Control Theory

Course ID: 322MI – Spring 2023

Felice Andrea Pellegrino

University of Trieste
Department of Engineering and Architecture

322MI –Spring 2023

Lecture 2: Solutions to linear systems

Continuous-time linear systems

Consider a *homogeneous* (i.e. having no input) continuous-time linear time-invariant system:

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ and the initial time is 0 with no loss of generality.
We want to find the solution $x(t)$, $t \geq 0$. To this aim, let's consider the scalar case (i.e., $n = 1$) first:

$$\dot{x}(t) = ax(t), \quad x(0) = x_0, \quad x, a \in \mathbb{R}. \tag{2}$$

The solution is easily proven to be:

$$x(t) = e^{at}x_0.$$

In the general case (i.e., $n \geq 1$), the solution takes the same form, as shown in the following.

Homogeneous systems (cont.)

Motivated by the analogy with the scalar

$$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!},$$

we can define the *matrix exponential* of a given $n \times n$ matrix A by

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

The above is a convergent series, since the factorial dominates the exponential for $k \to \infty$. Now, it is easy to prove that

$$x(t) = e^{At} x_0$$

is the solution of (1).

Indeed, it satisfies the initial condition:

$$x(0) = e^{A0}x_0 = Ix_0 = x_0;$$

moreover, by taking the derivative, we get:

$$\frac{d}{dt}\left(e^{At}x_0\right) = \frac{d}{dt}\left(\sum_{k=0}^{\infty}\frac{A^k t^k}{k!}\right) = \sum_{k=1}^{\infty} A\frac{A^{k-1}t^{k-1}}{(k-1)!} = A\left(e^{At}x_0\right),$$

thus it satisfies the differential equation.

Nonhomogeneous systems

Consider a *nonhomogeneous* continuous-time linear time-invariant system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{3}$$

It can be verified that the solution of (3) is:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau.$$

Indeed, by taking the Laplace transform of (3), we get

$$sX(s) - x_0 = AX(s) + BU(s) \quad \Longrightarrow \quad (sI - A)X(s) = x_0 + BU(s),$$

and, solving for $X(s)$:

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s),$$

which is the solution in terms of Laplace transforms.

Now, recalling that $\mathcal{L}\left(e^{At}\right) = (sI - A)^{-1}$, and that the Laplace transform of the convolution of two functions is the product of the individual Laplace transforms, we have

$$x(t) = \mathcal{L}^{-1}\left[X(s)\right] = \mathcal{L}^{-1}\left[(sI - A)^{-1}x_0\right] + \mathcal{L}^{-1}\left[(sI - A)^{-1}BU(s)\right] = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau.$$

We can thus state the following (where the initial time is now t_0)

Theorem

The solution of

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \quad x(t_0) = x_0$$

takes the form:

$$x(t) = \varphi\left(t, t_0, x_0, u(\cdot)\right) = e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau.$$

Nonhomogeneous systems (cont.)

In the right side of the expression

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

the first term depends on the initial state x_0, but not on the input, while the second depends on the input $u(\cdot)$, but not on the initial state. Thus, the whole solution can be decomposed as follows:

- *Natural (state) response*, i.e. the solution when the input is zero:

$$u(t) = 0,\, \forall t \geq t_0 \quad \Longrightarrow \quad x_N(t) = e^{A(t-t_0)}x_0$$

- *Forced (state) response*, i.e. the solution when the initial state is zero:

$$x_0 = 0 \quad \Longrightarrow \quad x_F(t) = \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

The whole solution is indeed

$$x(t) = x_N(t) + x_F(t).$$

Output response

Taking into account the output equation, we have:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t)\,, \quad x(t_0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

and by substituting the state response $x(t)$ in the output equation, we get:

$$y(t) = Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

- *Natural response*. By setting $u(t) = 0$, $t \geq t_0$ we get:

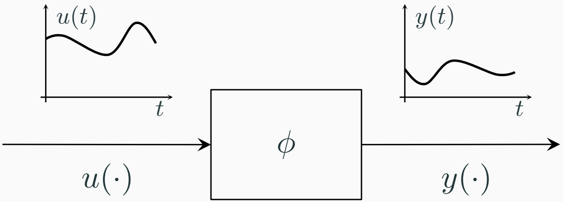
$$y(t) = y_N(t) = Ce^{A(t-t_0)}x_0$$

- *Forced response*. By setting $x_0 = 0$ we get:

$$y(t) = y_F(t) = \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

The whole output response is thus given by:

$$y(t) = y_N(t) + y_F(t).$$



A continuous-time linear system can be represented as linear *operator* ϕ mapping input signals to output signals. That representation is the *input-output representation* of linear systems:

$$\phi : \mathbb{U} \longrightarrow \mathbb{Y}$$

where $\mathbb{U}$ is a vector space of input signals

$$u(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}^m$$

and $\mathbb{Y}$ is a vector space of output signals

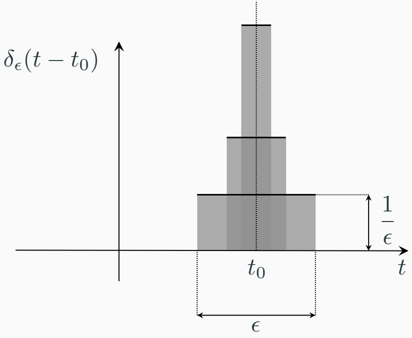
$$y(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}^p$$

The operator (and thus, the system) can be characterized by the *impulse response*, i.e. the response of the system to a particular input called the impulse. Although the concept is far more general, in the following we consider only the case of causal linear time-invariant systems.

The Dirac delta

An impulse is a phenomenon with high intensity and very short duration. To represent it mathematically, we can consider a function $\delta_\epsilon(t)$ defined as

$$\delta_\epsilon(t) = \begin{cases} 0 & \text{if} \quad t < -\frac{\epsilon}{2} \\ \frac{1}{\epsilon} & \text{if} \quad -\frac{\epsilon}{2} \leq t \leq \frac{\epsilon}{2} \\ 0 & \text{if} \quad t > \frac{\epsilon}{2} \end{cases}$$



The support of the function (namely, the interval where the function is non-zero) is $[-\epsilon/2, \epsilon/2]$. For decreasing ϵ, the interval becomes increasingly small, while the value taken by the function, i.e. $1/\epsilon$, becomes increasingly large. Note that the integral of the function remains equal to 1.

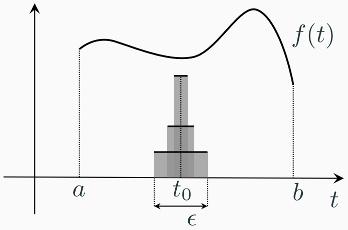
Let t_0 be a point in the interior of $[a, b]$ and ϵ be such that $[t_0 - \epsilon/2, t_0 + \epsilon/2] \subset [a, b]$. The impulse in t_0, $\delta(t - t_0)$, can be seen as the the "limit" for $\epsilon \to 0$ of the function $\delta_\epsilon(t - t_0)$. Intuitively (a formal treatment can be found in Antsaklis and Michel (2006)), consider the integral

$$\int_a^b f(t)\, \delta_\epsilon(t - t_0) dt,$$

where f is a continuous function. Then

$$\begin{aligned}\int_a^b f(t)\, \delta_\epsilon(t - t_0) dt &= \int_{t_0 - \frac{\epsilon}{2}}^{t_0 + \frac{\epsilon}{2}} f(t)\, \delta_\epsilon(t - t_0) dt = \\ &= \int_{t_0 - \frac{\epsilon}{2}}^{t_0 + \frac{\epsilon}{2}} f(t)\, \frac{1}{\epsilon} dt = \frac{1}{\epsilon} f(\tau)\epsilon = f(\tau)\end{aligned}$$

where τ (which exists for the mean value theorem) belongs to the interval $[t_0 - \frac{\epsilon}{2}, t_0 + \frac{\epsilon}{2}]$.



The Dirac delta (cont.)

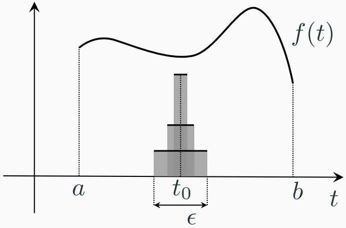
Being f continuous, when $\epsilon \to 0$ we have

$$\int_a^b f(t)\,\delta_\epsilon(t-t_0)dt \to f(t_0).$$

The *Dirac delta distribution* $\delta(t-t_0)$ is defined as the "function" such that for every continuous function f defined in $[a,b]$ containing t_0, we have that

$$\int_a^b f(t)\,\delta(t-t_0)dt = f(t_0). \tag{4}$$

Eq. (4) is called the *sifting property* of the impulse (or the *sampling property* of the impulse).



Consider the SISO system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}$$

and assume that $x(0) = 0$. By applying the unit impulse $\delta(t)$, we get

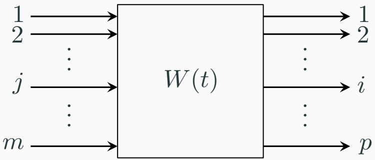
$$y(t) = \int_0^t Ce^{A(t-\tau)}B\delta(\tau)d\tau + D\delta(t) = Ce^{At}B + D\delta(t)$$

where the second equality follows from the sifting property. The function

$$W(t) \doteq Ce^{At}B + D\delta(t)$$

is called the *impulse response* of the system.

Impulse response of continuous-time LTI systems (cont.)



In the MIMO case, $W(t)$ is a $p \times m$ matrix: each element $w_{ij}(t)$ represents the ensuing response of the ith output at time t, due to an impulse applied at time 0 to the jth input, for zero initial condition.

Discrete-time linear systems

Homogeneous systems

Consider a homogeneous discrete-time linear time-invariant system:

$$x(k+1) = Ax(k)\,, \quad x(k_0) = x_0$$

where $x \in \mathbb{R}^n$.
Clearly, $x(k)$, $k > k_0$ can be determined by iterating the state equation:

$$\begin{aligned} x(k_0) &= x_0 \\ x(k_0+1) &= Ax(k_0) \\ x(k_0+2) &= Ax(k_0+1) = A^2 x(k_0) \\ &\vdots \\ x(k) &= A^{k-k_0} x(k_0) \end{aligned}$$

thus we have:

$$x(k) = A^{k-k_0} x_0.$$

Nonhomogeneous systems

Now, consider a nonhomogeneous linear discrete-time system:

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(k_0) = x_0$$

Clearly:

$$\begin{aligned}
x(k_0) &= x_0 \\
x(k_0+1) &= Ax(k_0) + Bu(k_0) \\
x(k_0+2) &= Ax(k_0+1) + Bu(k_0+1) \\
&= A[Ax(k_0) + Bu(k_0)] + Bu(k_0+1) \\
&= A^2x(k_0) + ABu(k_0) + Bu(k_0+1) \\
x(k_0+3) &= Ax(k_0+2) + B(k_0+2)u(k_0+2) \\
&= A^3x(k_0) + A^2Bu(k_0) + ABu(k_0+1) + Bu(k_0+2) \\
&\vdots \\
x(k) &= A^{(k-k_0)}x(k_0) + \textstyle\sum_{j=k_0}^{k-1} A^{(k-1-j)}Bu(j)
\end{aligned}$$

Nonhomogeneous systems (cont.)

We can thus state the following

Theorem

The solution of

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(k_0) = x_0$$

takes the form:

$$x(k) = \varphi\left(k, k_0, x_0, u(\cdot)\right) = A^{(k-k_0)}x_0 + \sum_{j=k_0}^{k-1} A^{(k-1-j)}Bu(j)$$

Output response

Taking into account the output equation, we have:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k)\,, \quad x(k_0) = x_0 \\ y(k) = Cx(k) + Du(k) \end{cases}$$

By substituting the state response $x(k)$ in the output equation we get:

$$y(k) = \quad CA^{(k-k_0)}x_0 + \sum_{j=k_0}^{k-1} CA^{k-1-j}Bu(j) + Du(k)\,, \quad k \geq k_0$$

- *Natural response*. By setting $u(k) = 0, \forall\, k \geq k_0$, we get:

$$y(k) = y_N(k) = CA^{(k-k_0)}x_0\,,\, k \geq k_0$$

- *Forced response*. By setting $x_0 = 0$, we get:

$$y(k) = y_F(k) = \sum_{j=k_0}^{k-1} CA^{k-1-j}Bu(j) + Du(k)\,,\, k \geq k_0$$

The whole response is thus given by:

$$y(k) = y_N(k) + y_F(k).$$

Input-output representation

A discrete-time linear system can be represented as linear *operator* ϕ mapping input signals to output signals. That representation is the input-output representation of linear systems:

$$\phi : \mathbb{U} \longrightarrow \mathbb{Y}$$

where $\mathbb{U}$ is a vector space of input signals

$$u(\cdot) : \mathbb{Z} \longrightarrow \mathbb{R}^m$$

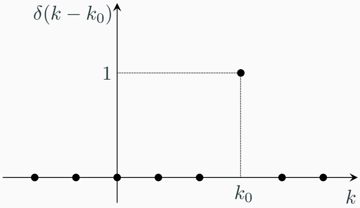
and $\mathbb{Y}$ is a vector space of output signals

$$y(\cdot) : \mathbb{Z} \longrightarrow \mathbb{R}^p$$

The operator (and thus, the system) can be characterized by the *impulse response*, i.e. the response of the system to a particular input called the impulse.

Although the concept is far more general, in the following we consider only the case of causal linear time-invariant systems.

Impulse response of discrete-time LTI systems



In the discrete-time case, the unit impulse at time k_0 is simply:

$$\delta(k-k_0) = \begin{cases} 0, & k \neq k_0,\ k \in \mathbb{Z} \\ 1, & k = k_0 \end{cases}$$

Impulse response of discrete-time LTI systems (cont.)

Consider the SISO system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

and assume that $x(0) = 0$. By applying the unit impulse $\delta(k)$ we get

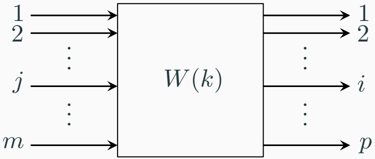
$$y(k) = \sum_{j=0}^{k-1} CA^{k-1-j} B\delta(j) + D\delta(k)$$

where the summation is assumed to be zero for $k = 0$. The function

$$W(k) \doteq \begin{cases} CA^{k-1}B, & k > 0 \\ D, & k = 0 \\ 0, & k < 0 \end{cases}$$

is called the *impulse response* of the system.

Impulse response of discrete-time LTI systems (cont.)



In the MIMO case, $W(k)$ is a $p \times m$ matrix: each element $w_{ij}(k)$ represents the ensuing response of the ith output at time k, due to an impulse applied at time 0 to the jth input, for zero initial condition.

Modal analysis (continuous-time)

State response

We have seen that the state response of the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

takes the form:

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Without loss of generality, we can take $t_0 = 0$ thus obtaining

$$x(t) = e^{At}x_0 + \int_{0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

The matrix A plays a fundamental role and is responsible of the qualitative behavior of the response. In the following we will analyze the qualitative behavior, starting with the simple case of A being diagonalizable.

Modal analysis, diagonalizable A

If A is diagonalizable by a similarity transformation we can write:

$$\begin{cases} A = T\Lambda T^{-1} \\ \Lambda = T^{-1}AT \end{cases}$$

where Λ is a diagonal matrix having diagonal elements $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where λ_i is the ith eigenvalue of A. The columns of the matrix T are eigenvectors t_i of matrix A. The inverse of T, $S = T^{-1}$ can be partitioned row-wise

$$T = [t_1\ t_2\ \ldots\ t_n], \quad T^{-1} = S = \begin{bmatrix} s_1^\top \\ s_2^\top \\ \vdots \\ s_n^\top \end{bmatrix}.$$

Thus, A may be rewritten as:

$$A = [t_1\ t_2\ \ldots\ t_n] \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & \ldots \\ 0 & \lambda_2 & 0 & 0 & \ldots \\ 0 & 0 & \lambda_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix} \begin{bmatrix} s_1^\top \\ s_2^\top \\ \vdots \\ s_n^\top \end{bmatrix}. \tag{5}$$

Modal analysis, diagonalizable A (cont.)

As a consequence, e^{At} can be written as:

$$\begin{aligned} e^{At} &= \sum_{k=0}^{\infty} A^k \frac{t^k}{k!} = \sum_{k=0}^{\infty} (T\Lambda T^{-1})^k \frac{t^k}{k!} = \sum_{k=0}^{\infty} \underbrace{(T\Lambda T^{-1} T\Lambda T^{-1} \dots T\Lambda T^{-1})}_{k \text{ times}} \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} T\Lambda^k T^{-1} \frac{t^k}{k!} = T(\sum_{k=0}^{\infty} \Lambda^k \frac{t^k}{k!})T^{-1} \end{aligned}$$

thus

$$e^{At} = Te^{\Lambda t}T^{-1}$$

By expliciting the columns t_i of T e the rows $s_i^\top$ of $S = T^{-1}$ we get:

$$e^{At} = [t_1 \ t_2 \ \dots \ t_n] \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 & \dots \\ 0 & e^{\lambda_2 t} & 0 & 0 & \dots \\ 0 & 0 & e^{\lambda_3 t} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} s_1^\top \\ s_2^\top \\ \vdots \\ s_n^\top \end{bmatrix} = \sum_{i=1}^{n} t_i s_i^\top \ e^{\lambda_i t}.$$

Modal analysis, diagonalizable A (cont.)

By defining n matrices of size $n \times n$

$$Z_i = t_i s_i^\top, \quad i = 1, \dots, n$$

we can state the following

Property

If A is diagonalizable, the state transition matrix e^{At} can be written as the sum of constant matrices Z_i, each multiplied by the function $e^{\lambda_i t}$

$$e^{At} = \sum_{i=1}^{n} Z_i \, e^{\lambda_i t}. \tag{6}$$

The natural state response can thus be expressed as

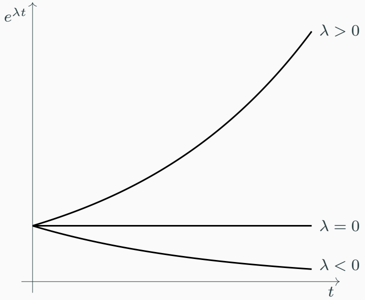
$$x_N(t) = \sum_{i=1}^{n} Z_i \, e^{\lambda_i t} x(0) = \sum_{i=1}^{n} t_i \, (s_i^\top x(0)) e^{\lambda_i t} = \sum_{i=1}^{n} t_i \, \alpha_i(x(0)) e^{\lambda_i t} \tag{7}$$

where $\alpha_i(x(0)) = s_i^\top x(0), \; i = 1, 2, \dots, n$ are scalars obtained as the dot product of each left eigenvector and the initial condition x_0. The functions

$$e^{\lambda_i t}$$

are the *modes* of the system.

Modal analysis, diagonalizable A (cont.)



If $\lambda \in \mathbb{R}$, the mode $e^{\lambda t}$ is an exponential mode that, for increasing t, has the following behavior:

- if $\lambda > 0$ the mode diverges;
- if $\lambda < 0$ the mode vanishes;
- if $\lambda = 0$ the mode is constant.

Modal analysis, diagonalizable A (cont.)

In general, A may have complex eigenvalues (i.e. eigenvalues whose imaginary part is non zero). It is well-known that if λ, v is an eigenpair of A, the complex conjugate pair λ^*, v^* is an eigenpair of A too.

Without loss of generality, assume that $\lambda_1, \lambda_2, \ldots, \lambda_r$, where $r \leq n$, are real numbers and $\lambda_{r+1}, \ldots, \lambda_n$ are complex numbers, ordered pairwise $\lambda_{i+1} = \lambda_i^*$, or

$$\sigma(A) = \{ \underbrace{\lambda_1, \lambda_2, \ldots \lambda_r}_{\text{real}}, \underbrace{\lambda_{r+1}, \lambda_{r+1}^*}_{\text{conjugate}}, \underbrace{\lambda_{r+3}, \lambda_{r+3}^*}_{\text{conjugate}}, \ldots, \underbrace{\lambda_{n-1}, \lambda_{n-1}^*}_{\text{conjugate}} \}.$$

Then it follows that Z_i e Z_{i+1} are conjugate if λ_i and λ_{i+1} are. As a consequence:

$$e^{At} = \sum_{i=1}^{r} Z_i e^{\lambda_i t} + \sum_{r+1 \text{ (step } 2)}^{n-1} (Z_i e^{\lambda_i t} + Z_i^* e^{\lambda_i^* t})$$

Decomposing λ_i and Z_i in real and imaginary part we get:

$$\begin{aligned} \lambda_i &= \mu_i + j\omega_i \\ Z_i &= M_i + jN_i. \end{aligned}$$

From Euler's formula

$$e^{\lambda_i t} = e^{\mu_i t}[\cos(\omega_i t) + j\sin(\omega_i t)]$$

and its simple to check that the imaginary contributions cancel each other thus

$$e^{At} = \sum_{i=1}^{r} Z_i e^{\mu_i t} +$$

$$+ \sum_{r+1 \text{ (step } 2)}^{n-1} 2e^{\mu_i t}[M_i \cos(\omega_i t) - N_i \sin(\omega_i t)].$$

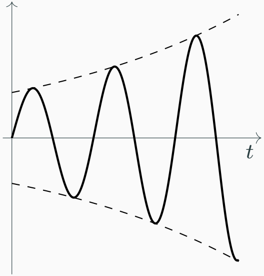
The following fundamental property holds.

Property

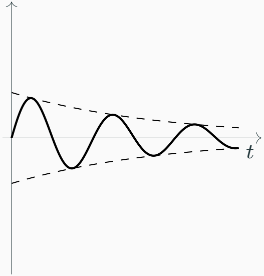
Each conjugate pair of eigenvalues $\lambda = \mu + j\omega$ and $\lambda^* = \mu - j\omega$ produces the complex modes $e^{\lambda t}$ and $e^{\lambda^* t}$, that result in real modes of the form:

$$e^{\mu t}\cos(\omega t) \quad \text{and} \quad e^{\mu t}\sin(\omega t).$$

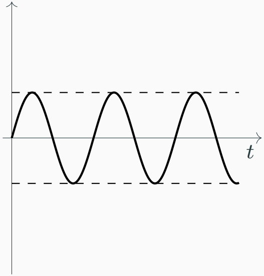
Modal analysis, diagonalizable A (cont.)



$e^{\mu t}\sin(\omega t)$ for $\mu > 0$.



$e^{\mu t}\sin(\omega t)$ for $\mu < 0$.



$e^{\mu t}\sin(\omega t)$ for $\mu = 0$.

The state transition matrix is thus governed by real exponential terms associated to real eigenvalues and oscillating ("*pseudo-periodic*") modes associated to the conjugate pairs of eigenvalues. Depending on the real part of the conjugate pairs the following behaviors may occur:

- if $\mu > 0$ the amplitude of the oscillation diverges;
- if $\mu < 0$ the amplitude of the oscillation vanishes;
- if $\mu = 0$ the amplitude of the oscillation is constant.

Modal analysis, non-diagonalizable A

If A is non-diagonalizable, we need to resort to the following

Theorem (Jordan normal form)

For every matrix $A \in \mathbb{C}^{n\times n}$, there exists a non-singular change of basis matrix $T \in \mathbb{C}^{n\times n}$ such that

$$J = T^{-1}AT = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_s \end{bmatrix}, \quad \text{where}$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots \\ 0 & \lambda_i & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & \dots & \dots & \lambda_i \end{bmatrix}$$

is a *Jordan block* with each λ_i an eigenvalue of A and s equal to the number of independent eigenvectors of A. The matrix J is unique up to a reordering of the blocks and is called the *Jordan normal form* of A.

Note

There may be several Jordan blocks associated to the same eigenvalue. Sometimes the J_i are referred to as the Jordan *mini*-blocks, and the diagonal block composed of all the mini-blocks associated to the same eigenvalue is called a Jordan block.

Notice that the ith block $J_i \in \mathbb{R}^{\nu_i \times \nu_i}$ may be written as

$$J_i = \underbrace{\begin{bmatrix} \lambda_i & 0 & 0 & 0 & \dots \\ 0 & \lambda_i & 0 & 0 & \dots \\ 0 & 0 & \lambda_i & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & \lambda_i \end{bmatrix}}_{=\lambda_i \ I} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}}_{\doteq J_{i0}}$$

thus

$$J_i = \Lambda_i + J_{i0}.$$

Modal analysis, non-diagonalizable A (cont.)

Since $J = T^{-1}AT$, the state transition matrix may be written as

$$e^{At} = Te^{Jt}T^{-1} \tag{8}$$

where

$$e^{Jt} = \begin{bmatrix} e^{J_1t} & 0 & 0 & 0 & \dots \\ 0 & e^{J_2t} & 0 & 0 & \dots \\ 0 & 0 & e^{J_3t} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e^{J_st} \end{bmatrix}.$$

Now, let us consider the block $e^{J_it} = e^{(\Lambda_i t + J_{i0} t)}$. It is easy to check that, if the square matrices M and N are such that $MN = NM$, i.e. if they commute, then $e^{(M+N)} = e^M e^N$. From the definition of Λ_i and J_{i0} it follows that they commute: $\Lambda_i J_{i0} = J_{i0}\Lambda_i$, hence

$$e^{J_it} = e^{\lambda_i It} e^{J_{i0}t}. \tag{9}$$

Modal analysis, non-diagonalizable A (cont.)

The powers of J_{i0} are obtained, by "moving upwards the 1s", for instance:

$$J_{i0} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad J_{i0}^2 = \begin{bmatrix} 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & 1 & : \\ 0 & 0 & 0 & \dots & : \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \dots$$

$$J_{i0}^{\nu_i - 1} \begin{bmatrix} 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad J_{i0}^{\nu_i} \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Moreover, $J_{i0}^p = 0, \quad \forall p \geq \nu_i$. Thus, the series corresponding to $e^{J_{i0}t}$ is actually a sum of a finite number of terms

$$e^{J_{i0}t} = \sum_{j=0}^{\nu_i - 1} \frac{1}{j!} J_{i0}^j t^j.$$

Modal analysis, non-diagonalizable A (cont.)

By inspecting the form of each of the terms of $e^{J_{i0}t} = \sum_{j=0}^{\nu_i-1} \frac{1}{j!} J_{i0}^j t^j$, it is easy to check that

$$e^{J_{i0}t} = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \dots & \frac{t^{\nu_i-1}}{(\nu_i-1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{\nu_i-2}}{(\nu_i-2)!} \\ 0 & 0 & 1 & t & \dots & \frac{t^{\nu_i-3}}{(\nu_i-3)!} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & t \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

On the other hand:

$$e^{\lambda_i I t} = \begin{bmatrix} e^{\lambda_i t} & 0 & 0 & \dots \\ 0 & e^{\lambda_i t} & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_i t} \end{bmatrix}.$$

Modal analysis, non-diagonalizable A (cont.)

Thus, the ith block of e^{Jt} takes the form

$$e^{J_i t} = \sum_{j=0}^{\nu_i - 1} J_{i0}^j \frac{t^j}{j!} e^{\lambda_i t} = e^{\lambda_i t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \dots & \frac{t^{\nu_i - 1}}{(\nu_i - 1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{\nu_i - 2}}{(\nu_i - 2)!} \\ 0 & 0 & 1 & t & \dots & \frac{t^{\nu_i - 3}}{(\nu_i - 3)!} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & t \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{10}$$

or

$$e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & t e^{\lambda_i t} & \frac{t^2}{2!} e^{\lambda_i t} & \frac{t^3}{3!} e^{\lambda_i t} & \dots & \frac{t^{\nu_i - 1}}{(\nu_i - 1)!} e^{\lambda_i t} \\ 0 & e^{\lambda_i t} & t e^{\lambda_i t} & \frac{t^2}{2!} e^{\lambda_i t} & \dots & \frac{t^{\nu_i - 2}}{(\nu_i - 2)!} e^{\lambda_i t} \\ 0 & 0 & e^{\lambda_i t} & t e^{\lambda_i t} & \dots & \frac{t^{\nu_i - 3}}{(\nu_i - 3)!} e^{\lambda_i t} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & t e^{\lambda_i t} \\ 0 & 0 & 0 & 0 & \dots & e^{\lambda_i t} \end{bmatrix} \tag{11}$$

Modal analysis, non-diagonalizable A (cont.)

Back to the exponential matrix, letting $S = T^{-1}$, by partitioning T (column-wise) and S (row-wise) – according to the size of the diagonal blocks – we get:

$$e^{At} = [T_1 \ T_2 \ \dots \ T_s] \begin{bmatrix} e^{J_1 t} & 0 & 0 & 0 & \dots \\ 0 & e^{J_2 t} & 0 & 0 & \dots \\ 0 & 0 & e^{J_3 t} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e^{J_s t} \end{bmatrix} \begin{bmatrix} S_1^\top \\ S_2^\top \\ \vdots \\ S_s^\top \end{bmatrix}, \tag{12}$$

where $T_i \in \mathbb{C}^{n \times \nu_i}$ e $S_i^\top \in \mathbb{C}^{\nu_i \times n}$.

Thus

$$e^{At} = \sum_{i=1}^{s} [T_i e^{J_i t} S_i^\top], \tag{13}$$

and, by using (10),

$$e^{At} = \sum_{i=1}^{s} \sum_{j=0}^{\nu_i - 1} T_i J_{i0}^j S_i^\top \frac{t^j}{j!} e^{\lambda_i t}.$$

Modal analysis, non-diagonalizable A (cont.)

Finally, by letting $Z_{ij} = T_i J_{i0}^j S_i^\top \frac{1}{j!}$ we obtain

$$e^{At} = \sum_{i=1}^{s} \sum_{j=0}^{\nu_i - 1} Z_{ij} t^j e^{\lambda_i t}. \tag{14}$$

If there exist Jordan blocks of size greater than one, associated to the eigenvalue λ, in the matrix e^{At}, the following modes will appear:

$$e^{\lambda t}, \;\; te^{\lambda t}, \;\; t^2 e^{\lambda t}, \ldots, t^{(\nu(\lambda)-1)} e^{\lambda t}$$

where $\nu(\lambda)$ denotes the *degree* of the eigenvalue λ, i.e. the size of the largest Jordan block associated to λ.

Modal analysis, non-diagonalizable A (cont.)

Example. Let the Jordan form of a matrix A be

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Then, its eigenvalues are 2 and 5, having degree $\nu(2) = 3$ and $\nu(5) = 2$, respectively. The modes of e^{At} are:

$$e^{2t},\ te^{2t},\ t^2e^{2t},\ e^{5t},\ te^{5t}.$$

Modal analysis (discrete-time)

State response

We have seen that the state response of the system

$$x(k+1) = Ax(k) + Bu(k)$$

takes the form:

$$x(k) = A^{(k-k_0)}x_0 + \sum_{j=k_0}^{k-1} A^{(k-1-j)}Bu(j)$$

Without loss of generality we can take $k_0 = 0$ thus obtaining

$$x(k) = A^k x_0 + \sum_{j=0}^{k-1} A^{(k-1-j)}Bu(j)$$

The transition matrix A^k can be computed simply as a matrix product repeated $k-1$ times, but this is of little interest. Instead, to reveal the properties of the response, we can perform a modal analysis, similarly to the continuous-time case. We will first consider the case of diagonalizable A.

Modal analysis, diagonalizable A

If A is diagonalizable by a similarity transformation, we can write:

$$\begin{cases} A = T\Lambda T^{-1} \\ \Lambda = T^{-1}AT \end{cases}$$

where Λ is a diagonal matrix having diagonal elements $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where λ_i is the ith eigenvalue of A. The columns of the matrix T are eigenvectors t_i of matrix A. The inverse of T, $S = T^{-1}$ can be partitioned row-wise

$$T = [t_1\ t_2\ \ldots\ t_n], \quad T^{-1} = S = \begin{bmatrix} s_1^\top \\ s_2^\top \\ \vdots \\ s_n^\top \end{bmatrix}.$$

Thus, A may be rewritten as:

$$A = [t_1\ t_2\ \ldots\ t_n] \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & \ldots \\ 0 & \lambda_2 & 0 & 0 & \ldots \\ 0 & 0 & \lambda_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix} \begin{bmatrix} s_1^\top \\ s_2^\top \\ \vdots \\ s_n^\top \end{bmatrix}. \tag{15}$$

Since

$$A^k = \underbrace{T\Lambda T^{-1} T\Lambda T^{-1} \dots T\Lambda T^{-1}}_{k \text{ times}} = T\Lambda^k T^{-1}, \tag{16}$$

we get

$$A^k = [t_1 \; t_2 \; \dots \; t_n] \begin{bmatrix} \lambda_1^k & 0 & 0 & 0 & \dots \\ 0 & \lambda_2^k & 0 & 0 & \dots \\ 0 & 0 & \lambda_3^k & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n^k \end{bmatrix} \begin{bmatrix} s_1^\top \\ s_2^\top \\ \vdots \\ s_n^\top \end{bmatrix}. \tag{17}$$

By defining the matrices

$$Z_h = t_h s_h^\top,$$

we can state the following

Property

If A is diagonalizable, the state transition matrix A^k can be written as the sum of constant matrices Z_h, each multiplied by the discrete mode λ_h^k

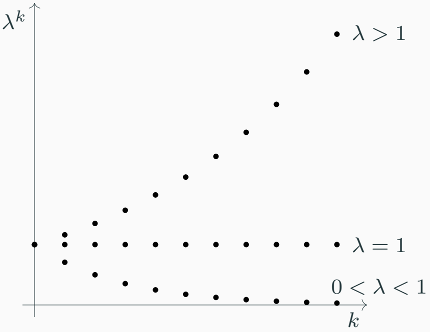
$$A^k = \sum_{h=1}^{n} t_h s_h^\top \lambda_h^k = \sum_{h=1}^{n} Z_h \lambda_h^k. \tag{18}$$

As in the continuous-time case, we can distinguish the two cases of the eigenvalue λ being real or complex.

Modal analysis, diagonalizable A (cont.)

For $\lambda \in \mathbb{R}$, the mode λ^k has the following behavior:

- if $|\lambda| > 1$ the mode diverges;
- if $|\lambda| < 1$ the mode vanishes;
- if $|\lambda| = 1$ the mode has constant amplitude.



λ^k for $0 < \lambda < 1$, $\lambda = 1$ and $\lambda > 1$.

Based on the sign of λ, there is the further distinction:

- if $\lambda > 0$ the mode is positive;
- if $\lambda < 0$ the mode has **alternated sign**;
- if $\lambda = 0$ the mode is null.

Modal analysis, diagonalizable A (cont.)

If some eigenvalue is complex we can, as before, order the eigenvalues:

$$\sigma(A) = \{ \underbrace{\lambda_1, \lambda_2, \ldots \lambda_r}_{\text{real}}, \underbrace{\lambda_{r+1}, \lambda^*_{r+1}}_{\text{conjugate}}, \underbrace{\lambda_{r+3}, \lambda^*_{r+3}}_{\text{conjugate}}, \ldots, \underbrace{\lambda_{n-1}, \lambda^*_{n-1}}_{\text{conjugate}} \}$$

By taking the real and imaginary part of Z_h and expressing λ_h in polar form we have

$$\begin{aligned} \lambda_h &= \rho_h \, e^{j\theta_h} \\ Z_h &= M_h + jN_h. \end{aligned}$$

From Euler's formula

$$\lambda_h^k = \rho_h^k e^{j\theta_h k} = \rho_h^k [\cos(\theta_h k) + j \sin(\theta_h k)]$$

it can be obtained (with a rather long maths)

$$A^k = \sum_{h=1}^{r} Z_h \lambda_h^k + \sum_{r+1 \text{ (step } 2)}^{n-1} 2\rho_h^k [M_h \cos(\theta_h k) - N_h \sin(\theta_h k)]$$

Modal analysis, diagonalizable A (cont.)

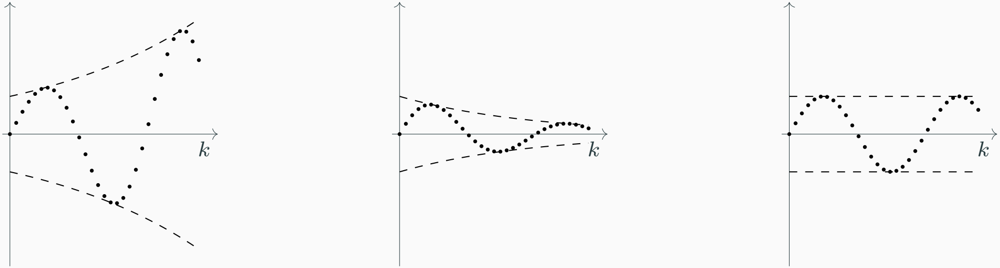
The following important property follows:

Property

Each conjugate pair of eigenvalues λ and λ^* produces complex modes that result in real sequences of the form:

$$\rho^k \cos(\theta k) \quad \text{and} \quad \rho^k \sin(\theta k).$$

Modal analysis, diagonalizable A (cont.)



$\rho^k \sin(\theta k)$ for $\rho > 1$. $\rho^k \sin(\theta k)$ for $0 < \rho < 1$. $\rho^k \sin(\theta k)$ for $\rho = 1$.

The state transition matrix is thus governed by real exponential terms associated to real eigenvalues and oscillating ("*pseudo-periodic*") modes associated to the conjugate pairs of eigenvalues. Depending on the modulus of the eigenvalue, the following behaviors may occur:

- if $\rho > 1$ the amplitude of the oscillation diverges;
- if $\rho < 1$ the amplitude of the oscillation vanishes;
- if $\rho = 1$ the amplitude of the oscillation is constant.

Modal analysis, non-diagonalizable A

As for the continuous-time case, if A is non-diagonalizable, we may resort to the Jordan form

$$A = TJT^{-1} \quad \Longrightarrow \quad A^k = TJ^kT^{-1} \tag{19}$$

where

$$J = \mathrm{diag}\{J_1, J_2, \ldots, J_s\}$$

is the Jordan normal form of A.

Recalling the definition of the binomial coefficient $\binom{k}{i} = \frac{k!}{i!(k-i)!}$, the kth power of the block J_h can be written as

$$\begin{aligned} J_h^k &= (\lambda_h I + J_{h0})^k = \\ &= \lambda_h^k I + \binom{k}{1}\lambda_h^{k-1} J_{h0} + \binom{k}{2}\lambda_h^{k-2} J_{h0}^2 + \cdots + \binom{k}{k-1}\lambda_h^1 J_{h0}^{k-1} + J_{h0}^k = \\ &= \sum_{i=0}^{k} \binom{k}{i} \lambda_h^{k-i} J_{h0}^i . \end{aligned}$$

Modal analysis, non-diagonalizable A (cont.)

Recall that $J_{h0}^k = 0 \; \forall k \geq \nu_h$ (ν_h being the size of the block J_{h0}). Moreover, by definition, $\binom{k}{i} = 0$ if $k < i$. Then

$$J_h^k = \sum_{i=0}^{\nu_h - 1} \binom{k}{i} \lambda_h^{k-i} J_{h0}^i. \tag{20}$$

Observe that, for $k \geq i$, the binomial coefficient

$$\binom{k}{i} = \frac{k!}{i!(k-i)!} = \frac{k(k-1)(k-2)\ldots(k-i+1)}{i!} \doteq p_i(k),$$

is **a polynomial of degree i in the variable k**. Thus, similarly to the continuous-time case, we get:

$$A^k = \sum_{h=1}^{s} \sum_{i=0}^{\nu_h - 1} Z_{hi} p_i(k) \lambda_h^{k-i}, \tag{21}$$

where $Z_{hi} = T_h J_{h0}^i S_h^\top$.
If there exist Jordan blocks of size ≥ 1, associated to λ, in the matrix A^k, the following modes will appear:

$$\lambda^k, \; \binom{k}{1}\lambda^{k-1}, \; \binom{k}{2}\lambda^{k-2}, \ldots, \binom{k}{\nu - 1}\lambda^{k-\nu+1}$$

where $\nu = \nu(\lambda)$ is the degree of λ.

Modal analysis, non-diagonalizable A (cont.)

Example. Let the Jordan form of a matrix A be

$$\begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Then, the sole eigenvalue $\lambda = 3$ has a degree $\deg(\lambda) = 3$ and, as a consequence, the modes of A^k are:

$$3^k, \quad \binom{k}{1} 3^{k-1}, \quad \binom{k}{2} 3^{k-2}$$

Transfer function (continuous-time)

Transfer function

Consider the time-invariant dynamic system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

and let $x(0) = x_0$. By applying the Laplace transform to both sides of the first equation we get:

$$sX(s) - x_0 = AX(s) + BU(s) \quad \Longrightarrow \quad (sI - A)X(s) = x_0 + BU(s)$$

which implies

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s) \tag{22}$$

By substituting $X(s)$ in the Laplace transform of the output equation we get

$$Y(s) = C(sI - A)^{-1}x_0 + \left[C(sI - A)^{-1}B + D\right]U(s) \tag{23}$$

Letting $x_0 = 0$, it follows that:

$$Y(s) = \left[C(sI - A)^{-1}B + D\right]U(s) = W(s)U(s)$$

and $W(s)$ is called the *transfer function*.

Transfer function (cont.)

Let's analyze the structure of the transfer function:

$$W(s) = \begin{bmatrix} w_{11}(s) & \cdots & w_{1m}(s) \\ \vdots & & \vdots \\ w_{i1}(s) & \cdots & w_{im}(s) \\ \vdots & & \vdots \\ w_{p1}(s) & \cdots & w_{pm}(s) \end{bmatrix}$$

$W(s)$ is a $p \times m$ matrix. If $x_0 = 0$, the ith component of the output vector is given by:

$$Y_i(s) = \sum_{r=1}^{m} w_{ir}(s)U_r(s) = w_{i1}(s)U_1(s) + w_{i2}(s)U_2(s) + \cdots$$

Thus:

$$\begin{aligned} &x(0) = 0 \\ &u_r(t) = 0,\ r \neq j \end{aligned} \quad \Longrightarrow \quad w_{ij}(s) = \frac{Y_i(s)}{U_j(s)}$$

Transfer function (cont.)

In particular, if we take $u_j(t) = \delta(t)$, we have

$$U_j(s) = \mathcal{L}\left[u_j(t)\right] = \mathcal{L}\left[\delta(t)\right] = 1$$

hence

$$w_{ij}(s) = \frac{Y_i(s)}{U_j(s)} = Y_i(s)$$

In other words, $w_{ij}(s)$ is the Laplace transform of the ith component of the output response to the unit impulse applied to the jth input. Thus

$$w_{ij}(s) = \mathcal{L}\left[w_{ij}(t)\right]$$

where $w_{ij}(t)$ is the ijth element of the impulse response matrix $W(t)$.

Since the above holds for any pair i, j, it follows that

$$W(s) = \mathcal{L}\left[W(t)\right]$$

hence, the transfer function is the Laplace transform of the impulse response.

Transfer function (cont.)

Impulse response and transfer function

The impulse response and transfer function of the system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

are given, respectively, by

$$W(t) = \mathcal{L}^{-1}\left[C(sI - A)^{-1}B + D\right]$$

and

$$W(s) = C(sI - A)^{-1}B + D$$

Transfer function (cont.)

In the following, we show that the entry $w_{ij}(s)$ of a transfer function is a **proper rational function** (a rational function is a ratio of polynomials; it is *proper* if the degree of the numerator is less than or equal to the degree of the denominator; it is *strictly proper* if strict inequality holds).

Indeed:

$$W(s) = C\,(sI - A)^{-1}\,B + D$$

and

$$(sI - A)^{-1} = \begin{bmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s - a_{22} & & \vdots \\ \vdots & & \ddots & \\ -a_{n1} & \cdots & & s - a_{nn} \end{bmatrix}^{-1}$$

Transfer function (cont.)

The inverse can be expressed as:

$$(sI - A)^{-1} = \frac{1}{\det (sI - A)} K(s)$$

where $K(s)$ is the matrix of the algebraic complements (each of which is the determinant of an $(n-1) \times (n-1)$ minor of $sI - A$).

Clearly:

- $\varphi(s) \doteq \det (sI - A)$ is a polynomial of degree n (the characteristic polynomial of A)
- $K(s) = [k_{ij}(s)]\,,\; i, j = 1, \ldots, n$, where $k_{ij}(s)$ is a polynomial of degree $< n$, $\forall\, i, j$

As a consequence,

$$(sI - A)^{-1} = \frac{K(s)}{\varphi(s)}$$

is an $n \times n$ matrix of strictly proper rational functions.

Transfer function (cont.)

Therefore:

$$W(s) = C(sI - A)^{-1} B + D = C\frac{K(s)}{\varphi(s)}B + D = \frac{M(s)}{\varphi(s)} + D = \frac{N(s)}{\varphi(s)}$$

where the entries of $N(s)$ are polynomials of degree $\leq n$:

$$\deg(n_{ij}(s)) \leq n$$

The strict inequality holds if and only if the corresponding entry of D is zero, i.e. $d_{ij} = 0$.

In summary, $W(s)$ is strictly proper (all its entries are strictly proper) if and only if $D = 0$ (i.e. the system is strictly proper).

Given

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

let $\hat{x} \doteq T^{-1}x$, where $T \in \mathbb{R}^{n\times n}$ is a generic non-singular $n \times n$ matrix ($\det(T) \neq 0$). Then, an equivalent state-space description is given by:

$$\begin{cases} \dot{\hat{x}}(t) = T^{-1}\dot{x}(t) = T^{-1}AT\hat{x}(t) + T^{-1}Bu(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \\ y(t) = CT\hat{x}(t) + Du(t) = \hat{C}\hat{x}(t) + Du(t) \end{cases}$$

In other words:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \iff \begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \\ y(t) = \hat{C}\hat{x}(t) + Du(t) \end{cases}$$

$$
\begin{aligned}
\hat{W}(s) &= \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D} \\
&= (CT)\left(sI - T^{-1}AT\right)^{-1}(T^{-1}B) + D \\
&= CT\left(sT^{-1}T - T^{-1}AT\right)^{-1}T^{-1}B + D \\
&= CT\left[T^{-1}(sI - A)T\right]^{-1}T^{-1}B + D \\
&= CT\left[T^{-1}(sI - A)^{-1}T\right]T^{-1}B + D \\
&= C(sI - A)^{-1}B + D \\
&= W(s)
\end{aligned}
$$

Hence, the transfer function is invariant to change of basis.

Transfer function (discrete-time)

Transfer function

Consider the time-invariant dynamic system:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

and let $x(0) = x_0$. By applying the $\mathcal{Z}$-transform to both sides of the first equation we get:

$$z\,[X(z) - x_0] = AX(z) + BU(z) \quad \Longrightarrow \quad (zI - A)X(z) = z\,x_0 + BU(z)$$

which implies

$$X(z) = (zI - A)^{-1} z\,x_0 + (zI - A)^{-1} BU(z) \tag{24}$$

By substituting $X(z)$ in the $\mathcal{Z}$-transform of the output equation we get

$$Y(z) = C(zI - A)^{-1} z\,x_0 + \left[C(zI - A)^{-1} B + D\right] U(z) \tag{25}$$

Letting $x_0 = 0$, it follows that:

$$Y(z) = \left[C(zI - A)^{-1} B + D\right] U(z) = W(z)U(z)$$

and $W(z)$ is called the *transfer function*.

Transfer function (cont.)

Let's analyze the structure of the transfer function:

$$W(z) = \begin{bmatrix} w_{11}(z) & \cdots & w_{1m}(z) \\ \vdots & & \vdots \\ w_{i1}(z) & \cdots & w_{im}(z) \\ \vdots & & \vdots \\ w_{p1}(z) & \cdots & w_{pm}(z) \end{bmatrix}$$

$W(z)$ is a $p \times m$ matrix. If $x_0 = 0$, the ith component of the output vector is given by:

$$Y_i(z) = \sum_{r=1}^{m} w_{ir}(z)U_r(z) = w_{i1}(z)U_1(z) + w_{i2}(z)U_2(z) + \cdots$$

Thus:

$$\begin{aligned} &x(0) = 0 \\ &u_r(k) = 0, \ r \neq j \end{aligned} \quad \Longrightarrow \quad w_{ij}(z) = \frac{Y_i(z)}{U_j(z)}$$

In particular, if we take $u_j(k) = \delta(k)$, we have

$$U_j(z) = \mathcal{Z}\left[u_j(k)\right] = \mathcal{Z}\left[\delta(k)\right] = 1$$

hence

$$w_{ij}(z) = \frac{Y_i(z)}{U_j(z)} = Y_i(z)$$

In other words, $w_{ij}(z)$ is the $\mathcal{Z}$-transform of the ith component of the output response to the unit impulse applied to the jth input. Thus

$$w_{ij}(z) = \mathcal{Z}\left[w_{ij}(k)\right]$$

where $w_{ij}(k)$ is the ijth element of the impulse response matrix $W(k)$.

Since the above holds for any pair i, j, it follows that

$$W(z) = \mathcal{Z}\left[W(k)\right]$$

hence, the transfer function is the $\mathcal{Z}$-transform of the impulse response.

Transfer function (cont.)

Impulse response and transfer function

The impulse response and transfer function of the system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

are given, respectively, by

$$W(k) = \mathcal{Z}^{-1}\left[C(zI - A)^{-1}B + D\right]$$

and

$$W(z) = C(zI - A)^{-1}B + D$$

Transfer function (cont.)

In the following, we show that the entry $w_{ij}(z)$ of a transfer function is a proper rational function (a rational function is a ratio of polynomials; it is *proper* if the degree of the numerator is less than or equal to the degree of the denominator; it is *strictly proper* if strict inequality holds).

Indeed:

$$W(z) = C\,(zI - A)^{-1}\,B + D$$

and

$$(zI - A)^{-1} = \begin{bmatrix} z - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & z - a_{22} & & \vdots \\ \vdots & & \ddots & \\ -a_{n1} & \cdots & & z - a_{nn} \end{bmatrix}^{-1}$$

Transfer function (cont.)

The inverse can be expressed as:

$$(zI - A)^{-1} = \frac{1}{\det(zI - A)} K(z)$$

where $K(z)$ is the matrix of the algebraic complements (each of which is the determinant of an $(n-1) \times (n-1)$ minor of $zI - A$).

Clearly:

- $\varphi(z) \doteq \det(zI - A)$ is a polynomial of degree n (the characteristic polynomial of A)
- $K(z) = [k_{ij}(z)]$, $i, j = 1, \ldots, n$, where $k_{ij}(z)$ is a polynomial of degree $< n$, $\forall\, i, j$

As a consequence,

$$(zI - A)^{-1} = \frac{K(z)}{\varphi(z)}$$

is an $n \times n$ matrix of strictly proper rational functions.

Transfer function (cont.)

Therefore:

$$W(z) = C\,(zI - A)^{-1}\,B + D = C\,\frac{K(z)}{\varphi(z)}B + D = \frac{M(z)}{\varphi(z)} + D = \frac{N(z)}{\varphi(z)}$$

where the entries of $N(z)$ are polynomials of degree $\leq n$:

$$\deg\left(n_{ij}(z)\right) \leq n$$

The strict inequality holds if and only if the corresponding entry of D is zero, i.e. $d_{ij} = 0$.

In summary, $W(z)$ is strictly proper (all its entries are strictly proper) if and only if $D = 0$ (i.e. the system is strictly proper).

Transfer function of equivalent dynamic systems

Given

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

Let $\hat{x} := T^{-1}x$, where $T \in \mathbb{R}^{n \times n}$ is a generic non-singular $n \times n$ matrix ($\det(T) \neq 0$). Then, the equivalent state-space description is given by:

$$\begin{cases} \hat{x}(k+1) = T^{-1}x(k+1) = T^{-1}AT\hat{x}(k) + T^{-1}Bu(k) = \hat{A}\hat{x}(k) + \hat{B}u(k) \\ y(k) = CT\hat{x}(k) + Du(k) = \hat{C}\hat{x}(k) + Du(k) \end{cases}$$

In other words:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases} \iff \begin{cases} \hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}u(k) \\ y(k) = \hat{C}\hat{x}(k) + Du(k) \end{cases}$$

Transfer function of equivalent dynamic systems (cont.)

$$
\begin{aligned}
\hat{W}(z) &= \hat{C}(zI - \hat{A})^{-1}\hat{B} + \hat{D} \\
&= (CT)\left(zI - T^{-1}AT\right)^{-1}(T^{-1}B) + D \\
&= CT\left(zT^{-1}T - T^{-1}AT\right)^{-1}T^{-1}B + D \\
&= CT\left[T^{-1}(zI - A)T\right]^{-1}T^{-1}B + D \\
&= CT\left[T^{-1}(zI - A)^{-1}T\right]T^{-1}B + D \\
&= C(zI - A)^{-1}B + D \\
&= W(z)
\end{aligned}
$$

Hence, the transfer function is invariant to change of basis.

Example

Given the LTI discrete-time system, having two inputs and one output:

$$\begin{cases} x(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix} u(k) \\ \\ y(k) = [-3 \;\; 3]\, x(k) \end{cases}$$

the transfer function is a 1×2 matrix:

$$\begin{aligned} W(z) &= [-3 \;\; 3] \begin{bmatrix} z & -1 \\ 1 & z+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix} \\ &= [-3 \;\; 3] \frac{1}{(z+1)^2} \begin{bmatrix} z+2 & 1 \\ -1 & z \end{bmatrix} \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix} \\ &= \left[-\frac{3}{z+1} \;\; \frac{3(z-1)}{(z+1)^2} \right] \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix} = \left[\frac{3(z-1)}{(z+1)^2} \;\; \frac{3}{z+1} \right] \end{aligned}$$

References

References

Antsaklis, P. J. and Michel, A. N. (2006). *Linear Systems*. Springer Science & Business Media.

322MI –Spring 2023

Lecture 2
Solutions to linear systems

END