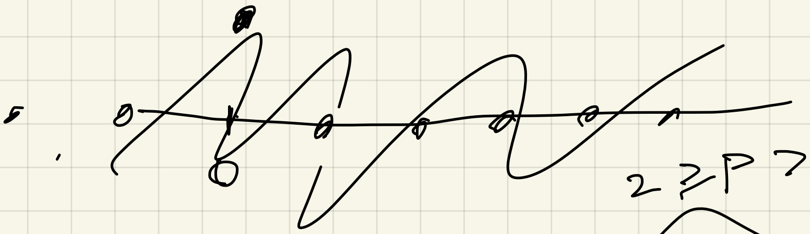
$\ni$

$f \not\equiv 0$

$\ell \quad \mathbb{Z} \longrightarrow \mathbb{R}$

$f(\cdot - n) \longrightarrow 0 \quad \ell^2(\mathbb{Z})$

$f(\cdot - n) \longrightarrow 0$ in $\ell^\infty(\mathbb{Z})$



$$\|f(\cdot - n)\|_{\ell^\infty(\mathbb{Z})} = \|f\|_{\ell^\infty(\mathbb{Z})}$$

$2 \geq p > 1$

$$\begin{tikzcd} L^p(\mathbb{T}) \arrow[r, hook] & L^1(\mathbb{T}) \\ L^2(\mathbb{T}) \arrow[u] \arrow[ur] & \end{tikzcd}$$

$p > 2$

$$L^p(\mathbb{T}) \longrightarrow L^1(\mathbb{T})$$

$$\{ e^{inx} f \} \qquad |e^{inx} f|_{L^p} = |f|_{L^p}$$

Suppose $\{ e^{in_k x} f \}$ converges in $L^1(\mathbb{T})$

$\forall \varepsilon > 0 \quad \exists \; N_\varepsilon \;$ s.t. $\; k, \ell > N_\varepsilon$

$$\int_{\mathbb{T}} | e^{in_k x} - e^{in_\ell x} | \, |f| \, dx < \varepsilon$$

$$= \int_{\mathbb{T}} | e^{i(n_k - n_\ell)x} - 1 | \, |f| \, dx$$

$$= \int_{\mathbb{T}} | \cos(n_k - n_\ell) - 1 + i \sin(n_k - n_\ell) x | \, |f| \, dx$$

$$\leq \int_{\mathbb{T}} | \sin(n_k - n_\ell) x | \, |f| \, dx < \varepsilon$$

$\downarrow$

$$\varepsilon \geq \frac{2}{\pi} \|f\|_{L^1} > 0 \qquad \forall \varepsilon > 0$$

$$k \in L^q(\mathbb{R}) \qquad f \in L^p(\mathbb{R}) \qquad \infty \geq r \geq 1$$

$$Tf = k * f$$

$$L^p(\mathbb{R}) \longrightarrow L^r(\mathbb{R}) \qquad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$$

not compact

$$\tau_{x_0} f = f(\cdot - x_0)$$

$$\tau_{x_0} T = T \tau_{x_0} \qquad \forall \; x_0 \in \mathbb{R}$$

$$(\tau_{x_0} T_k f)(x) = T_k f(x - x_0) = \int_{\mathbb{R}} k(x - x_0 - y) f(y)\,dy$$

$$y = z - x_0$$
$$z = y + x_0$$
$$dz = dy$$

$$= \int_{\mathbb{R}} k(x - z) \; f(z - x_0)\,dz$$

$$= \int_{\mathbb{R}} k(x - y) \; (\tau_{x_0} f)(y)\,dy = (T_k \tau_{x_0} f)(x)$$

$$\tau_{x_0} T_k = T_k \tau_{x_0}$$

$$T_k : L^p(\mathbb{R}) \to L^r(\mathbb{R}) \qquad 1 < p < +\infty$$

$$r \neq +\infty \qquad q < \infty$$

$\tau_n f = f(\cdot - n) \longrightarrow 0$ in $L^p(\mathbb{R})$

$\downarrow$

$\underline{T_K \tau_n f \longrightarrow 0}$ strongly in $L^r(\mathbb{R})$

$|T_K \tau_n f|_{L^r} = \| \tau_n T_K f \|_{L^r} = \| T_K f \|_{L^r} > 0$

$\downarrow$

0

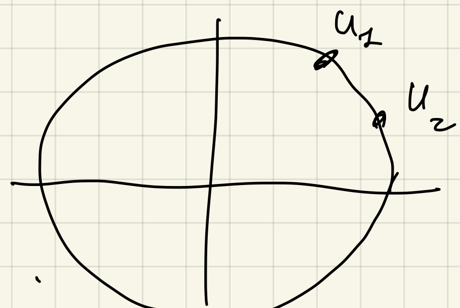
$\mathbb{T}$ $\qquad K \in L^q(\mathbb{T})$

$L^p(\mathbb{T}) \longrightarrow L^r(\mathbb{T})$ $\qquad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$

$f \longrightarrow K * f$

$\int_{\mathbb{T}} K(z w^{-1}) f(w)\, d\cancel{w}$ $\qquad \cancel{e^{it}}$ $\quad \cancel{u_1} \in$

$S^1 =$



$|u_1 u_2| = |u_1||u_2| = 1$

$\int K(\vartheta - t) f(t)\, dt$

$L^p \longrightarrow L^r$ $\qquad K \in L^q(\mathbb{T})$ $\qquad q < +\infty$

$K_n \in C^0(\mathbb{T})$

$\| K * f - K_n * f \|_{L^r} = \| (K - K_n) * f \|_{L^r} \le |K - K_n|_{L^q} |f|_{L^p}$

$\downarrow^{n \to +\infty}$
0

$K \quad \forall \varepsilon > 0 \quad \exists\, \delta_\varepsilon > 0 \quad$ s.t.

$I \subseteq \mathbb{T} \qquad \operatorname{diam} I < \delta_\varepsilon$

$\Rightarrow \operatorname{osc}_I K < \varepsilon$

$$K * f = \int_{-\pi}^{\pi} K(x-y)\, f(y)\, dy =$$

$y_j \in I_j$

$$= \sum_j \int_{I_j} K(x-y)\, f(y)\, dy =$$

$$= \sum_j K(x-y_j) \int_{I_j} f(y)\, dy +$$

$$+ \sum \int_{I_j} \left(K(x-y) - K(x-y_j) \right) f(y)\, dy$$

$$\varepsilon \sum \int_{I_j} |f(y)|\, dy = \varepsilon \, \|f\|_{L^1}$$

$$\Rightarrow \left| \sum \int_{I_j} (\ldots) f(y)\, dy \right|_{L^r}$$

$$\lesssim \left| \quad \ldots \quad \right|_{L^\infty} \lesssim \varepsilon \|f\|_{L^1}$$

$$W^{1,1}(I) = \{ f \in L^1(I) : f' \in L^1(I) \}$$

$$H^s(\mathbb{T}^d) \hookrightarrow L^q(\mathbb{T}^d)$$

$$s < \frac{d}{2}$$

$$\frac{1}{q} = \frac{1}{2} - \frac{s}{d}$$

$$W^{p,s}(\Omega) \hookrightarrow L^q(\Omega)$$

$s < \frac{d}{p}$ $\qquad$ $p < \frac{d}{s}$ $\qquad$ $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$

$$\frac{1}{p} > \frac{s}{d}$$

$W^{p,s}(I) \hookrightarrow L^q(I)$ $\qquad$ $d = 1$

compact

$W^{1,1}(I) \hookrightarrow L^\infty(I)$ $\qquad$ $L^{\tilde{q}}(I)$

$$\tilde{q} < q$$

$W^{p,}$

$W^{p,s}(I)$ $\qquad$ $L^q(I)$ $\qquad$ $[0,1]$

$\frac{1}{q} = \frac{1}{p} - s$ $\qquad$ $\delta_\lambda f = \lambda^{\frac{1}{q}} f(\lambda \cdot)$

$$\lambda \geq 1$$

$$\underbrace{\delta_n f}_{\Downarrow \atop 0}$$

$$\underbrace{\|\delta_n f\|_{L^q}}_{\downarrow \atop 0} = \|f\|_{L^1} \geq \delta_0^{\lambda}$$