Control Theory

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Lecture 5: Realization

Realization

Realization

The *realization* problem, in general, is the problem of associating a state-space representation to a given input-output description.

In the following, we consider the case of linear time-invariant systems, whose input-output behavior is described by a transfer function matrix.

Definition

Given a transfer function matrix $W(s)$ we say that a linear time-invariant state-space system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}$$

is a *realization of* $W(s)$ if

$$C(sI - A)^{-1}B + D = W(s).$$

Transfer functions

A first question is, What type of transfer functions can be realized by LTI state-space systems?

For a linear time-invariant system $\Sigma(A, B, C, D)$, where the matrices are respectively of size $n \times n$, $n \times m$, $p \times n$ and $p \times m$, the transfer function matrix takes the form

$$W(s) = C(sI - A)^{-1}B + D = \frac{N(s)}{d(s)} = \frac{N_0 + N_1 s + N_2 s^2 + \cdots + N_\nu s^\nu}{d_0 + d_1 s + d_2 s^2 + \cdots + d_\nu s^\nu},$$

where $\nu \leq n$ and N_k are matrices of dimension $p \times m$. Notice that the entries of $W(s)$,

$$[w_{ij}(s)] = \left[\frac{n_{ij}(s)}{d(s)}\right],$$

are proper rational functions.

Transfer functions (cont.)

Theorem

A transfer function $W(s)$ can be realized by an LTI state-space system if and only if all its entries are proper rational functions.

Proof.

The fact that LTI systems can realize only proper rational functions only has already been proven. The converse will be proven constructively in the following. □

Time delay

The time delay of τ time units has transfer function

$$W(s) = e^{-\tau s},$$

which is not a rational function, thus it cannot be realized by a system $\Sigma(A, B, C, D)$.

PID controllers

The PID controllers given by transfer functions of the type

$$W(s) = K_D s + K_P + \frac{K_I}{s} = \frac{K_D s^2 + K_P s + K_I}{s}$$

do not admit a realization $\Sigma(A, B, C, D)$ because the transfer function is not proper. A realization is possible by introducing an additional pole, obtaining

$$W(s) = \frac{K_D s^2 + K_P s + K_I}{s(1 + \tau s)}.$$

A realization procedure

It is easy to recognize that if the realization problem admits a solution, then it admits infinite solutions: indeed, the transfer function is invariant under change of basis. In the following, we provide a procedure for realizing an arbitrary proper rational transfer function $W(s)$. The steps of the procedure are:

1. write $W(s)$ as the sum of a strictly proper function $\tilde{W}(s)$ and a constant matrix D:

$$W(s) = \tilde{W}(s) + D.$$

   This can always be done, because if $W(s)$ is strictly proper, then $D = 0$ and $\tilde{W}(s) = W(s)$, otherwise we can add and subtract $N_\nu \left(s^\nu + \cdots + d_1 s + d_0\right)$ to the numerator, as below:

$$\begin{aligned} W(s) &= \frac{N_\nu s^\nu + N_{\nu-1} s^{\nu-1} + \cdots + N_0}{s^\nu + \cdots + d_1 s + d_0} = \\ &= \frac{N_\nu \left(d_0 + d_1 s + \cdots + s^\nu\right) + \left(N_{\nu-1} - N_\nu d_{\nu-1}\right) s^{\nu-1} + \cdots + \left(N_0 - N_\nu d_0\right)}{s^\nu + \cdots + d_1 s + d_0} \\ &= N_\nu + \underbrace{\frac{\tilde{N}_{\nu-1} s^{\nu-1} + \cdots + \tilde{N}_0}{s^\nu + \cdots + d_1 s + d_0}}_{\text{strictly proper}} = D + \tilde{W}(s) \end{aligned}$$

A realization procedure (cont.)

For example:

$$\frac{2s^2+10s+1}{s^2+3s-2} = \frac{2(s^2+3s-2)+2s^2+10s+1-2(s^2+3s-2)}{s^2+3s-2} =$$
$$= 2 + \frac{4s+5}{s^2+3s-2}.$$

Observe that we have assumed that the denominator of $W(s)$ is monic, i.e. $d_\nu = 1$. This is not a restriction, because if $d_\nu \neq 1$ we can divide numerator and denominator by d_ν.

Observe also that D can be easily found as

$$D = \lim_{s\to\infty} W(s).$$

2. Find the monic least common denominator of all the entries of $\tilde{W}(s)$:

$$d(s) = s^\nu + \cdots + d_1 s + d_0.$$

3. Write $\tilde{W}(s)$ as

$$\tilde{W}(s) = \frac{\tilde{N}_{\nu-1}s^{\nu-1} + \cdots + \tilde{N}_0}{s^\nu + \cdots + d_1 s + d_0}.$$

A realization procedure (cont.)

4. Take the matrices A,B and C as

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 \\ 0 & 0 & 0 & \dots & 0 & I \\ -d_0 I & -d_1 I & -d_2 I & \dots & -d_{\nu-2} I & -d_{\nu-1} I \end{bmatrix} B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ I \end{bmatrix} \\
C &= \begin{bmatrix} \tilde{N}_0 & \tilde{N}_1 & \tilde{N}_2 & \dots & \tilde{N}_{\nu-2} & \tilde{N}_{\nu-1} \end{bmatrix}
\end{aligned}
$$

where I is the identity matrix of dimension $m \times m$. Thus, A is of dimension $m\nu \times m\nu$.

A realization procedure (cont.)

To show that the above is a realization of $\tilde{W}(s)$ and, in turn, the system $\Sigma(A, B, C, D)$ is a realization of $W(s)$, we start by computing the matrix

$$\phi(s) = (sI - A)^{-1} B, \qquad \text{which is a solution to} \qquad (sI - A)\,\phi(s) = B.$$

By partitioning $\phi(s)$ conveniently, the latter can be written as

$$\begin{bmatrix} sI & -I & 0 & \dots & 0 & 0 \\ 0 & sI & -I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & sI & -I \\ d_0 I & d_1 I & d_2 I & \dots & d_{\nu-2} I & (s + d_{\nu-1}) I \end{bmatrix} \begin{bmatrix} \phi_0(s) \\ \phi_1(s) \\ \vdots \\ \vdots \\ \vdots \\ \phi_{\nu-1}(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ I \end{bmatrix}.$$

A realization procedure (cont.)

The previous equality is equivalent to the following equations:

$$\begin{cases} 0 &= s\phi_0(s) - \phi_1(s) \\ 0 &= s\phi_1(s) - \phi_2(s) \\ &\vdots \\ 0 &= s\phi_{\nu-2}(s) - \phi_{\nu-1}(s) \\ I &= d_0\phi_0(s) + d_1\phi_1(s) + \dots d_{\nu-2}\phi_{\nu-2} + (s + d_{\nu-1})\phi_{\nu-1}(s) \end{cases}$$

From the first $\nu - 1$ equations we get

$$\phi_i(s) = s^i\phi_0(s), \quad i = 1, \dots, \nu - 1,$$

and, by substituting in the last, we obtain

$$I = d_0\phi_0(s) + d_1 s\phi_0(s) + \dots + d_{\nu-2}s^{\nu-2}\phi_0(s) + (s + d_{\nu-1})s^{\nu-1}\phi_0(s) = d(s)\phi_0(s).$$

As a consequence, $\phi(s)$ is given by

$$\phi(s) = \frac{1}{d(s)} \begin{bmatrix} I \\ sI \\ s^2 I \\ \vdots \\ s^{\nu-1} I \end{bmatrix}.$$

A realization procedure (cont.)

We conclude observing that

$$
\begin{aligned}
C\,(sI - A)^{-1}\,B = C\phi(s) &= \begin{bmatrix} \tilde{N}_0 & \tilde{N}_1 & \dots & \tilde{N}_{\nu-1} \end{bmatrix} \frac{1}{d(s)} \begin{bmatrix} I \\ sI \\ \vdots \\ s^{\nu-1} I \end{bmatrix} \\
&= \frac{\tilde{N}_0 + \tilde{N}_1 s + \cdots + \tilde{N}_{\nu-1} s^{\nu-1}}{d(s)} \\
&= \tilde{W}(s).
\end{aligned}
$$

By duality, a realization can be found as follows. Consider the transfer function matrix $W^\top(s)$ of the dual system and find a realization (A_*, B_*, C_*, D_*). Such a realization has dimension $p\nu \times p\nu$. Now, the dual realization $(A_*^\top, C_*^\top, B_*^\top, D_*^\top)$ is a realization of $W(s)$. The procedure is convenient if $p < m$.

Example

We found the primal and dual realizations for the transfer function

$$W(s) = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{2}{(s+1)(s+2)} \end{bmatrix} = \frac{1}{s^2+3s+2}\left(\begin{bmatrix} 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} s\right).$$

The primal realization has order $m \times 2 = 4$:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -3 & 0 \\ 0 & -2 & 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 2 & 1 & 0 \end{bmatrix}.$$

Dual realization (cont.)

The dual realization, obtained by $W^\top(s)$, is of order $p \times 2 = 2$:

$$A_* = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad B_* = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C_* = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

The procedure described above becomes extremely simple for SISO strictly proper systems.

Theorem

The strictly proper SISO transfer function

$$W(s) = \frac{n_0 + n_1 s + \cdots + n_{\nu-1} s^{\nu-1}}{d_0 + d_1 s + \cdots + s^{\nu}}$$

admits the following realizations:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -d_0 & -d_1 & -d_2 & \dots & -d_{\nu-2} & -d_{\nu-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

$$C = \begin{bmatrix} n_0 & n_1 & n_2 & \dots & n_{\nu-2} & n_{\nu-1} \end{bmatrix} \tag{2}$$

which is called the *controllable canonical form*,

Theorem (cont.)

and

$$A_* = \begin{bmatrix} 0 & 0 & 0 & \dots & -d_0 \\ 1 & 0 & 0 & \dots & -d_1 \\ 0 & 1 & 0 & \dots & -d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -d_{\nu-1} \end{bmatrix} \quad B_* = \begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ \vdots \\ n_{\nu-2} \\ n_{\nu-1} \end{bmatrix} \tag{3}$$

$$C_* = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \tag{4}$$

which is called the *observable canonical form*.

The controllable and observable forms are, respectively, controllable and observable by construction (as it is easy to prove by means of, e.g., the PBH test).

The controllable and observable canonical forms turn out to be useful in the feedback control of SISO systems (as we will see in the following). The following results are important.

Lemma

Suppose that the SISO system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}, \qquad x \in \mathbb{R}^n, \quad u, y \in \mathbb{R}$$

is reachable. Then, there exists a matrix T such that the system

$$\begin{cases} \dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}u(t) \\ y(t) &= \hat{C}\hat{x}(t) + \hat{D}u(t) \end{cases},$$

where $\hat{A} = T^{-1}AT$, $\hat{B} = T^{-1}B$, $\hat{C} = CT$, and $\hat{D} = D$, is in controllable canonical form.

A constructive proof is reported in Antsaklis and Michel (2006).

Transformations towards controllable and observable forms (SISO) (cont.)

Lemma

Suppose that the SISO system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}, \qquad x \in \mathbb{R}^n, \quad u, y \in \mathbb{R}$$

is observable. Then, there exists a matrix T such that the system

$$\begin{cases} \dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}u(t) \\ y(t) &= \hat{C}\hat{x}(t) + \hat{D}u(t) \end{cases},$$

where $\hat{A} = T^{-1}AT$, $\hat{B} = T^{-1}B$, $\hat{C} = CT$, and $\hat{D} = D$, is in observable canonical form.

The result follows by duality from the previous.

Transformations towards controllable and observable forms (SISO) (cont.)

The transformations toward the controllable and observable forms can be found easily. Suppose we are given a reachable system $\Sigma(A, B, C, D)$. Then, by the previous theorem, a transformation T exists that puts the system in controllable canonical form. Observe that for any transformation T, the reachability matrix in the new state form is

$$\hat{R} = \begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \dots & \hat{A}^{n-1}\hat{B} \end{bmatrix} = \begin{bmatrix} T^{-1}B & T^{-1}AB & \dots & T^{-1}A^{n-1}B \end{bmatrix} = T^{-1}R$$

thus

$$\hat{R} = T^{-1}R.$$

Now, since the system is single input and reachable, R must be square and invertible and $\hat{R}$ as well (because it is the product of two invertible matrices). It follows that

$$T = R\hat{R}^{-1}.$$

Notice that $\hat{R}$ can be easily computed from the controller canonical form, which is known in advance without the need of computing T.

Transformations towards controllable and observable forms (SISO) (cont.)

To summarize, the steps for computing T are:

1. compute $\det(sI - A)$ and form the matrix $\hat{A}$ with its coefficients;
2. compute R from A and B;
3. compute $\hat{R}$ from $\hat{A}$ and $\hat{B} = [0 \;\; 0 \;\; \ldots \;\; 0 \;\; 1]^\top$;
4. compute $T = R\hat{R}^{-1}$.

As for the observable canonical form, it follows, by duality that

$$T = O^{-1}\hat{O}.$$

Example

Let

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

The characteristic polynomial is

$$\det(sI - A) = (s+1)(s-1)(s+2) = s^3 + 2s^2 - s - 2,$$

thus the state and input matrices of the controllable canonical form are:

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}, \qquad \hat{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Moreover:

$$R = \begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -2 & 4 \end{bmatrix} \quad \text{and} \quad \hat{R} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 5 \end{bmatrix}.$$

Transformations towards controllable and observable forms (SISO) (cont.)

Finally, the change of basis is given by:

$$T = R\hat{R}^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ -2 & -3 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Minimal realizations

Minimal realization

Suppose that the state-space system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}, \qquad x \in \mathbb{R}^n, \;\; u \in \mathbb{R}^m, \;\; y \in \mathbb{R}^p$$

is a realization of the transfer matrix $W(s)$. As a consequence we have:

$$C(sI - A)^{-1}B + D = W(s).$$

Observe that m is necessarily equal to the number of columns of $W(s)$. Likewise, p is equal to the number of rows of $W(s)$. On the contrary, the order n of the state-space representation is not uniquely determined.

It is easy to check that n can be arbitrarily large. Indeed, the extended system

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B \\ G \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + Du(t) \end{cases},$$

where F is $q \times q$, and G is $q \times m$, is a realization of $W(s)$, for any $q > 0$.

Minimal realization (cont.)

There is, however a minimal possible value of n that allows to realize a given transfer matrix $W(s)$.

Definition

A realization of $W(s)$ is called *minimal* if there is no realization of $W(s)$ of smaller order.

The minimality of a realization is strictly related to reachability and observability, as expressed by the following fundamental result.

Theorem

A realization is minimal if and only if it is both reachable and observable.

Proof of necessity.

The fact reachability and observability are necessary for minimality is easy to prove by contradiction: assume that a realization is either not reachable or not observable. Then, by the Kalman decomposition one could find another realization of smaller order that realizes the same transfer function, which would contradict minimality. □

Markov parameters

To prove the converse, i.e. that reachability and observability are also sufficient for minimality, we need some preliminary results.

First, observe that by the properties of the Laplace transform (and, in particular, the identity $\mathcal{L}\left[tf(t)\right] = -\dfrac{d\mathcal{L}[f(t)]}{ds}$), we have:

$$\begin{aligned}(sI - A)^{-1} &= \mathcal{L}\left[e^{At}\right] = \mathcal{L}\left[\sum_{i=0}^{\infty} \frac{t^i}{i!} A^i\right] \\ &= \sum_{i=0}^{\infty} \mathcal{L}\left[\frac{t^i}{i!}\right] A^i = \sum_{i=0}^{\infty} s^{-(i+1)} A^i.\end{aligned}$$

Thus we can write

$$W(s) = C\,(sI - A)^{-1}\,B + D = D + \sum_{i=0}^{\infty} s^{-(i+1)} C A^i B. \tag{5}$$

The matrices

$$D, \quad CA^iB, \quad i \geq 0$$

are called the *Markov parameters*.

Markov parameters (cont.)

The Markov parameters are also related to the impulse response. Indeed, we can write

$$W(t) = Ce^{At}B + D\delta(t),$$

and, by taking the derivatives of the right-hand side, we get

$$\frac{d^i W(t)}{dt^i} = CA^i e^{At} B, \quad \forall i \geq 0, \quad t > 0.$$

Thus, since $\lim_{t \to 0} e^{At} = I$,

$$\lim_{t \to 0^+} \frac{d^i W(t)}{dt^i} = CA^i B, \quad \forall i \geq 0. \tag{6}$$

Markov parameters (cont.)

Theorem

Two systems

$$\begin{cases} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{cases} \quad \text{and} \quad \begin{cases} \dot{\bar{x}}(t) & = & \bar{A}\bar{x}(t) + \bar{B}u(t) \\ y(t) & = & \bar{C}\bar{x}(t) + \bar{D}u(t) \end{cases}$$

are realizations of the same transfer function $W(s)$ if and only if they have the same Markov parameters, i.e.

$$D = \bar{D}, \quad CA^iB = \bar{C}\bar{A}^i\bar{B}, \quad \forall i \geq 0.$$

Proof.
From (5) we conclude that if the Markov parameters are the same, then the transfer function is the same. Conversely, if the transfer function is the same, the two systems must have the same D matrix since this matrix is equal to $\lim_{s\to\infty} W(s)$. Moreover, they must have the same impulse response, thus from (6) it follows that the Markov parameters are the same. □

Proof of sufficiency

We are now ready to prove the sufficiency.

Proof of sufficiency.
Suppose by contradiction that the realization of $W(s)$

$$(A, B, C, D), \qquad A \in \mathbb{R}^{n \times n} \tag{7}$$

is both reachable and observable, but not minimal. Therefore, there exists a realization

$$(\bar{A}, \bar{B}, \bar{C}, \bar{D}), \qquad \bar{A} \in \mathbb{R}^{\bar{n} \times \bar{n}} \tag{8}$$

of the same transfer function $W(s)$ of order $\bar{n} < n$. Consider now the reachability and observability matrices of (7):

$$R = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}, \qquad O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Proof of sufficiency (cont.)

By taking the product we get the $pn \times mn$ matrix

$$
\begin{aligned}
OR &= \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} CB & CAB & CA^2B & \dots & CA^{n-1}B \\ CAB & & \ddots & & CA^nB \\ \vdots & & \ddots & & \vdots \\ CA^{n-1}B & CA^nB & \dots & & CA^{2n-2}B \end{bmatrix}}_{\text{Markov parameters}};
\end{aligned} \tag{9}
$$

thus, OR is a block matrix whose blocks are Markov parameters.

Proof of sufficiency (cont.)

Suppose now that we compute

$$
\begin{aligned}
\bar{O}_n \bar{R}_n &= \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \bar{C}\bar{A}^2 \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix} \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \dots & \bar{A}^{n-1}B \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} \bar{C}\bar{B} & \bar{C}\bar{A}\bar{B} & \bar{C}\bar{A}^2\bar{B} & \dots & \bar{C}\bar{A}^{n-1}\bar{B} \\ \bar{C}\bar{A}\bar{B} & & \ddots & & \bar{C}\bar{A}^{n}\bar{B} \\ \vdots & & \ddots & & \vdots \\ \bar{C}\bar{A}^{n-1}\bar{B} & \bar{C}\bar{A}^{n}\bar{B} & \dots & & \bar{C}\bar{A}^{2n-2}\bar{B} \end{bmatrix}}_{\text{Markov parameters}},
\end{aligned}
\tag{10}
$$

which is also of size $pn \times mn$.

Proof of sufficiency (cont.)

Since (7) and (8) realize the same transfer function, by the previous theorem they must have the same Markov parameters, therefore we have

$$OR = \bar{O}_n \bar{R}_n.$$

Now we recall that given two matrices $A^{p\times q}$ and $B^{q\times v}$, the following inequalities hold:

$$\operatorname{rank} AB \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}$$
$$\operatorname{rank} AB \geq \operatorname{rank} A + \operatorname{rank} B - q$$

the latter being the *Sylvester's inequality*. Therefore, we have

$$\operatorname{rank} \bar{O}_n \bar{R}_n \leq \min\{\operatorname{rank} \bar{O}_n, \underbrace{\operatorname{rank} \bar{R}_n}_{\leq \bar{n}}\} \leq \bar{n} < n$$

and

$$\operatorname{rank} \bar{O}_n \bar{R}_n = \operatorname{rank} OR \geq \underbrace{\operatorname{rank} O}_{n} + \underbrace{\operatorname{rank} R}_{n} - n = n.$$

Hence, the rank of $\bar{O}_n \bar{R}_n$ must be both $< n$ and $\geq n$, which is clearly absurd.

$\square$

Order of a minimal realization

The proof of the previous theorem suggests a way to compute the *minimum order* of the systems having the same transfer function $W(s)$ of a given system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ of order $\bar{n}$. Let (A, B, C, D) be a minimal realization of $W(s)$, of order n. Since the Markov coefficients must be the same, we have

$$O_{\bar{n}} R_{\bar{n}} = \bar{O}\bar{R},$$

hence the rank is the same:

$$\operatorname{rank} O_{\bar{n}} R_{\bar{n}} = \operatorname{rank} \bar{O}\bar{R}.$$

On the other hand, $\operatorname{rank} O_{\bar{n}} R_{\bar{n}} = n$, i.e., the minimum order, because both factors are rank n, and have, respectively, n columns and n rows. Thus, the minimum order is

$$\operatorname{rank} \bar{O}\bar{R}.$$

As an alternative, the Kalman decomposition can be computed and the minimum order is the order of the reachable and observable part.

Order of a minimal realization (cont.)

Example

Consider the system:

$$\begin{cases} \dot{x} &= \begin{bmatrix} -1 & -4 & 0 \\ 0 & -\frac{7}{2} & 0 \\ -1 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} x \end{cases}$$

The order of the system is 3. However, by computing

$$OR = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

whose rank is zero, we obtain the minimum order is zero. Thus, there are no modes that are both reachable and observable.

Similarity of minimal realizations

We have seen that all minimal realizations have the same order, but it turns out that minimal realizations are even more closely related. The following theorem holds (we do not report the proof).

Theorem

All minimal realizations of a transfer function are similar.
In other words, if

$$(A, B, C, D) \quad \text{and} \quad (\bar{A}, \bar{B}, \bar{C}, \bar{D})$$

are minimal realizations of the same transfer function, then there exists T such that

$$(\bar{A}, \bar{B}, \bar{C}, \bar{D}) = (T^{-1}AT, T^{-1}B, CT, D).$$

Note that the transformations T can be easily found, in the SISO case. Indeed, the reachability matrices of two similar systems are related by $\bar{R} = T^{-1}R$, and if the two SISO systems are minimal, R and $\bar{R}$ are square and invertible, thus

$$T = R\bar{R}^{-1}.$$

Similarly, we get

$$T = O^{-1}\bar{O}.$$

References

References

Antsaklis, P. J. and Michel, A. N. (2006). *Linear Systems*. Springer Science & Business Media.

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Lecture 5
Realization

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