Control Theory

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Lecture 3: Stability

When dealing with stability in the context of dynamic systems we consider three different cases:

1. Stability of state movements
2. Stability of equilibrium states
3. Stability of linear systems

Remark. Concerning the first case, we provide definitions and concepts in the context of general abstract dynamic systems, thus the treatment is the same for both continuous-time and discrete-time systems.

Stability of state movements

Stability of state movements

Consider a general abstract dynamic system characterised by the state-transition function

$$\varphi(t, t_0, x_0, u(\cdot))$$

Then, consider a generic nominal state movement for a given initial state $\bar{x}_0$ and a given input function $\bar{u}(\cdot)$:

$$\bar{x}(\cdot) = \varphi(\cdot, t_0, \bar{x}_0, \bar{u}(\cdot))$$

Now, consider the perturbed state movement generated by a perturbation of the initial state and a perturbation of the input function:

$$\begin{aligned} x(t_0) &= \bar{x}_0 + \delta\bar{x} \\ u(\cdot) &= \bar{u}(\cdot) + \delta u(\cdot) \end{aligned} \implies \varphi(t, t_0, \bar{x}_0 + \delta\bar{x}, \bar{u}(\cdot) + \delta u(\cdot))$$

perturbed state movement

Stability with respect to perturbations of the initial state

Definition

The nominal state movement

$$\bar{x}(\cdot) = \varphi(\cdot, t_0, \bar{x}_0, \bar{u}(\cdot))$$

is *stable* with respect to perturbations of the initial state $\bar{x}_0$ if $\forall\, \varepsilon > 0\,,\, \forall t_0 > 0$ there exists $\delta(\varepsilon, t_0) > 0$ such that

$$\|\delta\bar{x}\| < \delta(\varepsilon, t_0) \quad \Longrightarrow \quad \|\varphi(t, t_0, \bar{x}_0 + \delta\bar{x}, \bar{u}(\cdot)) - \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))\| < \varepsilon\,, \forall\, t \geq t_0$$

Definition

The nominal state movement

$$\bar{x}(\cdot) = \varphi(\cdot, t_0, \bar{x}_0, \bar{u}(\cdot))$$

is *attractive* if $\forall t_0 > 0$ there exists $\eta(t_0) > 0$ such that

$$\lim_{t \to +\infty} \|\varphi(t, t_0, \bar{x}_0 + \delta\bar{x}, \bar{u}(\cdot)) - \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))\| = 0\,,\ \ \forall\, \|\delta\bar{x}\| < \eta(t_0)$$

Asymptotic stability with respect to perturbations of the initial state

Definition

The nominal state movement

$$\bar{x}(\cdot) = \varphi(\cdot, t_0, \bar{x}_0, \bar{u}(\cdot))$$

is *asymptotically stable* with respect to perturbations of the initial state $\bar{x}_0$ if:

- it is stable;
- it is attractive.

Instability with respect to perturbations of the initial state

Definition

The nominal state movement

$$\bar{x}(\cdot) = \varphi(\cdot, t_0, \bar{x}_0, \bar{u}(\cdot))$$

is *unstable* with respect to perturbations of the initial state $\bar{x}_0$ if it is not stable with respect to such a kind of perturbations.

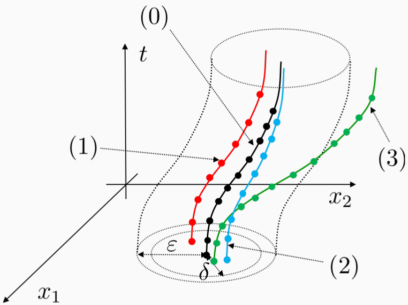
Geometrical interpretation

(0) nominal state movement

(1) perturbed state movement remaining confined in the "tube" of radius ε

(2) perturbed state movement remaining confined in the "tube" of radius ε and asymptotically converging to the nominal movement

(3) perturbed state movement crossing the "tube" of radius ε



Stability with respect to perturbations of the input function

Definition

The nominal state movement

$$\bar{x}(\cdot) = \varphi(\cdot, t_0, \bar{x}_0, \bar{u}(\cdot))$$

is *stable* with respect to perturbations of the input function $\bar{u}(\cdot)$ if $\forall\, \varepsilon > 0$, $\forall t_0 > 0$, there exist $\delta(\varepsilon, t_0) > 0$ such that

$$\|\delta\bar{u}(\cdot)\| < \delta(\varepsilon, t_0) \quad \Longrightarrow \quad \|\varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot) + \delta\bar{u}(\cdot)) - \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))\| < \varepsilon\,, \forall\, t \geq t_0$$

where the norm in $\|\delta\bar{u}(\cdot)\|$ is any function norm.

Instability with respect to perturbations of the input function

Definition

The nominal state movement

$$\bar{x}(\cdot) = \varphi(\cdot, t_0, \bar{x}_0, \bar{u}(\cdot))$$

is *unstable* with respect to perturbations of the input function $\bar{u}(\cdot)$ if it is not stable with respect to such a kind of perturbations.

Stability of equilibrium

Stability of equilibrium

- An equilibrium is a movement which is constant with respect to time.
- The stability of equilibrium is the stability of such constant movement.
- Thus, all the definitions related to the stability of movements apply to the equilibrium case.

Stability of equilibrium (cont.)

Definition

The equilibrium state $\bar{x}$ corresponding to the input $\bar{u}(\cdot)$ is said to be *stable* with respect to a state perturbation if $\forall \epsilon > 0, \forall t_0 > 0$ there exists $\delta(\epsilon, t_0) > 0$ such that

$$\|\delta\bar{x}\| < \delta(\epsilon, t_0) \quad \Longrightarrow \quad \|\varphi(t, t_0, \bar{x} + \delta\bar{x}, \bar{u}(\cdot)) - \bar{x}\| < \epsilon, \ \forall t \geq t_0$$

Definition

The equilibrium state $\bar{x}$ corresponding to the input $\bar{u}(\cdot)$ is said to be *asymptotically stable* with respect to state perturbations if

- it is stable;
- it is attractive, i.e. $\forall t_0 > 0$ there exist $\eta(t_0) > 0$ such that

$$\lim_{t \to +\infty} \|\varphi(t, t_0, \bar{x}_0 + \delta\bar{x}, \bar{u}(\cdot)) - \bar{x}\| = 0, \ \forall \|\delta\bar{x}\| < \eta$$

Uniform stability

In the previous definitions δ may depend on the initial time:

$$\delta = \delta(\epsilon, t_0)$$

If δ is not dependent on the initial time

$$\delta = \delta(\epsilon),$$

the stability is said to be *uniform*.

The concept of uniform asymptotically stability follows accordingly.

Instability and divergence of trajectories

- It is important to notice that **instability does not imply divergent trajectories**.
- Systems do exist that admit unstable equilibrium states but do not have divergent trajectories.
- The Van der Pol oscillator is a significant example.

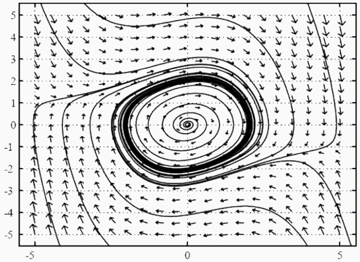
The Van der Pol oscillator is described by a second order non-linear equation corresponding to an RLC circuit in which the resistive element has a nonlinear characteristic:

$$\frac{x(t)}{L} + \dot{x}(t)(3x^2(t) - \alpha) + \ddot{x}(t)C = 0,$$

where $\alpha > 0$. A state-space representation is:

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\dfrac{x_1(t)}{LC} - \dfrac{x_2(t)}{C}(3x_1^2(t) - \alpha) \end{cases}$$

The system admits the origin as unique equilibrium state and such equilibrium state is unstable. However, no trajectory diverges. Indeed all the trajectories, for any initial state, converge to a curve called *limit cycle*.



Stability and convergence of trajectories

- Conversely, **the convergence of all trajectories of a system to an equilibrium state doesn't imply the stability of that state**.
- There exist systems whose trajectories are all convergent to one (ore more) unstable equilibrium states. For example:

$$x(k+1) = \begin{cases} 2x(k) & \text{if } \|x(k)\| < 1 \\ 0 & \text{otherwise} \end{cases}$$

- The origin is an equilibrium state (the sole). Furthermore all the trajectories converge to the origin. However the origin is unstable (why?).

Stability of regular systems of finite dimension

Given the system

$$\begin{cases} \dot{x}(t) & = & f(x(t), u(t), t) \\ y(t) & = & g(x(t), u(t), t) \end{cases}$$

we want to study the stability, w.r.t. the initial state perturbation, of the movement $\bar{x}(\cdot) = \varphi(\cdot, t_0, \bar{x}_0, \bar{u}(\cdot))$.

The difference between the perturbed and the nominal movement $\bar{x}(\cdot)$ is

$$z(t) \doteq x(t) - \bar{x}(t)$$

From which:

$$\begin{aligned} \dot{z}(t) = \dot{x}(t) - \dot{\bar{x}}(t) \quad & = \quad f(z(t) + \bar{x}(t), \bar{u}(t), t) - f(\bar{x}(t), \bar{u}(t), t) \\ & \doteq \quad w(z(t), \bar{x}(t), \bar{u}(t), t) \end{aligned}$$

Where the function w enjoys the property:

$$w(0, \bar{x}(t), \bar{u}(t), t) = 0 \; \forall t \geq t_0$$

From movement stability to equilibrium stability

The dynamic system $\dot{z}(t) = w(z(t), \bar{x}(t), \bar{u}(t), t)$ describes the dynamic of perturbed movements w.r.t. a nominal movement.

For a given movement, i.e., for given $\bar{x}(t)$ and $\bar{u}(t)$, the function w depends on z and t.

The property

$$w(0, \bar{x}(t), \bar{u}(t), t) = 0, \quad \forall t \geq t_0$$

shows that the system admits the equilibrium state (i.e. the constant movement):

$$z(t) = 0, \quad \forall t \geq t_0$$

Therefore, the stability analysis of a movement has been cast as the stability analysis of an equilibrium of a suitable dynamic system (the one describing the perturbed movement).

As a consequence, it is not restrictive to treat the equilibrium stability only.

The Lyapunov method is a fundamental tool for system analysis and control.

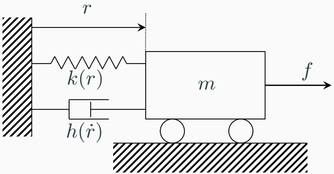
It originates from the following observation in physics:

If the total energy of a system (for example, a mechanical system) is continuously dissipated, then such a system must necessarily approach an equilibrium state.

The Lyapunov method generalizes the above observation by employing a suitable positive scalar function of the state, which acts as an "energy".

Example

Consider a **nonlinear** mechanical system



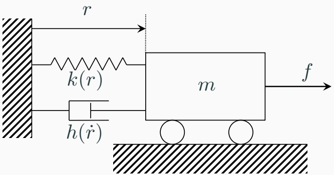
$$f = 0$$
$$k(r) = k_0 r + k_1 r^3$$
$$h(\dot r) = b\dot r|\dot r|$$
$$k_0, k_1, b > 0$$
$$m\ddot r + b\dot r|\dot r| + k_0 r + k_1 r^3 = 0$$

Letting $x_1 = r$ and $x_2 = \dot r$, we get the state equations:

$$\begin{cases} \dot x_1(t) = x_2(t) \\ \dot x_2(t) = -\dfrac{k_0}{m} x_1(t) - \dfrac{k_1}{m} x_1^3(t) - \dfrac{b}{m} x_2(t)|x_2(t)| \end{cases}$$

It is immediate to check that $\bar x = [0 \;\; 0]^\top$ is an equilibrium state.

Example (cont.)



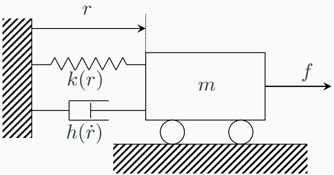
$$f = 0$$
$$k(r) = k_0 r + k_1 r^3$$
$$h(\dot{r}) = b\dot{r}|\dot{r}|$$
$$k_0, k_1, b > 0$$
$$m\ddot{r} + b\dot{r}|\dot{r}| + k_0 r + k_1 r^3 = 0$$

The total mechanical energy is the sum of the kinetic energy and the elastic potential energy:

$$V(x_1, x_2) = \frac{1}{2} m x_2^2 + \int_0^{x_1} k(\xi) d\xi = \frac{1}{2} m x_2^2 + \frac{1}{2} k_0 x_1^2 + \frac{1}{4} k_1 x_1^4 \quad (1)$$

Clearly, the function $V(x_1, x_2)$ is a positive scalar function having the state as argument. It is zero if and only if $x_1 = 0$, $x_2 = 0$.

Example (cont.)



$f = 0$

$k(r) = k_0 r + k_1 r^3$

$h(\dot{r}) = b\dot{r}|\dot{r}|$

$k_0, k_1, b > 0$

$m\ddot{r} + b\dot{r}|\dot{r}| + k_0 r + k_1 r^3 = 0$

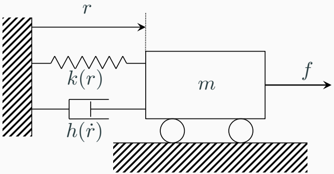
How does the total mechanical energy change in time?

By deriving (1) with respect to time and taking into account the state equation, we get:

$$\dot{V}(x_1, x_2) = \frac{dV(x_1, x_2)}{dt} = m x_2 \dot{x}_2 + k_0 x_1 \dot{x}_1 + k_1 x_1^3 \dot{x}_1 = -b|x_2|^3$$

Clearly:

- The function $\dot{V}(x_1, x_2)$ is not an explicit function of time but only of the state. Hence, for a given state $x = [x_1 \; x_2]^\top$ the rate of variation of $V(x_1, x_2)$ is fixed.
- The mechanical energy is continuously dissipated, except when $x_2 = 0$, i.e., when the cart velocity is zero.



$$f = 0$$
$$k(r) = k_0 r + k_1 r^3$$
$$h(\dot r) = b\dot r|\dot r|$$
$$k_0, k_1, b > 0$$
$$m\ddot r + b\dot r|\dot r| + k_0 r + k_1 r^3 = 0$$

Question:

Is it possible to exploit the condition

$$\dot V(x_1, x_2) \leq 0$$

to draw conclusions about the stability of the equilibrium state $\bar x = [0 \;\; 0]^\top$?

Answer:

Yes, it is! $\longrightarrow$ Lyapunov Stability Theory

The direct Lyapunov method consists of

- associating to the system state, a suitable positive scalar function that plays the role of “energy”;
- (trying to) prove that such function is decreasing along the trajectories of the system.

Preliminary notions:

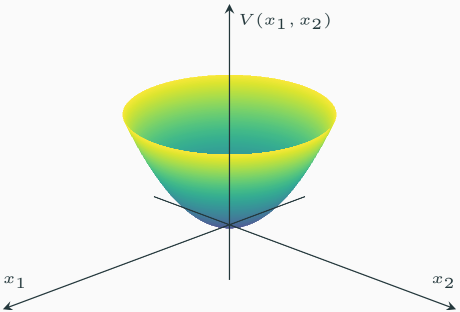
- positive (semi)definite and negative (semi)definite functions
- computation of $\dot{V}(x)$

Positive definite functions

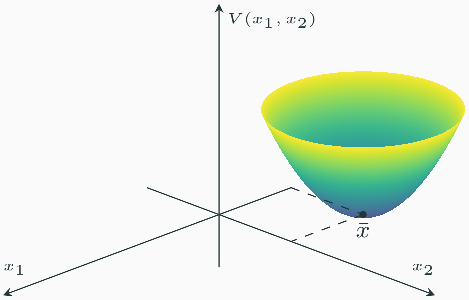
Definition

A function $V(\cdot):\ \mathbb{R}^n \to \mathbb{R}$ is said to be *positive definite* in $\bar{x}$ if:

- $V(\bar{x}) = 0$
- $\exists\, \xi > 0 :\ V(x) > 0\,, \quad \forall x :\ \|x - \bar{x}\| < \xi,\ x \neq \bar{x}$



Positive definite in 0



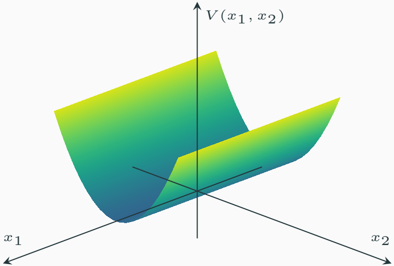
Positive definite in $\bar{x}$

Positive semi-definite functions

Definition

A function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is said to be *positive semi-definite* in $\bar{x}$ if:

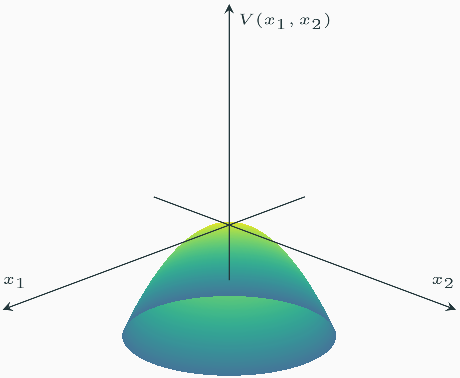
- $V(\bar{x}) = 0$
- $\exists\, \xi > 0 : V(x) \geq 0\,, \quad \forall\, x \,:\, \|x - \bar{x}\| < \xi,\, x \neq \bar{x}$



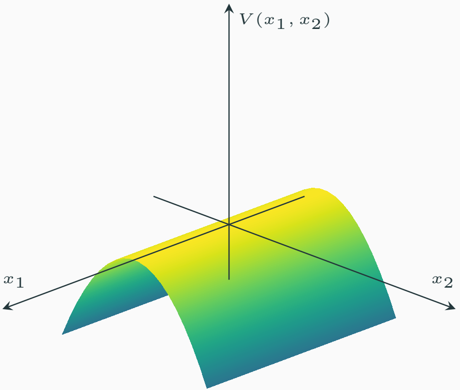
Negative definite and semi-definite functions

Definition

A function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is said to be *negative definite* (*negative semi-definite*) in $\bar{x}$ if $-V(\cdot)$ is positive definite (positive semi-definite).



Negative definite in 0



Negative semi-definite in 0

Quadratic functions

Particularly useful for the applications are the quadratic functions of the form:

$$V(x) = x^\top A x = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

where A is a symmetric matrix.

Definition

The matrix A is said to be positive definite if the quadratic form $V(x) = x^\top A x$ is positive definite in the origin.

Analogous definitions can be given for A positive semi-definite, negative definite, and so on.

If A is not symmetric, it can be easily shown that only its "symmetric part" contributes to the quadratic form $V(x) = x^\top A x$. In that case, matrix A can be replaced by its symmetric part A^S whose entries are given by:

$$a_{ij}^S = \frac{a_{ij} + a_{ji}}{2}$$

Criteria for checking the definiteness of a matrix

Recall that all eigenvalues of a symmetric matrix are real. The following results are useful.

Lemma

A symmetric matrix A is positive (negative) definite if and only if all its eigenvalues are strictly positive (negative):

$$\lambda_i \underset{(<)}{>} 0\,, \quad i = 1, \ldots, n$$

where λ_i denotes the i-th eigenvalue of A.

Lemma

A symmetric matrix A is positive (negative) semi-definite if and only if all its eigenvalues are non-negative (non-positive):

$$\lambda_i \underset{(\leq)}{\geq} 0\,, \quad i = 1, \ldots, n$$

Observe that, for non-symmetric matrices, the eigenvalue check must be carried out on the symmetric part.

Computation of $\dot{V}(x)$

Given a nonlinear system

$$\dot{x}(t) = f(x(t))$$

and a scalar function of the state

$$V(\cdot) : \mathbb{R}^n \to \mathbb{R},$$

how does V vary along the system trajectories?

$$V(x(t)) : \mathbb{R} \to \mathbb{R}^n \to \mathbb{R}$$

$$t \to x(t) \to V(x(t))$$

Using the rule for the derivative of compound functions (chain rule) we get:

$$\dot{V}(x) = \frac{dV(x(t))}{dt} = \nabla V(x)\dot{x} = \nabla V(x) f(x)$$

where the gradient $\nabla V(x)$ is a row vector:

$$\nabla V(x) = \left[\frac{\partial V(x)}{\partial x_1} \dots \frac{\partial V(x)}{\partial x_n} \right].$$

Notice that $\dot{V}(x)$ is a function of the state only (as a consequence, there's no need to solve the differential equations, to compute it).

Lyapunov Theorem

Theorem

Given the system $\dot{x}(t) = f(x(t))$, $x \in \mathbb{R}^n$, having $\bar{x}$ as an equilibrium state.

Given a function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ which is positive definite in $\bar{x}$ and of class $\mathcal{C}^1$ (i.e. $V(\cdot)$ is continuous along with its partial derivatives).

Then, the following implications hold:

$\dot{V}(x)$ negative semi-definite in $\bar{x} \implies \bar{x}$ is a stable equilibrium state

$\dot{V}(x)$ negative definite in $\bar{x} \implies \bar{x}$ is an asymptotically stable equilibrium state

$\dot{V}(x)$ positive definite in $\bar{x} \implies \bar{x}$ is an unstable equilibrium state

Before showing the proof, some remarks are in order.

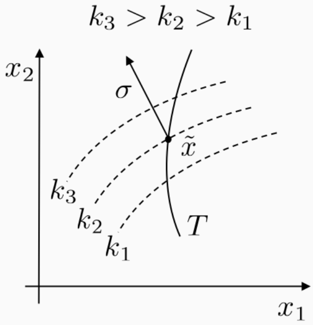
Remarks

1. The Lyapunov Theorem is of key importance since it allows to analyze the stability of equilibrium states (and hence of arbitrary nominal state movements) without the need to determine the explicit solutions of the state equations
2. For the above reason, the Lyapunov Theorem is also called the *Direct Lyapunov Method*
3. The Lyapunov Theorem provides **sufficient** conditions for the stability of equilibrium states. In other words: if a positive definite function $V(\cdot)$ does not satisfy some of the conditions on $\dot{V}(\cdot)$, no conclusion can be drawn about the stability of the equilibrium state.

Intuition

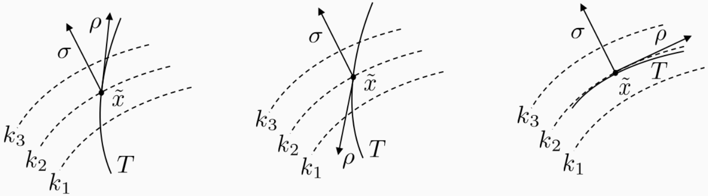
Consider a second-order nonlinear system:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$



- Denote by σ the vector orthogonal to the level curve $k_2 = V(\tilde{x})$ evaluated on the state $\tilde{x}$ belonging to the trajectory T
- Given $\tilde{x}$, the value of $\dot{V}(\tilde{x})$ is uniquely determined
- The knowledge of $\dot{V}(\tilde{x})$ allows to establish whether the state is evolving towards increasing or decreasing values of $V(\cdot)$

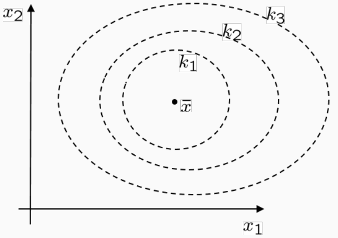
Denoting by ρ the tangent vector to the state trajectory, the following three cases may occur:



$\dot{V}(\tilde{x}) > 0$ $\qquad$ $\dot{V}(\tilde{x}) < 0$ $\qquad$ $\dot{V}(\tilde{x}) = 0$

Intuition

Since $V(x)$ is continuous and positive definite, the level curves are closed, at least in a neighborhood of $\bar{x}$, and contain the state $\bar{x}$



The compact set delimited by a level curve k is called k *sublevel set*. The sublevel sets allow to understand the meaning of the conditions stated by the Lyapunov Theorem.

For example, it is clear that the condition

$$\dot{V}(x) \text{ negative definite}$$

implies that the system state can only evolve entering in a lower sublevel set and then it converges to the equilibrium state.

Proof

We first prove the implication:

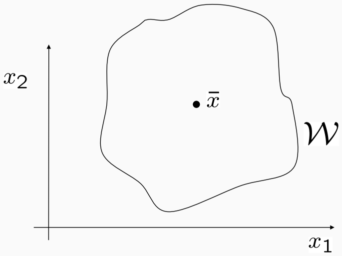
$$\dot{V}(x) \text{ negative semi-definite in } \bar{x} \implies \bar{x} \text{ is stable.}$$

Observe that:

- $V(x)$ positive definite in $\bar{x}$
- $\dot{V}(x)$ negative semi-definite in $\bar{x}$

imply that there exists a neighborhood $\mathcal{W}$ of $\bar{x}$ in which

$$V(x) > 0, \dot{V}(x) \leq 0 \quad \forall x \neq \bar{x}$$



Proof (cont.)

To prove the stability of $\bar{x}$ it is necessary to show that:

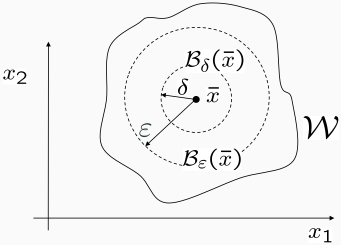
$$\forall \epsilon > 0 \; \exists \delta(\epsilon) > 0 : \|x(0) - \bar{x}\| < \delta \quad \Longrightarrow \quad \|x(t) - \bar{x}\| < \epsilon, \forall t \geq 0$$

Let $\mathcal{B}_\epsilon(\bar{x})$ be a ball of radius ϵ centered in $\bar{x}$. It is not restrictive to suppose that $\mathcal{B}_\epsilon(\bar{x})$ is contained in $\mathcal{W}$: indeed, if δ guarantees the previous condition for $\epsilon : \mathcal{B}_\epsilon(\bar{x}) \subset \mathcal{W}$, the same δ will guarantee the condition for any $\bar{\epsilon} > \epsilon$.

Let

$$m = \min_{x \in \partial \mathcal{B}_\epsilon(\bar{x})} V(x)$$

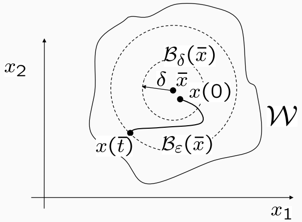
which exists because $V(x)$ is continuous and the set is closed and bounded.



Choose δ such that $\max_{x \in \mathcal{B}_\delta(\bar{x})} V(x) \doteq M < m$ (this is possible because $V(x)$ is continuous and null in $\bar{x}$).

Then, take $x(0) \in \mathcal{B}_\delta(\bar{x})$ and suppose that the subsequent trajectory goes out from the sphere $\mathcal{B}_\epsilon(\bar{x})$. Let $\bar{t}$ be the first time instant in which the trajectory intersects the boundary $\mathcal{B}_\epsilon(\bar{x})$

$$\bar{t} = \min\{t \geq 0 : x(t) \in \partial\mathcal{B}_\epsilon(\bar{x})\}$$

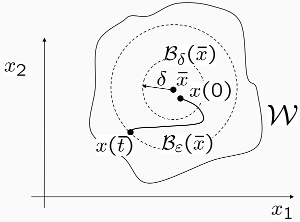


For the particular choice of $x(0)$ and $\bar{t}$:

$$V(x(0)) \leq M < m \leq V(x(\bar{t}))$$

from which

$$V(x(0)) < V(x(\bar{t})).$$



That result contradicts the hypothesis of $\dot{V}$ being negative semi-definite. Indeed V can't grow up along system trajectories:

$$V(x(\bar{t})) - V(x(0)) = \int_0^{\bar{t}} \underbrace{\dot{V}(x(t))}_{\leq 0 \quad \forall x \in \mathcal{W}} \, dt \leq 0$$

Therefore, the trajectory never leaves the sphere $\mathcal{B}_\epsilon(\bar{x})$. □

The proofs of the remaining implications are similar.

Consider the first order system

$$\dot{x} = (1 - x)^5$$

whose sole equilibrium state is $\bar{x} = 1$. Take the Lyapunov function

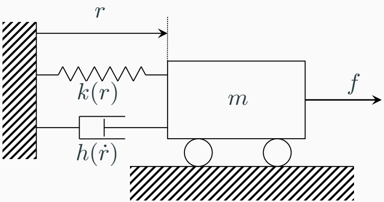
$$V(x) = (1 - x)^2$$

that is continuous and positive definite in $\bar{x} = 1$. We get

$$\dot{V}(x) = \frac{dV(x)}{dx} f(x) = -2(1 - x)(1 - x)^5 = -2(1 - x)^6$$

that is negative definite in $\bar{x}$ because $\dot{V}(\bar{x}) = 0$ and $\dot{V}(x) < 0 \; \forall x \neq \bar{x}$.

Thus, the equilibrium state is asymptotically stable.



$$k(r) = k_0 r + k_1 r^3$$
$$h(\dot r) = b\dot r|\dot r|$$

Consider the total energy of the nonlinear system shown in figure:

$$V(x_1, x_2) = \frac{1}{2} m x_2^2 + \frac{1}{2} k_0 x_1^2 + \frac{1}{4} k_1 x_1^4$$

which belongs to $\mathcal{C}^1$ and is positive definite in the origin. Its time-derivative is

$$\dot V(x_1, x_2) = m x_2 \dot x_2 + k_0 x_1 \dot x_1 + k_1 x_1^3 \dot x_1 = -b|x_2|^3$$

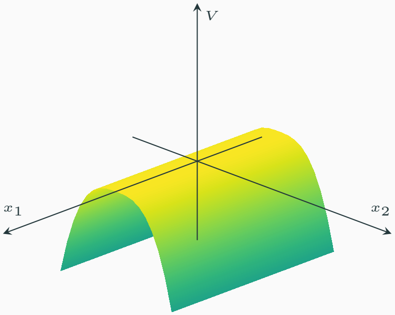
which is negative semi-definite in the origin.

Examples (cont.)

$$\dot{V}(x_1, x_2) = m x_2 \dot{x}_2 \; + \; k_0 x_1 \dot{x}_1 \; + \; k_1 x_1^3 \dot{x}_1 = -b|x_2|^3$$

Indeed, it is equal to zero as long as $x_2 = 0$, i.e.

$$\dot{V}(x_1, 0) = 0, \quad \forall x_1$$



From the Lyapunov theorem, we can conclude that the equilibrium state is stable.

Counterexample (Van der Pol oscillator)

The state equation of the Van der Pol oscillator is

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{x_1}{LC} - \frac{x_2}{C}(3x_1^2 - \alpha)\end{aligned}$$

where $L, C, \alpha > 0$. The origin is the unique equilibrium state:

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Take the function

$$V(x_1, x_2) = x_1^2 + LCx_2^2$$

which belongs to $\mathcal{C}^1$ and is positive definite in the origin. Thus, we get

$$\dot{V}(x) = [2x_1 \;\; 2LCx_2] \begin{bmatrix} x_2 \\ -\frac{x_1}{LC} - \frac{x_2}{C}(3x_1^2 - \alpha) \end{bmatrix} = -2Lx_2^2(3x_1^2 - \alpha)$$

that is positive semi-definite (it is equal to zero if $x_2 = 0$ and it is closely positive for x_1 sufficiently small). Therefore, no conclusions can be drawn based on the Lyapunov theorem.

The following result is useful to conclude asymptotic stability even when $\dot{V}$ is negative semi-definite.

Theorem

Given the nonlinear system $\dot{x} = f(x)$ having $\bar{x}$ as an equilibrium state.

Given a function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ which is positive definite in $\bar{x}$ and of class $\mathcal{C}^1$ and such that $\dot{V}(x)$ is negative semi-definite in $\bar{x}$.

Let $\mathcal{W}$ be a neighborhood of $\bar{x}$ such that $V(x) > 0, \;\; \dot{V}(x) \leq 0 \;\; \forall x \in \mathcal{W} \setminus \bar{x}$.

Let

$$\mathcal{N} = \left\{ x \in \mathcal{W} \setminus \bar{x} : \dot{V}(x) = 0 \right\}.$$

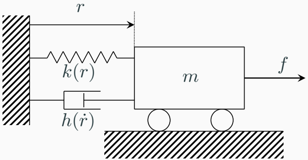
If there are no trajectories entirely contained in $\mathcal{N}$, i.e.

$$\nexists \tilde{x} \in \mathcal{N} : x(0) = \tilde{x} \Longrightarrow x(t) \in \mathcal{N} \;\; \forall t \geq 0$$

then $\bar{x}$ is asymptotically stable.

Intuitively, the conditions of the Krasowskii criterion prevent the state to be stucked in the region where no dissipation occurs.

Let consider again the mechanical system



$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k_0}{m}x_1 - \frac{k_1}{m}x_1^3 - \frac{b}{m}x_2|x_2| \end{cases}$$

By employing the total energy as Lyapunov function, we previously obtained a negative semi-definite

$$\dot{V}(x_1, x_2) = -b|x_2|^3.$$

Clearly, $\mathcal{N} = \{[x_1 \ \ 0]^\top : x_1 \neq 0\}$ as $\mathcal{W}$ can be taken arbitrarily large. For initial states belonging to $\mathcal{N}$ we have

$$\begin{cases} \dot{x}_1(0) = 0 \\ \dot{x}_2(0) = -\frac{k_0}{m}x_1(0) - \frac{k_1}{m}x_1^3(0) \neq 0 \end{cases}$$

Thus, after an arbitrarily small time interval, the state will leave the set $\mathcal{N}$. As a consequence, by the Krasowskii criterion, the origin is asymptotically stable.

For discrete-time systems, we can state the following result.

Theorem

Given the nonlinear system $x(k+1) = f(x(k))$, $x \in \mathbb{R}^n$, having the equilibrium state $\bar{x}$.

Given a continuous function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ which is positive definite in $\bar{x}$.

Let

$$\Delta V(x) \doteq V(f(x)) - V(x)$$

Then, the following implications hold:

$\Delta V(x)$ negative semi-definite in $\bar{x} \implies \bar{x}$ is a stable equilibrium state

$\Delta V(x)$ negative definite in $\bar{x} \implies \bar{x}$ is an asymptotically stable equilibrium state

$\Delta V(x)$ positive definite in $\bar{x} \implies \bar{x}$ is an unstable equilibrium state

Examples

Consider the second-order nonlinear system:

$$\begin{cases} x_1(k+1) = \dfrac{x_2(k)}{1+x_2^2(k)} \\ x_2(k+1) = \dfrac{x_1(k)}{1+x_2^2(k)} \end{cases}$$

Clearly, $\bar{x} = [0\ 0]^\top$ is an equilibrium state.

The function $V(x_1, x_2) = x_1^2 + x_2^2$ is continuous and positive definite.

It follows that:

$$\begin{aligned} \Delta V(x) = V(f(x)) - V(x) &= \left(\frac{x_2}{1+x_2^2}\right)^2 + \left(\frac{x_1}{1+x_2^2}\right)^2 - x_1^2 - x_2^2 \\ &= \frac{-2x_2^2 - x_2^4}{(1+x_2^2)^2}(x_1^2 + x_2^2) \end{aligned}$$

which is negative semi-definite. Therefore, the equilibrium state $\bar{x} = [0\ 0]^\top$ is stable.

Examples (cont.)

Consider the same second-order nonlinear system as before:

$$\begin{cases} x_1(k+1) = \dfrac{x_2(k)}{1+x_2^2(k)} \\ x_2(k+1) = \dfrac{x_1(k)}{1+x_2^2(k)} \end{cases}$$

but let us choose a different candidate Lyapunov function:

$$V(x) = (x_1^2 + x_2^2)\left(1 + \frac{1}{(1+x_2^2)^2}\right)$$

After some algebra, we obtain:

$$\Delta V(x) = \underbrace{\frac{x_1^2 + x_2^2}{\left[(1+x_2^2)^2 + x_1^2\right]^2}}_{=0 \text{ for } x=[0\ 0]^\top \text{ and } >0 \text{ elsewhere}} \quad \underbrace{\left\{(1+x_2^2)^2 - \left[(1+x_2^2)^2 + x_1^2\right]^2\right\}}_{=0 \text{ for } x=[0\ 0]^\top \text{ and } <0 \text{ elsewhere}}$$

Thus, the equilibrium state $\bar{x} = [0\ 0]^\top$ is asymptotically stable.

Stability of linear continuous-time systems

Stability of linear systems

Consider a generic regular linear system of finite dimension:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases}$$

We want to study the stability of the generic movement

$$\bar{x}(t) = \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))$$

with respect to perturbations of the initial state $\bar{x}_0 = x(t_0)$.

By defining the difference between the perturbed and the unperturbed movement (nominal movement):

$$z(t) \doteq x(t) - \bar{x}(t)$$

we get, by linearity:

$$\begin{aligned} \dot{z}(t) &= A(t)(z(t) + \bar{x}(t)) + B(t)\bar{u}(t) - A(t)\bar{x}(t) - B(t)\bar{u}(t) \\ &= A(t)z(t) \end{aligned}$$

Stability of linear systems (cont.)

The difference between the perturbed and the unperturbed movement evolves according to:

$$\dot{z}(t) = A(t)z(t)$$

and for this reason it **doesn't depend on the initial state** (but only on the perturbation $z(t_0)$ of the initial state).

Property

If a system is linear **all** the movements (and therefore, in particular, the constant movements: the equilibria) share the same stability properties. In other words, for a linear system, the stability is a **global** property. We can thus refer to the stability as **a property of the system**.

Stability of linear continuous-time invariant systems

Stability and eigenvalues

The following theorem establishes a relationship between the stability of a LTI system and the eigenvalues of the state matrix.

Theorem

Given the LTI system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

let $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ be the spectrum of A.

(a) The system is asymptotically stable if and only if for any $\lambda \in \sigma(A)$, we have $\mathrm{Re}(\lambda) < 0$;

(b) The system is stable if and only if for any $\lambda \in \sigma(A)$, we have $\mathrm{Re}(\lambda) \leq 0$ and, moreover, if $\mathrm{Re}(\lambda) = 0$ then $\deg(\lambda) = 1$.

Stability and eigenvalues (cont.)

Before proceeding with proof, we recall the definition and some properties of matrix norms.

Definition

Given $A^{m\times n}$ and a vector norm $||\cdot|| : \mathbb{R}^m \to \mathbb{R}$, the following is the *induced matrix norm*:

$$||A|| = \max_{x\neq 0} \frac{||Ax||}{||x||}$$

Properties

Induced matrix norms are *submultiplicative*, i.e.

$$||MN|| \leq ||M|| ||N||.$$

Moreover, if M is square of size $n \times n$, then

$$||M|| \leq n \max_{i,j} |m_{ij}|.$$

Proof. We first prove the implication ($\Leftarrow$) of statement (b). The norm of the state is

$$\|x(t)\| = \|e^{At}x(0)\| = \|Te^{Jt}T^{-1}x(0)\| \le \|T\|\|e^{Jt}\|\|T^{-1}\|\|x(0)\|,$$

where $J = T^{-1}AT$ is the Jordan form of A. From the lecture on the solution to LTI systems we know that

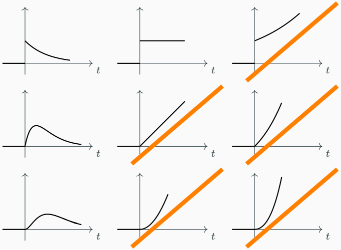
$$e^{Jt} = \begin{bmatrix} e^{J_1 t} & 0 & \dots & 0 \\ 0 & e^{J_2 t} & & \\ \vdots & & \ddots & \\ 0 & \dots & \dots & e^{J_s t} \end{bmatrix}, \qquad \text{where}$$

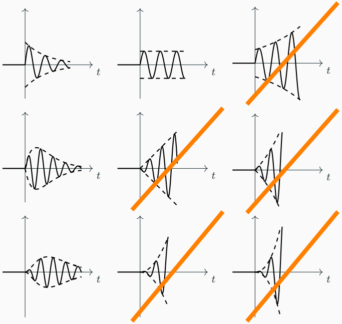
$$e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \frac{t^2}{2!}e^{\lambda_i t} & \frac{t^3}{3!}e^{\lambda_i t} & \dots & \frac{t^{\nu_i-1}}{(\nu_i-1)!}e^{\lambda_i t} \\ 0 & e^{\lambda_i t} & te^{\lambda_i t} & \frac{t^2}{2!}e^{\lambda_i t} & \dots & \frac{t^{\nu_i-2}}{(\nu_i-2)!}e^{\lambda_i t} \\ 0 & 0 & e^{\lambda_i t} & te^{\lambda_i t} & \dots & \frac{t^{\nu_i-3}}{(\nu_i-3)!}e^{\lambda_i t} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & te^{\lambda_i t} \\ 0 & 0 & 0 & 0 & \dots & e^{\lambda_i t} \end{bmatrix}. \qquad (2)$$

Stability and eigenvalues (cont.)

The condition on the eigenvalues required by statement (b) implies that the matrix e^{Jt} is bounded:

$$\exists k > 0 : [e^{Jt}]_{ij} \leq k, \ \forall t \geq 0, \ \forall i, j$$





Indeed, with reference to the figures above:

- if $\mathrm{Re}(\lambda) < 0$, the modes $e^{\lambda t}, te^{\lambda t}, \ldots$ are bounded (left columns of the figures);
- if $\mathrm{Re}(\lambda) = 0$, the unbounded modes $te^{\lambda t}, t^2 e^{\lambda t}, \ldots$ are excluded (central columns of the figures), because $\deg(\lambda) = 1$ (hence there are no Jordan blocks of size greater than one associated to λ);
- the case $\mathrm{Re}(\lambda) > 0$ (right columns) is excluded.

It follows that $\forall t \geq 0$:

$$\|x(t)\| \leq \|T\|\|e^{Jt}\|\|T^{-1}\|\|x(0)\| \leq nk\|T\|\|T^{-1}\|\|x(0)\|. \tag{3}$$

Given an arbitrary $\epsilon > 0$, if we take δ as

$$\delta = \frac{\epsilon}{nk\|T\|\|T^{-1}\|}$$

it follows that any trajectory starting in $\mathcal{B}_\delta(0)$ is included in $\mathcal{B}_\epsilon(0)$.

Indeed, inequality (3) guarantees that $\|x(t)\| \leq \frac{\epsilon}{\delta}\|x(0)\|$, so

$$\|x(0)\| \leq \delta \quad \Longrightarrow \quad \|x(t)\| \leq \epsilon, \ \forall t \geq 0.$$

As a consequence, the origin (and therefore the system) is stable.

Stability and eigenvalues (cont.)

(⇒) We now show that whenever the condition on the eigenvalues is violated, the system is unstable. If $\mathrm{Re}(\lambda) > 0$ or $\mathrm{Re}(\lambda) = 0, \deg(\lambda) > 1$, there is at least one unbounded entry of e^{Jt}, say $\left[e^{Jt}\right]_{ij}$. Take

$$x(0) = T\alpha = T\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_n \end{bmatrix} = T\begin{bmatrix} 0 \\ \vdots \\ \alpha_j \\ \vdots \\ 0 \end{bmatrix}$$

as initial condition. Thus, we get:

$$x(t) = Te^{Jt}T^{-1}x(0) = Te^{Jt}T^{-1}T\alpha = T\alpha_j \overbrace{\begin{bmatrix} \vdots \\ \left[e^{Jt}\right]_{ij} \\ \vdots \end{bmatrix}}^{j\text{th column of } e^{Jt}}$$

which is unbounded $\forall \alpha_j \neq 0$. Therefore the origin (and thus, the system) is not stable.

Stability and eigenvalues (cont.)

As for the statement (a), which concerns the asymptotic stability, the proof is as follows.

($\Leftarrow$)
We need to show that, if $\mathrm{Re}(\lambda_i) < 0 \; \forall i$, then the origin is both stable and attractive. The stability is guaranteed by the statement (b). The origin is attractive because all the modes converge to zero.

($\Rightarrow$)
Suppose that a non-convergent mode exists (necessarily due to an eigenvalue having non strictly-negative real part). In the same way as before, we can find an initial state that generates a non-convergent natural movement. □

Stability and eigenvalues (cont.)

Corollary

Given the LTI system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

let $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ be the spectrum of A.
The system is unstable if and only if there exist at least one $\lambda \in \sigma(A)$ such that $\mathrm{Re}(\lambda) > 0$ or $\mathrm{Re}(\lambda) = 0$ and $\deg(\lambda) > 1$.

Lyapunov Theorem for linear continuous-time systems

Theorem

The system

$$\dot{x}(t) = Ax(t) \tag{4}$$

is asymptotically stable if and only if for every symmetric positive-definite matrix Q, there exists a unique symmetric and positive-definite matrix P, that solves the following *Lyapunov equation*:

$$A^\top P + PA = -Q. \tag{5}$$

Proof.
$(\Leftarrow)$

Given the symmetric and positive definite matrices P, Q that satisfy the Lyapunov equation, let

$$V(x) = x^\top P x.$$

By taking the derivative w.r.t. time we get:

$$\dot{V}(x) = \dot{x}^\top P x + x^\top P \dot{x} = (Ax)^\top P x + x^\top P A x = x^\top (A^\top P + PA) x = -x^\top Q x,$$

where $-x^\top Q x < 0, \ \forall x \neq 0.$

Therefore, $\dot{V}(x)$ is negative-definite and by the Lyapunov theorem the origin is an asymptotically stable equilibrium state. Consequently, the system is asymptotically stable.

$(\Rightarrow)$

Let the system (4) be asymptotically stable and Q be symmetric and positive definite. We need to show that a symmetric and positive definite matrix P exists that satisfies the Lyapunov equation. Let

$$P = \int_0^{+\infty} e^{A^\top t} Q e^{At} dt.$$

Such a matrix is well-defined because, being the system asymptotically stable, all the entries of $e^{A^\top t} Q e^{At}$ converge exponentially to zero making the improper integral well-defined.

Moreover, P is indeed a solution of the Lyapunov equation. By substitution:

$$A^\top P + PA = A^\top \int_0^{+\infty} e^{A^\top t} Q e^{At} dt + \int_0^{+\infty} e^{A^\top t} Q e^{At} dt A =$$

$$= \int_0^{+\infty} \frac{d}{dt}(e^{A^\top t} Q e^{At}) dt = \left[e^{A^\top t} Q e^{At} \right]_0^{\infty} = -Q$$

where the second equality follows from the fact that $\frac{d}{dt} e^{At} = Ae^{At} = e^{At}A$.

Furthermore, a P of the form

$$P = \int_0^{+\infty} e^{A^\top t} Q e^{At} dt$$

- is symmetric, being the integral of symmetric matrices:

$$(e^{A^\top t} Q e^{At})^\top = e^{A^\top t} Q e^{At};$$

- is positive definite, being the integral of positive definite matrices:

$$x^\top (e^{A^\top t} Q e^{At}) x = (e^{At} x)^\top Q (e^{At} x) > 0, \forall x \neq 0$$

Indeed, Q is positive definite by hypothesis and e^{At} (that has as eigenvalues $e^{\lambda_i t}$) has trivial kernel:

$$\ker[e^{At}] = \{0\}, \forall t.$$

It remains to show that the solution is unique. Suppose, by contradiction, that there exists another solution $\bar{P}$ to (9), i.e.

$$A^\top P + PA = -Q, \qquad \text{and} \qquad A^\top \bar{P} + \bar{P}A = -Q.$$

Then, by subtracting

$$A^\top (P - \bar{P}) + (P - \bar{P})A = 0.$$

and by multiplying on the left and right by $e^{A^\top t}$ and e^{At} respectively, we get

$$\underbrace{e^{A^\top t} A^\top (P - \bar{P}) e^{At} + e^{A^\top t} (P - \bar{P}) A e^{At}}_{\frac{d}{dt}\left(e^{A^\top t}(P - \bar{P})e^{At}\right)} = 0, \qquad \forall t \geq 0.$$

Therefore $e^{A^\top t}(P - \bar{P})e^{At}$ is constant for all times. On the other hand it must converge to zero as $t \to \infty$ because of stability, hence it must be always zero. Then, since e^{At} is nonsingular, this implies $P = \bar{P}$. $\square$

Observations

1. The Lyapunov equation

$$A^\top P + PA = -Q$$

is linear in P. For given A and Q, finding P amounts to solving a linear system of equations, whose unknowns are the entries of P. Actually, only the entries above (or below) the main diagonal of P have to be considered (including the diagonal), because P is symmetric. The total number of unknowns and equations is then

$$n + (n-1) + (n-2) + \cdots + 1 = \frac{n(n+1)}{2}.$$

2. The theorem suggests a stability test for linear systems: choose a symmetric positive definite Q and solve the Lyapunov equation for symmetric P. If a solution exists such as P is positive definite, then the system is asymptotically stable.
3. The proof of the theorem shows that, if the test is successful, then the function

$$V(x) = x^\top P x$$

is a Lyapunov function for the system. Thus, the theorem provides a **method for finding a Lyapunov function** (for linear systems).

Example

Consider the state matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

which yields an asymptotically stable system. By taking $Q = I$, the Lyapunov equation is

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} + \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

thus

$$\begin{bmatrix} -2\beta & \alpha - \beta - \gamma \\ \alpha - \beta - \gamma & 2(\beta - \gamma) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

which, by equating the corresponding entries, yields $\beta = 1/2$, $\gamma = 1$ and $\alpha = 3/2$. Thus, the resulting matrix P is

$$P = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1 \end{bmatrix}.$$

Kronecker product and vector operator

Definition

Given the matrices $A^{m\times n}$ and $B^{p\times q}$, the Kronecker product is the $mp \times nq$ matrix:

$$A \otimes B = \begin{bmatrix} A_{11}B & \dots & A_{1n}B \\ \vdots & & \vdots \\ A_{m1}B & \dots & A_{mn}B \end{bmatrix}$$

The product is defined for any pair of matrices, is non commutative, and enjoys the following properties:

1. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$;
2. $(A \otimes B)^\top = A^\top \otimes B^\top$;
3. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;
4. For square A and B: if λ_A is an eigenvalue of A and λ_B is an eigenvalue of B, then $\lambda_A \lambda_B$ is an eigenvalue of $A \otimes B$;
5. $\mathrm{rank}(A \otimes B) = \mathrm{rank}(A)\,\mathrm{rank}(B)$.

To state another property we must introduce the *vector operator*.

Definition

Given a matrix $X = [x_1 \; x_2 \; \dots x_n]$, the vector operator vec is the operator that stacks the columns one underneath the other to a single vector:

$$\mathrm{vec}\,(X) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The following useful identity holds:

$$\mathrm{vec}\,(AXB) = \left(B^\top \otimes A\right) \mathrm{vec}\,(X)\,. \tag{6}$$

The above identity is useful for "pulling out" the unknown X from a matrix equation.

Kronecker product and vector operator (cont.)

Example: consider the following linear system of equation in the unknown X

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By taking the vec of both sides and applying the previous identity we get:

$$[b_1 \quad b_2] \otimes \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

thus we obtain an easy to manage equation:

$$\begin{bmatrix} b_1 a_{11} & b_1 a_{12} & b_2 a_{11} & b_2 a_{12} \\ b_1 a_{21} & b_1 a_{22} & b_2 a_{21} & b_2 a_{22} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Kronecker product and vector operator (cont.)

Consider the Lyapunov equation:

$$A^\top P + PA = -Q.$$

By applying the vec operator to both sides, and the identity (6), we get

$$\left(I \otimes A^\top + A^\top \otimes I\right) \mathrm{vec}\,(P) = -\,\mathrm{vec}\,(Q)\,.$$

Thus, provided that the matrix on the left side is invertible, we have

$$\mathrm{vec}\,(P) = -\left(I \otimes A^\top + A^\top \otimes I\right)^{-1} \mathrm{vec}\,(Q)\,.$$

Note

In Matlab, the vec operator can be computed with `b=B(:)` and the inverse operation can be computed with `B=reshape(b,n,length(b)/n)`. The Kronecker product is implemented in `kron(A,B)`.

Stability of linear discrete-time systems

Stability of linear discrete-time systems

Consider the general discrete-time linear dynamic system

$$x(k+1) = A(k)x(k) + B(k)u(k),$$

and define a nominal state movement

$$\bar{x}(k) = \varphi(k, k_0, \bar{x}_0, \{\bar{u}(k_0), \ldots, \bar{u}(k-1)\})$$

starting from the initial state $\bar{x}(k_0) = \bar{x}_0$.

We analyze the stability of the nominal movement $\bar{x}(k)$ with respect to perturbations of the initial state $\bar{x}_0$, that is, we consider the perturbed state movement

$$x(k) = \varphi(k, k_0, x_0, \{\bar{u}(k_0), \ldots, \bar{u}(k-1)\})$$

starting from the perturbed initial state $x_0 \neq \bar{x}_0$.

Hence, by defining the difference between the perturbed and the nominal state movement as follows:

$$z(k) \doteq x(k) - \bar{x}(k)\,,$$

we get:

$$\begin{aligned} z(k+1) &= x(k+1) - \bar{x}(k+1) \\ &= A(k)[z(k) + \bar{x}(k)] + B(k)\bar{u}(k) - A(k)\bar{x}(k) - B(k)\bar{u}(k) \\ &= A(k)z(k) \end{aligned}$$

Thus, the dynamics of $z(k)$ is described by the system

$$z(k+1) = A(k)z(k). \tag{7}$$

Stability of linear discrete-time systems (cont.)

Therefore, we can draw the following conclusions, analogously to the continuous-time case

- for linear systems, the dynamics of the difference between the perturbed and the nominal state movement $z(k) = x(k) - \bar{x}(k)$ satisfies:
$$z(k+1) = A(k)z(k)\,;$$
- the dynamics of $z(k)$ does not depend on the specific initial state $\bar{x}_0$, but on the initial state perturbation $z(k_0) = x(k) - \bar{x}_0$;
- all state movements have the same stability properties or, in other terms, stability is not a property of a specific nominal state movement but, instead, is a global property of the system.

Stability of linear discrete-time invariant systems

The following theorem is analogous to that of the continuous-time case, and can be proven similarly, based on the Jordan normal form of A.
However, it is easily understood recalling that the modes associated to the eigenvalue λ have the form:

$$p_i(k)\lambda^{k-i} = p_i(k)|\lambda|^{k-i}\left(\cos(\theta(k-i)) + j\sin(\theta(k-i))\right), \qquad i = 0\ldots \deg(\lambda)-1,$$

where $p_i(k)$ is a polynomial of degree i in the variable k.

Theorem

Given the LTI system

$$x(k+1) = Ax(k) + Bu(k)$$

let $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ be the spectrum of A.

(a) The system is asymptotically stable if and only if for any $\lambda \in \sigma(A)$, we have $|\lambda| < 1$;

(b) The system is stable if and only if for any $\lambda \in \sigma(A)$, we have $|\lambda| \leq 1$ and, if $|\lambda| = 1$, then $\deg(\lambda) = 1$.

Corollary

Given the LTI system

$$x(k+1) = Ax(k) + Bu(k)$$

let $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ be the spectrum of A.
The system is unstable if and only if there exist at least one $\lambda \in \sigma(A)$ such that $|\lambda| > 1$ or $|\lambda| = 1$ and $\deg(\lambda) > 1$.

Theorem

The system

$$x(k+1) = Ax(k) \tag{8}$$

is asymptotically stable if and only if for every symmetric positive-definite matrix Q, there exists a unique symmetric and positive-definite matrix P, that solves the following *discrete-time Lyapunov equation*:

$$A^\top PA - P = -Q. \tag{9}$$

Proof.
$(\Leftarrow)$
Take two symmetric positive definite matrices $P, Q \in \mathbb{R}^{n \times n}$ that satisfy the Lyapunov equation

$$A^\top PA - P = -Q.$$

To show that the system $x(k+1) = Ax(k)$ is asymptotically stable, let

$$V(x) = x^\top Px$$

thus obtaining:

$$\begin{aligned} \Delta V(x) &= (Ax)^\top PAx - x^\top Px = x^\top A^\top PAx - x^\top Px \\ &= x^\top (A^\top PA - P)x = -x^\top Qx < 0\,,\ \forall x \neq 0 \end{aligned}$$

Hence, $\Delta V(x)$ is negative definite, which implies that the origin is an asymptotically stable equilibrium state of the system.

($\Rightarrow$)

Suppose that the system $x(k+1) = Ax(k)$ is asymptotically stable. Given a symmetric positive definite matrix Q, consider the Lyapunov equation:

$$A^\top PA - P = -Q.$$

By multiplying both members to the left by $(A^\top)^i$, and to the right by A^i, we get

$$(A^\top)^{i+1}PA^{i+1} - (A^\top)^i PA^i = -(A^\top)^i QA^i.$$

Then, summing from $i = 0$ to l we get

$$A^\top PA - P + (A^\top)^2PA^2 - A^\top PA + \cdots = -\sum_{i=0}^{l}(A^\top)^i QA^i$$

or

$$(A^\top)^{l+1}PA^{l+1} - P = -\sum_{i=0}^{l}(A^\top)^i QA^i.$$

Letting $l \longrightarrow \infty$, the first term on the left vanishes and we get

$$P = \sum_{i=0}^{+\infty}(A^\top)^i QA^i. \tag{10}$$

Notice that the series on the right side converges due to the asymptotic stability assumption, and that P is symmetric positive definite, being the sum of a symmetric positive definite matrix (Q) and symmetric positive semi-definite matrices ($(A^\top)^i Q A^i, \quad i \geq 1$).

Thus, we have proven that, if the system is asymptotically stable, a matrix P satisfying the Lyapunov equation must necessarily satisfy the condition (10), which defines P uniquely. In other words, we have proven its uniqueness.

It can be easily shown, by substitution, that such a P is indeed a solution of the Lyapunov equation:

$$\begin{aligned} A^\top \left(\sum_{i=0}^{+\infty} \left(A^\top\right)^i Q A^i \right) A - \sum_{i=0}^{+\infty} \left(A^\top\right)^i Q A^i \\ = \sum_{i=0}^{+\infty} \left(A^\top\right)^{i+1} Q A^{i+1} - \sum_{i=0}^{+\infty} \left(A^\top\right)^i Q A^i \\ = \left[\left(A^\top\right) Q A + \left(A^\top\right)^2 Q A^2 + \dots \right] - \left[Q + \left(A^\top\right) Q A + \dots \right] = -Q \end{aligned}$$

which concludes the proof. □

Observations

1. The Lyapunov equation
$$A^\top PA - P = -Q$$
is linear in P. For given A and Q, finding P amounts to solving a linear system of equations, whose unknowns are the entries of P. Actually, only the entries above (or below) the main diagonal of P have to be considered (including the diagonal), because P is symmetric. The total number of unknowns and equations is then
$$n + (n-1) + (n-2) + \cdots + 1 = \frac{n(n+1)}{2}.$$
2. The theorem suggests a stability test for linear systems: choose a symmetric positive definite Q and solve the Lyapunov equation for symmetric P. If a solution exists such as P is positive definite, then the system is asymptotically stable.
3. The proof of the theorem shows that, if the test is successful, then the function
$$V(x) = x^\top Px$$
is a Lyapunov function for the system. Thus, the theorem provides a **method for finding a Lyapunov function** (for linear systems).

Stability analysis via linearization

Stability analysis via linearization

In many cases, the stability of equilibria of nonlinear systems can be studied resorting to a suitable linearized system.

Consider the system:

$$\dot{x}(t) = f(x(t))$$

and suppose that $f(\bar{x}) = 0$, i.e. $\bar{x}$ is an equilibrium state.

Then, under regularity assumptions on f, we can define a linearized system:

$$\dot{\delta x}(t) = A\delta x(t),$$

where $\delta x = x - \bar{x}$ and

$$A = f_x(\bar{x}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}_{\bar{x}}.$$

Indirect Lyapunov method

Theorem

Given the system:

$$\dot{x}(t) = f(x(t))$$

having $\bar{x}$ as an equilibrium state, let $A = f_x(\bar{x})$. The following implications hold.

(1)	All the eigenvalues of A have strictly negative real part	$\Longrightarrow$	$\bar{x}$ is an **asymptotically stable** equilibrium state
(2)	At least one eigenvalue of A has strictly positive real part	$\Longrightarrow$	$\bar{x}$ is an **unstable** equilibrium state

Indirect Lyapunov method (cont.)

Proof.
We only prove statement (1). First, we recall some properties that will be employed:

- As a consequence of the definition of matrix norm, we have

$$\|Mx\| \leq \|M\|\|x\|$$

  and

$$\|MNx\| \leq \|M\|\|Nx\| \leq \|M\|\|N\|\|x\|$$

- by the Schwartz inequality we have

$$|x^\top y| \leq \|x\|\|y\|.$$

With no loss of generality we take $\bar{x} = 0$, thus $\delta x = x$ (otherwise, a coordinate change can be applied). If A has all the eigenvalues with strictly negative real part, then a matrix P exists such that

$$A^\top P + PA = -I.$$

Thus

$$V(x) = x^\top Px$$

is a Lyapunov function for the linearized system. We now prove that the same V is a Lyapunov function for (the equilibrium state of) the nonlinear system as well.

Let

$$f(x) = Ax + h(x),$$

where $h(x)$ is the residual of the Taylor series expansion.

$$\begin{aligned}\dot{V}(x) = \nabla V(x) f(x) = 2x^\top P(Ax + h(x)) = 2x^\top PAx + 2x^\top Ph(x) = \\ = x^\top (PA + A^\top P)x + 2x^\top Ph(x) = -x^\top x + 2x^\top Ph(x) = \\ = -\|x\|^2 + 2x^\top Ph(x)\end{aligned}$$

Recall that $h(x)$ is an infinitesimal of a higher order than x, thus, for any $\epsilon > 0$:

$$\exists \delta > 0 : \frac{\|h(x)\|}{\|x\|} < \epsilon, \forall \|x\| < \delta.$$

In particular, for $\epsilon = \dfrac{1}{2\|P\|}$:

$$\exists \delta > 0 : \|h(x)\| < \frac{\|x\|}{2\|P\|}, \forall \|x\| < \delta.$$

Using the Schwartz inequality

$$|x^\top y| \leq \|x\| \, \|y\|,$$

we get, $\forall \|x\| < \delta$:

$$\begin{aligned} |2x^\top Ph(x)| &\leq \|x\| \|2Ph(x)\| \\ &\leq 2\|x\| \|P\| \|h(x)\| \\ &< \|x\|^2 \end{aligned}$$

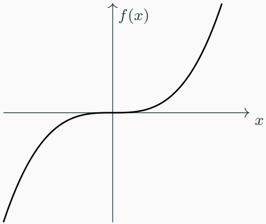
We have thus proven that $\dot{V}(x) < 0, \forall \|x\| < \delta$. Therefore, by the Lyapunov theorem, the equilibrium state is asymptotically stable. □

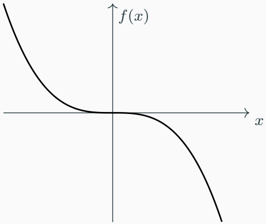
Remark

The Lyapunov indirect method doesn't cover all the possible cases.

Indeed, if the linearized system has all the eigenvalues with real part less or equal to zero and some eigenvalues with null real part, no conclusions can be drawn about the stability of the equilibrium based on the sole linearized system.

In these cases the linearized system doesn't capture the dynamics of the nonlinear one because that dynamics is related to the higher order terms of the Taylor series of $f(x)$.





For example, in the cases shown above, the linearized system around the origin is the same (having a null eigenvalue in both cases).

Examples

When the Lyapunov indirect method does not allow to draw conclusions about the equilibrium stability, it is still possible to employ the direct method. For example, consider the system:

$$\begin{cases} \dot{x}_1 = -x_1^3 + x_2 \\ \dot{x}_2 = -x_1^3 - x_2^3 \end{cases}$$

The origin is an equilibrium state and $A = f_x(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

The indirect method does not allow to draw conclusions. However, by taking the candidate Lyapunov function

$$V(x) = x_1^4 + 2x_2^2$$

we get

$$\dot{V}(x) = \begin{bmatrix} 4x_1^3 & 4x_2 \end{bmatrix} \begin{bmatrix} -x_1^3 + x_2 \\ x_1^3 - x_2^3 \end{bmatrix} = -4x_1^6 - 4x_2^4$$

that is negative definite. Therefore the origin is an asymptotically stable equilibrium state.

Examples (cont.)

As a further example, consider the system

$$\begin{cases} \dot{x}_1 = x_1^3 + x_2 \\ \dot{x}_2 = -x_1^3 - x_2^3 \end{cases}$$

The origin is an equilibrium state and $A = f_x(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Again, the indirect method does not provide information about the stability. However, by taking the candidate Lyapunov function

$$V(x) = x_1^4 + 2x_2^2$$

we get

$$\dot{V}(x) = \begin{bmatrix} 4x_1^3 & 4x_2 \end{bmatrix} \begin{bmatrix} x_1^3 + x_2 \\ -x_1^3 + x_2^3 \end{bmatrix} = 4x_1^6 + 4x_2^4$$

that is positive definite. Therefore the origin is an unstable equilibrium state.

For discrete-time systems, an analogous result can be proven.

Theorem

Given the system:

$$x(k+1) = f(x(k))$$

having $\bar{x}$ as an equilibrium state, let $A = f_x(\bar{x})$. The following implications hold.

(1) All the eigenvalues of A have a module strictly less than one $\implies$ $\bar{x}$ is an **asymptotically stable** equilibrium state

(2) At least one eigenvalue of A has module strictly greater than one $\implies$ $\bar{x}$ is an **unstable** equilibrium state

Input-output stability

The stability discussed so far (*internal stability* or *Lyapunov stability*) is concerned with the effect of initial conditions on the response of the system.

We now consider a different notion of stability: the *input-output stability* ignores the initial conditions and is concerned on **the effect of the input on the forced response**. Although the concept is general, we will deal with LTI systems only.

Definition

The system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

is said to be *BIBO stable* (Bounded Input Bounded Output) if there exists $M > 0$ such that, for every input $u(\cdot)$:

$$\|u(t)\| \leq 1 \ \forall t \geq 0 \quad \Longrightarrow \quad \|y_F(t)\| \leq M \ \forall t \geq 0.$$

where y_F is the forced response, i.e. the response for $x(0) = 0$.

Time-domain conditions for BIBO stability

Theorem

The system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

is BIBO stable if and only if:

$$\int_0^\infty \|W(t)\| dt < \infty. \tag{11}$$

where $W(t) = Ce^{At}B + D\delta(t)$ is the impulse response matrix and $\|\cdot\|$ is any induced matrix norm.

Before proving the theorem, we observe that condition (11) is equivalent to

$$\exists L > 0 : \quad \int_0^t \|W(\tau)\| d\tau \leq L \quad \forall t \geq 0. \tag{12}$$

Indeed, the function $\int_0^t \|W(\tau)\| d\tau$ is positive and monotonically increasing with respect to t. In order to admit a finite limit for $t \to \infty$, it must be bounded from above. Conversely, if it is bounded from above, being monotonically increasing, it must admit a finite limit.

Proof.

($\Leftarrow$)

Assuming that $\|u(t)\| \leq 1 \; \forall t \geq 0$, we have, for null initial conditions:

$$\begin{aligned}\|y(t)\| = \left\| \int_0^t W(t-\tau)u(\tau)d\tau \right\| &\leq \\ \leq \int_0^t \|W(t-\tau)u(\tau)\| d\tau &\leq \quad \text{(by definition of induced norm)} \\ \leq \int_0^t \|W(t-\tau)\| \underbrace{\|u(\tau)\|}_{\leq 1} d\tau &\leq \int_0^t \|W(t-\tau)\| d\tau \leq L, \quad \forall t \geq 0\end{aligned}$$

thus the system is BIBO stable.

($\Rightarrow$)

Let the system be BIBO stable. Assume by contradiction that (12) (and, equivalently, (11)) does not hold. We recall that for any induced matrix norm it holds[1] that

$$\|M\| \leq \sum_{i,j} |m_{ij}|.$$

[1]see for instance Section 1.10.A of Antsaklis and Michel (2006)

Thus, we can write

$$\int_0^t \|W(\tau)\| d\tau \leq \int_0^t \sum_{i,j} |w_{ij}(\tau)| d\tau = \sum_{i,j} \int_0^t |w_{ij}(\tau)| d\tau$$

and observe that, for $\int_0^t \|W(\tau)\| d\tau$ to be unbounded, there must exist i, j such that

$$\int_0^t |w_{ij}(\tau)| d\tau \tag{13}$$

is unbounded.

Pick an arbitrary time T and consider the following input, whose nonzero entry is that of index j

$$\tilde{u}(t) = \begin{bmatrix} 0 \\ \vdots \\ \operatorname{sign}\left(w_{ij}(T-t)\right) \\ \vdots \\ 0 \end{bmatrix}, \quad \forall t \geq 0.$$

For that input, the ith component of the forced response, at time T, is

$$y_i(T) = \int_0^T w_{ij}(T-\tau)\underbrace{\mathrm{sign}\left(w_{ij}(T-\tau)\right)}_{\tilde{u}_j(\tau)} d\tau = \int_0^T |w_{ij}(T-\tau)| d\tau$$

Since (13) is unbounded, by a proper choice of T, we can make $|y_i(T)|$ (and thus $\|y(T)\|$) arbitrarily large by means of inputs such that $\|\tilde{u}(t)\| \leq 1 \; \forall t \geq 0$. Thus the system is not BIBO stable. □

It can be easily shown that **condition (11) is equivalent to the absolute integrability of each entry of the impulse response matrix** $W(t)$, thus the following corollary holds true.

Corollary

The system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

is BIBO stable if and only if:

$$\int_0^{\infty} |w_{ij}(t)| dt < \infty \qquad \forall i, j, \tag{14}$$

where $w_{ij}(t) = [W(t)]_{ij}$ is the ijth entry of the impulse response matrix.

Frequency-domain conditions for BIBO stability

The following result provides a condition for BIBO stability based on the poles of transfer function.

Theorem

The system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{15}$$

is BIBO stable if and only if all the poles of all the entries of the rational matrix $W(s) = C(sI - A)^{-1}B + D$ have strictly negative real part.

Proof.
($\Leftarrow$)
If all the poles of $W(s)$ have strictly negative real part, then each entry of the impulse response matrix $W(t) = Ce^{At}B + D\delta(t)$ is the summation of exponential terms converging to zero plus possibly an impulse at time $t = 0$. Thus, the integral $\int_0^\infty \|W(t)\|dt$ is finite and therefore the system is BIBO stable.

($\Rightarrow$)
Conversely, if $\int_0^\infty \|W(t)\|dt$ is finite, the exponential terms that compose the entries of $W(t)$ must converge to zero. Thus, the poles of $W(s)$ must have strictly negative real part.

□

Frequency-domain conditions for BIBO stability (cont.)

Since the poles of $W(s)$ are eigenvalues of A, the following corollary holds.

Corollary

For continuous-time LTI systems:

$$\text{Asymptotic stability} \quad \Longrightarrow \quad \text{BIBO stability.}$$

It easy to recognize that the the converse is not true, in general. Indeed, not all the eigenvalues of A appear as poles of $W(s)$.

Example:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \end{aligned}$$

we have $\sigma(A) = \{-1, 1\}$, but the transfer function is

$$W(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B = \frac{1}{s+1}.$$

Definition

The system:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

is said to be *BIBO stable* (Bounded Input Bounded Output) if there exists $M > 0$ such that, for every input $u(\cdot)$:

$$\|u(k)\| \leq 1 \ \forall k \geq 0 \quad \Longrightarrow \quad \|y_F(k)\| \leq M \ \forall k \geq 0.$$

where y_F is the forced response, i.e. the response for $x(0) = 0$.

Analogously to the continuous-time case, the following necessary and sufficient condition for BIBO stability of discrete-time systems can be stated, based on the impulse response matrix $W(k)$.

Theorem

The system:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

is BIBO stable if and only if the following **equivalent** conditions hold:

1. $\sum_{k=0}^{\infty} \|W(k)\| < \infty$
2. $\sum_{k=0}^{\infty} |w_{ij}(k)| < \infty \qquad \forall i, j$

where $\|\cdot\|$ is any induced matrix norm and $w_{ij}(k) = [W(k)]_{ij}$ is the ijth entry of the impulse response matrix.

Frequency-domain conditions for BIBO stability

As for the frequency-domain conditions, we have the following result.

Theorem

The system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

is BIBO stable if and only if all the poles of all the entries of the transfer matrix $W(z) = C(zI - A)^{-1}B + D$ have module strictly less than one.

Corollary

For discrete-time LTI systems:

$$\text{Asymptotic stability} \quad \Longrightarrow \quad \text{BIBO stability.}$$

References

References

Antsaklis, P. J. and Michel, A. N. (2006). *Linear Systems*. Springer Science & Business Media.

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Lecture 3
Stability

END