Control Theory

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Lecture 6: State feedback and output feedback

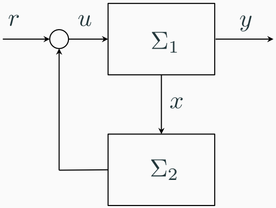
State feedback

Feedback

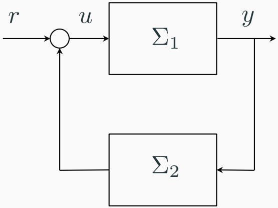
We talk about *feedback* when a quantity that influences another is in turn influenced by it. It is a mechanism that can be observed in many natural phenomena (for example the sodium-potassium pump that governs the transport process through the cell membrane, the glucose-insulin dynamics in the blood, the regulation of size of the pupil).

Feedback is of fundamental importance in the control of dynamical systems.

State feedback and output feedback



Example of state feedback.



Example of output feedback.

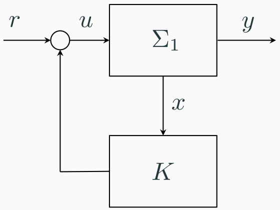
Typically, when the entire state is not accessible, the feedback block includes a sub-system, called the state observer, whose objective is to estimate the state (in order to exploit this information for feedback purposes)

Linear and static state feedback for LTI systems

Consider the system:

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

and take the input as $u = Kx + r$, where $K \in \mathbb{R}^{m \times n}$ is a *gain matrix*.



Static state feedback.

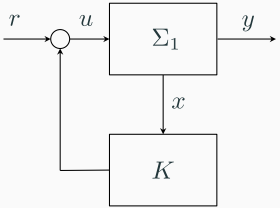
By substituting u is the state equation, we get:

$$\begin{cases} \dot{x} &= (A + BK)x + B\,r \\ y &= (C + DK)x + D\,r \end{cases}$$

which is the the state-space representation of the closed-loop system. Clearly, the closed-loop system has $A + BK$ as state matrix.

Linear and static state feedback for LTI systems (cont.)

$$\begin{cases} \dot{x} &= (A+BK)x + B\,r \\ y &= (C+DK)x + D\,r \end{cases}$$

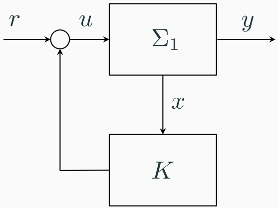


The gain matrix K affects the dynamics of the closed-loop system. In general it should be chosen to obtain desirable properties of the closed-loop system.

Here, we are mainly concerned on the **stability** of the closed-loop system, thus we will answer the question:

"Under what conditions, and how, is it possible to guarantee the closed-loop stability by a proper choice of K?"

$$\begin{cases} \dot{x} &= (A+BK)x + B\,r \\ y &= (C+DK)x + D\,r \end{cases}$$



The input r is called the *external input* (or *command input* or *reference input*). It has the same number of components of u and acts as an input for the closed-loop system.

Notice that the stability of the closed-loop depends only on the closed-loop matrix

$$A_{cl} \doteq A + BK$$

Thus, in particular, it does not depend on r (which, for simplicity, is usually set to zero when analyzing the stability).

Open-loop and closed-loop

The movement of the state under the control law

$$u = Kx + r$$

can be written, in terms of the initial state $x(0) = x_0$ and the external input $r(\cdot)$, as follows:

$$x(t) = e^{(A+BK)t}x_0 + \int_0^{\infty} e^{(A+BK)(t-\tau)}Br(\tau)d\tau.$$

Substituting in $u(t) = Kx(t) + r(t)$, we get:

$$u(t) = Ke^{(A+BK)t}x_0 + K\int_0^{\infty} e^{(A+BK)(t-\tau)}Br(\tau)d\tau + r(t),$$

which is an **open-loop control law**, because it does not require the knowledge of the current state (only the initial state is needed). In other words, it could be pre-computed, based on x_0 and $r(\cdot)$, and applied in an open-loop fashion.

Open-loop and closed-loop (cont.)

The same concept can be explained in terms of transfer functions:

$$u(t) = Kx(t) + r(t) \quad \xrightarrow{\mathcal{L}} \quad U(s) = KX(s) + R(s)$$

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \xrightarrow{\mathcal{L}} \quad sX(s) - x_0 = AX(s) + BU(s)$$

Solving the second equation for $X(s)$, substituting in the first, and using the matrix identities

$$[I - K(sI - A)^{-1}B]^{-1}K(sI - A)^{-1} = K(sI - A)^{-1}[I - BK(sI - A)^{-1}]^{-1} = K[sI - (A + BK)]^{-1},$$

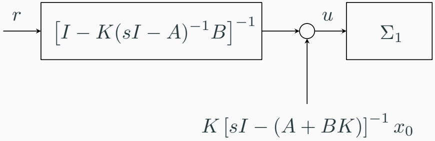
we get:

$$U(s) = K\left[sI - (A + BK)\right]^{-1} x_0 + \left[I - K(sI - A)^{-1}B\right]^{-1} R(s),$$

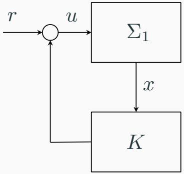
which is an open-loop control law, because it does not require the knowledge of the current state (only the initial state is needed).

Open-loop and closed-loop (cont.)

Thus, for the same initial state x_0, the following schemes result in the same input $u(t)$, $t \geq 0$.



Open-loop.



Closed-loop.

Open-loop and closed-loop (cont.)

However, the open loop control scheme has little chance of being effective in practice. Indeed:

- the initial state is not known exactly in practice, but is affected by **measurement noise**;
- the dynamics of the system (and in particular the matrices A and B) is not exactly known in practice, but is affected by **uncertainty**.

On the contrary, the closed-loop scheme does not require knowledge of the initial conditions and moreover, through feedback, it is able to regulate the input based on the current state of the system.

The two schemes are therefore equivalent only in the absence of noise and model uncertainties (a situation that never occurs in practice).

Eigenvalue assignment

Theorem

Given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, the eigenvalues of the matrix $A + BK$ can be assigned to arbitrary (real or complex *conjugate*) locations, by a properly chosen $K \in \mathbb{R}^{m\times n}$, if and only if the pair (A, B) is reachable.

Proof.

The sufficiency of the condition will be proved in a while, in a constructive way (i.e. by providing some algorithms to arbitrarily assign the eigenvalues, provided that the system is reachable).

The necessity can be proven as follows. Assume that the pair (A, B) is not reachable. Thus, there exists a similarity transformation T such that:

$$T^{-1}AT = \begin{bmatrix} \hat{A}_{1,1} & \hat{A}_{1,2} \\ 0 & \hat{A}_{2,2} \end{bmatrix} \qquad T^{-1}B = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}$$

where the null blocks are structural.

Eigenvalue assignment (cont.)

By applying the same transformation to the closed-loop matrix $A + BK$ we get:

$$T^{-1}(A+BK)T = T^{-1}AT + (T^{-1}B)\overbrace{(KT)}^{\begin{bmatrix}\hat{K}_1 & \hat{K}_2\end{bmatrix}} =$$

$$= \begin{bmatrix} \hat{A}_{1,1} & \hat{A}_{1,2} \\ 0 & \hat{A}_{2,2} \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} \begin{bmatrix}\hat{K}_1 & \hat{K}_2\end{bmatrix} =$$

$$= \begin{bmatrix} \hat{A}_{1,1} + \hat{B}_1\hat{K}_1 & \hat{A}_{1,2} + \hat{B}_1\hat{K}_2 \\ 0 & \hat{A}_{2,2} \end{bmatrix}.$$

This matrix is similar to $A + BK$ and therefore has the same eigenvalues. It can therefore be seen that regardless of the choice of K, the eigenvalues of the unreachable part are eigenvalues of the closed-loop matrix. That is, it is not possible to arbitrarily assign all the closed-loop eigenvalues.

Eigenvalue assignment (cont.)

In other words:

$$\sigma(\hat{A}_{2,2}) \subset \sigma(A + BK), \quad \forall K \in \mathbb{R}^{m \times n}$$

that is, the eigenvalues of the unreachable part are a subset of the eigenvalues of the closed-loop matrix, for each choice of K. □

Observations:

- the necessity proof shows an important fact, namely that **the unreachable eigenvalues are invariant under state feedback**: they cannot be changed by a state feedback;
- as stated by the theorem, the non real eigenvalues must be assigned as conjugate pairs (i.e. if $\lambda = \sigma + j\omega$ is to be assigned, then the eigenvalue $\lambda = \sigma - j\omega$ must be assigned too). Indeed, since $A + BK$ has real entries, its characteristic polynomial has real coefficients: as a consequence, non-real roots can only appear as conjugate pairs.

Stabilizable systems

Consider again the expression

$$T^{-1}(A+BK)T = \begin{bmatrix} \hat{A}_{1,1}+\hat{B}_1\hat{K}_1 & \hat{A}_{1,2}+\hat{B}_1\hat{K}_2 \\ 0 & \hat{A}_{2,2} \end{bmatrix}.$$

and observe that the pair $(\hat{A}_{1,1}, \hat{B}_1)$ is reachable by construction (thus the eigenvalues of $\hat{A}_{1,1}+\hat{B}_1\hat{K}_1$ can be arbitrarily assigned).

The remaining eigenvalues are fixed, thus a control law of the form

$$u = Kx + r$$

can produce an asymptotically stable closed-loop system if and only if the unreachable eigenvalues have strictly negative real part.

Definition

The system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is said to be *stabilizable* (and likewise the pair (A, B)) if its unreachable eigenvalues (if any) have strictly negative real part.

Stabilizable systems (cont.)

The following is the Popov-Belevitch-Hautus (PBH) test for stabilizability.

Theorem

The continuous-time system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

is stabilizable if and only if

$$\operatorname{rank}\,[\lambda I - A \quad B] = n, \quad \forall \lambda : \operatorname{Re}\lambda \geq 0.$$

The proof is analogous to that of PBH test for reachability, except that here we restrict the attention to the unstable region of $\mathbb{C}$.

An analogous result holds for discrete-time systems (where the condition $\operatorname{Re}\lambda \geq 0$ is substituted with $|\lambda| \geq 1$).

Eigenvalue assigment problem

The eigenvalue assignment problem can be formulated as follows:

Eigenvalue assignment problem

Given the reachable pair (A, B), and the set $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ (where the non-real elements, if any, appear as conjugate pairs), find K such that

$$\sigma(A + BK) = \Lambda.$$

A straightforward approach to solve the problem is the *direct method*, illustrated by the following example.

Example

Let

$$A = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \Lambda = \{-1+j, -1-j\}\,.$$

The sought matrix K must have one row and two columns: $K = [k_0 \;\; k_1]$. The characteristic polynomial of $A + BK$, as a function of k_0 and k_1 is thus

$$\begin{aligned}\alpha(\lambda) = \det(\lambda I - (A + BK)) &= \det\left(\begin{bmatrix} \lambda - \frac{1}{2} & -1 \\ -1 & \lambda - 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}[k_0 \;\; k_1]\right) \\ &= \det\begin{bmatrix} \lambda - \frac{1}{2} - k_0 & -1 - k_1 \\ -1 - k_1 & \lambda - 2 - k_2 \end{bmatrix} \\ &= \lambda^2 + \lambda\left(-\frac{5}{2} - k_0 - k_1\right) + k_0 - \frac{1}{2}k_1\end{aligned}$$

Direct method (cont.)

On the other hand, the desired eigenvalues are the roots of the polynomial:

$$\alpha_d(\lambda) = (\lambda - (-1+j))\,(\lambda - (-1-j)) = \lambda^2 + 2\lambda + 2.$$

By equating $\alpha(\lambda)$ to $\alpha_d(\lambda)$, we get a linear system of two equations and two unknowns:

$$\begin{cases} -\dfrac{5}{2} - k_0 - k_1 = 2 \\ k_0 - \dfrac{1}{2}k_1 = 2 \end{cases}$$

whose solution is

$$K = [k_0 \;\; k_1] = \begin{bmatrix} -\dfrac{1}{6} & -\dfrac{13}{3} \end{bmatrix}.$$

Observe that:

- it can be shown that if $m = 1$, i.e. when B is a column vector, the obtained system of equations is linear (the proof is based on the *Matrix determinant lemma* that states that

$$\det\left(M + uv^\top\right) = \left(1 + v^\top M^{-1} u\right) \det M$$

  where M is invertible and u and v are column vectors); if $m > 1$ the obtained system of equation will be, in general, non-linear;
- the reachability test must be performed in advance; if $m = 1$ and (A, B) is not reachable, it can be shown that the matrix associated to the linear system is singular and a solution exists only if the unreachable eigenvalues belong to Λ.

Methods based on controller form

Let T be the similarity transformation to the controller form:

$$A_c = T^{-1}AT, \quad B_c = T^{-1}B.$$

The matrices $A + BK$ and

$$T^{-1}(A + BK)T = T^{-1}AT + T^{-1}B \underbrace{KT}_{\doteq K_c} = A_c + B_cK_c$$

have the same eigenvalues thus the problem becomes that of assigning the eigenvalues to $A_c + B_cK_c$ by a proper choice of K_c.

Once the latter problem has been solved, the sought K can be obtained as $K = K_cT^{-1}$.

In the single input case ($m = 1$) we have

$$K_c = [k_0 \;\; k_1 \;\; \cdots \;\; k_{n-1}].$$

The closed-loop matrix is thus:

$$\begin{aligned}
A_{cK} &\doteq A_c + B_c K_c \\
&= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [k_0 \;\; k_1 \;\; \cdots \;\; k_{n-1}] \\
&= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -(a_0 - k_0) & -(a_1 - k_1) & \cdots & -(a_{n-1} - k_{n-1}) \end{bmatrix}
\end{aligned}$$

Methods based on controller form (cont.)

The matrix A_{cK} is in Frobenius form, hence its characteristic polynomial is

$$\det(\lambda I - A_{cK}) = \lambda^n + (a_{n-1} - k_{n-1})\lambda^{n-1} + \cdots + (a_1 - k_1)\lambda + (a_0 - k_0).$$

If the eigenvalues to be assigned are the roots of the polynomial:

$$\alpha_d(\lambda) = \lambda^n + d_{n-1}\lambda^{n-1} + \cdots + d_1\lambda + d_0,$$

by equating the coefficients of the same degree, we get the equations

$$k_i = a_i - d_i, \quad i = 0, \ldots, n-1$$

that provide the entries of K_c.

Notice that since the above equations admit a unique solution, we have proven the uniqueness of the solution in the case $m = 1$. It can be shown that, for $m > 1$, multiple solutions may exist.

Example

Let

$$A = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \Lambda = \{-1+j, -1-j\}.$$

We have

$$R = \left[\begin{array}{c|c} B & AB \end{array}\right] = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & 3 \end{bmatrix}$$

which is full rank. Moreover:

$$A_c = T^{-1}AT$$

where

$$T = \begin{bmatrix} -1 & 1 \\ \frac{1}{2} & 1 \end{bmatrix} \quad \text{and} \quad T^{-1} = \frac{1}{3}\begin{bmatrix} -2 & 2 \\ 1 & 2 \end{bmatrix}.$$

Methods based on controller form (cont.)

Thus

$$A_c = T^{-1}AT = \begin{array}{c} \begin{bmatrix} 0 & 1 \\ 0 & \frac{5}{2} \end{bmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ -a_0 & -a_1 \end{array} \end{array}, \qquad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

On the other hand, the desired polynomial is

$$\alpha_d(\lambda) = (\lambda - (-1+j))(\lambda - (-1-j)) = \lambda^2 + \underset{d_1}{\underset{\uparrow}{2}}\lambda + \underset{d_0}{\underset{\uparrow}{2}}$$

hence we get the system of equations

$$\begin{cases} k_0 = a_0 - d_0 = -2 \\ k_1 = a_1 - d_1 = -\frac{9}{2} \end{cases} \implies \qquad K_c = \begin{bmatrix} -2 & -\frac{9}{2} \end{bmatrix}$$

and, finally:

$$K = K_c T^{-1} = \begin{bmatrix} -2 & -\frac{9}{2} \end{bmatrix} \frac{1}{3} \begin{bmatrix} -2 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & -\frac{13}{3} \end{bmatrix}.$$

Ackermann's formula

In the single input case ($m = 1$), one can determine the elements of the gain matrix (which is a row matrix) directly, that is, without passing through the controller form. This is made possible by *Ackermann's formula*:

$$K = -e_n^\top R^{-1} \alpha_d(A)$$

where

$$e_n = [0, \dots, 0, 1]^\top \in \mathbb{R}^n, \qquad R = \left[\begin{array}{c|c|c|c} B & AB & \cdots & A^{n-1}B \end{array} \right]$$

and $\alpha_d(A)$ is a matrix obtained by evaluating the polynomial $\alpha_d(\lambda)$ for $\lambda = A$. In the previous example we would have:

$$\alpha_d(A) = A^2 + 2A + 2I.$$

Proof of the Ackermann's formula

Evaluating $\alpha_d(\lambda)$ for $\lambda = A_c$ we get:

$$\alpha_d(A_c) = A_c^n + d_{n-1}A_c^{n-1} + \cdots + d_1 A_c + d_0 I.$$

On the other hand, by the Cayley-Hamilton theorem:

$$A_c^n = -a_{n-1}A_c^{n-1} - \cdots - a_1 A_c - a_0 I.$$

By substitution in the first equation we get:

$$\alpha_d(A_c) = (d_{n-1} - a_{n-1})A_c^{n-1} + \cdots + (d_1 - a_1)A_c + (d_0 - a_0)I$$

It easy to check that, due to the particular structure of A_c and its powers, the first row of $\alpha_d(A_c)$ is

$$[\,1 \;\; 0 \;\; \ldots \;\; 0\,]\alpha_d(A_c) = [(d_0 - a_0) \;\; \ldots \;\; (d_{n-1} - a_{n-1})] = -K_c$$

i.e. the state-feedback matrix that assigns the desired eigenvalues to the system in controller form.

Ackermann's formula (cont.)

Since $K = K_c T^{-1}$, and observing that

$$\alpha_d(A_c) = \alpha_d(T^{-1}AT) = T^{-1}\alpha_d(A)T$$

we can write

$$K = -[\,1\;\;0\ldots\;0\,]T^{-1}\alpha_d(A)TT^{-1} = -[\,1\;\;0\ldots\;0\,]T^{-1}\alpha_d(A).$$

Now, recalling from the lecture on realizations, that

$$T^{-1} = R_c R^{-1}$$

where

$$R_c = \begin{bmatrix} B_c & A_c B_c & \cdots & A_c^{n-1}B_c \end{bmatrix}, \quad R = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

we have

$$K = -\underbrace{[\,1\;\;0\ldots\;0\,]R_c}_{\text{first row of } R_c} R^{-1}\alpha_d(A),$$

but the first row of R_c is easily checked to be $[0\;\ldots 0\;1] = e_n^\top$. $\square$

Ackermann's formula (cont.)

Example

Applying the Ackermann's formula to the same eigenvalue assignment problem previously seen, we have

$$
\begin{aligned}
K &= -e_2^\top R^{-1}\alpha_d(A) \\
&= -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \left(\begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{bmatrix}^2 + 2 \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\
&= \begin{bmatrix} -\frac{1}{6} & -\frac{13}{3} \end{bmatrix}
\end{aligned}
$$

Multiple input case

If $m > 1$, the solution of the eigenvalue assignment problem is not unique, in general.

We distinguish two cases:

(A) the system is reachable by a single input, i.e.

$$\exists i : \mathrm{rank}\left(\begin{bmatrix} b_i & Ab_i & \cdots & A^{n-1}b_i \end{bmatrix}\right) = n$$

where b_i is the ith column of the input matrix B;

(B) the system is not reachable by a single input.

Multiple input case (cont.)

Case (A): in this case, the eigenvalues can be assigned by resorting to the ith input only, i.e. solving the problem

$$\sigma(A + b_i K_i) = \Lambda$$

instead of $\sigma(A + BK) = \Lambda$.

Once K_i has been found, it becomes the ith row of the whole feedback matrix K:

$$K = \begin{bmatrix} 0 \\ \vdots \\ K_i \\ 0 \\ \vdots \end{bmatrix} \begin{matrix} \\ \\ \longleftarrow i\text{th row} \\ \\ \\ \end{matrix}$$

Multiple input case (cont.)

Example

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad B = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow \\ b_1 & b_2 \end{matrix} \end{array}$$

The system is reachable by each of the two inputs separately, since:

$$\operatorname{rank}\left(\begin{bmatrix} b_1 & Ab_1 & \cdots & A^{n-1}b_1 \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} b_2 & Ab_2 & \cdots & A^{n-1}b_2 \end{bmatrix}\right) = 3.$$

Thus, the eigenvalues can be assigned by any of the following control laws, for properly chosen K_1 and K_2:

$$u = \begin{bmatrix} K_1 \\ 0 \end{bmatrix} x + r \quad \text{and} \quad u = \begin{bmatrix} 0 \\ K_2 \end{bmatrix} x + r.$$

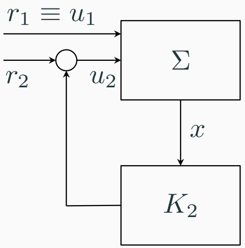
For instance, by applying the control law

$$u = \begin{bmatrix} 0 \\ K_2 \end{bmatrix} x + r,$$

where K_2 is a row matrix found by solving the assignment problem $\sigma(A + b_2 K_2) = \Lambda$, we have

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ k_{21} & k_{22} & k_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}.$$

The block scheme of the control system is represented in the figure. Notice that the first input channel is employed for the reference signal only. Possible problem: unbalanced exploitation of the actuators.



Heymann's lemma

Case (B): when the system is reachable but

$$\nexists i : \operatorname{rank}\left(\begin{bmatrix} b_i & Ab_i & \cdots & A^{n-1}b_i \end{bmatrix}\right) = n,$$

the eigenvalue assignment problem can be solved in two steps:

1. apply a state-feedback to render the closed-loop system reachable by a single input;
2. apply a single input assignment method to the obtained system.

The previous is possible thanks to the following lemma (*Heymann's lemma*).

Lemma

Let (A, B) be reachable, and $b_i \neq 0$ be a column of B. Then, there exists M_i such that $(A + BM_i, b_i)$ is reachable.

In other words, the Heymann's lemma states that a reachable system can be always rendered reachable by a single input, by a proper state feedback.

Heymann's lemma (cont.)

Proof of the Heymann's lemma

The proof is constructive. Recall that R has rank n and takes the form

$$R = \left[\begin{array}{cccc|cccc|c|ccc} b_1 & b_2 & \dots & b_m & Ab_1 & Ab_2 & \cdots & Ab_m & \dots & A^{n-1}b_1 & \dots & A^{n-1}b_m \end{array}\right]$$

Suppose $b_1 \neq 0$ and extract $\nu_1 + \cdots + \nu_l = n$ linearly independent columns, in the following order:

$$Q = \left[\begin{array}{cccc|ccc|c|ccc} b_1 & Ab_1 & \dots & A^{\nu_1 - 1}b_1 & b_2 & \cdots & A^{\nu_2 - 1}b_2 & \dots & b_l & \dots & A^{\nu_l - 1}b_l \end{array}\right].$$

Define $S \in \mathbb{R}^{m \times n}$ as

$$S = \left[\begin{array}{ccc|ccc|c|ccc|ccc} 0 & \dots & e_2 & 0 & \dots & e_3 & \dots & 0 & \dots & e_l & 0 & \dots & 0 \end{array}\right]$$

ν_1th column $\qquad$ $(\nu_1 + \nu_2)$th column $\qquad$ $(n - \nu_l)$th column

where e_i is the ith column of the identity matrix I_m.

Heymann's lemma (cont.)

Now, let the feedback matrix M_1 be defined as

$$M_1 = SQ^{-1}.$$

We now prove that $(A + BM_1, b_1)$ is reachable.

By construction of M_1, it follows that

$$\begin{aligned} M_1 Q &= M_1 \left[\begin{array}{cccc|c|cccc} b_1 & Ab_1 & \dots & A^{\nu_1 - 1} b_1 & \dots & b_l & Ab_l & \dots & A^{\nu_l - 1} b_l \end{array}\right] \\ &= \left[\begin{array}{ccc|ccc|c|ccc|ccc} 0 & \dots & e_2 & 0 & \dots & e_3 & \dots & 0 & \dots & e_l & 0 & \dots & 0 \end{array}\right]. \end{aligned} \tag{1}$$

We need to prove that the columns of the reachability matrix

$$R_M = \left[\begin{array}{cccc} b_1 & (A + BM_1) b_1 & \dots & (A + BM_1)^{n-1} b_1 \end{array}\right]$$

are linearly independent.

The first column b_1 is linearly independent since $b_1 \neq 0$. We now consider the subsequent columns.

$$
\begin{aligned}
(A+BM_1)b_1 &= Ab_1 + B\overbrace{M_1 b_1}^{=0 \text{ by (1)}} = Ab_1 \\
(A+BM_1)^2 b_1 &= (A+BM_1)Ab_1 = A^2 b_1 \\
&\vdots \\
(A+BM_1)^{\nu_1-1} b_1 &= (A+BM_1)A^{\nu_1-2} b_1 = A^{\nu_1-1} b_1 \\
(A+BM_1)^{\nu_1} b_1 &= (A+BM_1)A^{\nu_1-1} b_1 = A^{\nu_1} b_1 + Be_2 = A^{\nu_1} b_1 + b_2 = b_2 + \dots \\
(A+BM_1)^{\nu_1+1} b_1 &= (A+BM_1)(A^{\nu_1} b_1 + b_2) = Ab_2 + \dots \\
&\vdots \\
(A+BM_1)^{n-1} b_1 &= (A+BM_1)(A^{\nu_l-2} b_l + \dots) = A^{\nu_l-1} b_l + \dots
\end{aligned}
$$

The colored vectors are linearly independent and the dots denote vectors linearly dependent from the previous. $\square$

In Hautus (1977) a different, non-constructive, proof is provided.

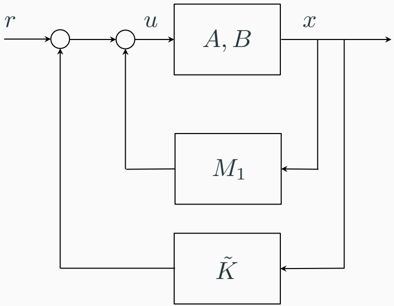
Heymann's lemma (cont.)

The whole scheme is reported in the figure, where

$$\tilde{K} = \begin{bmatrix} K_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The two feedback matrices can be combined as

$$K = M_1 + \tilde{K}.$$



Example

Consider the pair:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The reachability matrix is:

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

whose rank is 4, thus the system is reachable.

Heymann's lemma (cont.)

The reachability matrices from the first input and second input

$$R(b_1) = \begin{bmatrix} b_1 & Ab_1 & A^2b_1 & A^3b_1 \end{bmatrix}, \qquad R(b_2) = \begin{bmatrix} b_2 & Ab_2 & A^2b_2 & A^3b_2 \end{bmatrix}$$

are respectively

$$R(b_1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \qquad R(b_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

thus the system is not reachable from solely the first input, nor from the second.

By applying the procedure seen above we have:

$$Q = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right], \qquad S = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Heymann's lemma (cont.)

$$M_1 = SQ^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

By the Heymann's lemma, the pair $(A + BM_1, b_1)$

$$A + BM_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad b_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

is reachable. Observing that the pair is in controller form, we have

$$a_0 = 0, \ \ a_1 = 0, \ \ a_2 = -1, \ \ a_3 = 0.$$

If the polynomial to be assigned is

$$\alpha_d(\lambda) = \tilde{a}_0 + \tilde{a}_1\lambda + \tilde{a}_2\lambda^2 + \tilde{a}_3\lambda^3 + \lambda^4,$$

Heymann's lemma (cont.)

we need to take

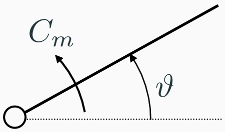
$$K_1 = \begin{bmatrix} -\tilde{a}_0 & -\tilde{a}_1 & -1-\tilde{a}_2 & \tilde{a}_3 \end{bmatrix}$$

thus, the whole feedback matrix K is

$$K = M_1 + \begin{bmatrix} K_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -\tilde{a}_0 & -\tilde{a}_1 & -1-\tilde{a}_2 & \tilde{a}_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -\tilde{a}_0 & -\tilde{a}_1 & -1-\tilde{a}_2 & \tilde{a}_3 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Example: servomechanism

Let J be the inertia of the beam, and $x_1 = \vartheta$, $x_2 = \dot{\vartheta}$, $u = C_m$ and $\beta = \dfrac{1}{J}$.



The system is described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u$$

The system is in a “quasi-controller form” (it would be in controller form for $\beta = 1$). However, we have:

$$A + BK = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} [k_1 \;\; k_2] = \begin{bmatrix} 0 & 1 \\ \beta k_1 & \beta k_2 \end{bmatrix}$$

whose characteristic polynomial is $\alpha(\lambda) = \lambda^2 - \beta k_2 \lambda - \beta k_1$.

Example: servomechanism (cont.)

Let $\Lambda = \{\lambda_1, \lambda_2\}$ be the set of eigenvalues to be assigned. Then:

$$\alpha_d(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

and to choose K it is sufficient equating the corresponding coefficients:

$$\begin{aligned} -\beta k_1 &= \lambda_1\lambda_2 \\ -\beta k_2 &= -(\lambda_1 + \lambda_2) \end{aligned}$$

thus obtaining:

$$\begin{aligned} k_1 &= -\frac{\lambda_1\lambda_2}{\beta} \\ k_2 &= \frac{\lambda_1 + \lambda_2}{\beta} \end{aligned}.$$

Example: servomechanism (cont.)

How should we choose the eigenvalues?

- they must of course have strictly negative real part, to obtain an asymptotically stable closed-loop system;
- depending on the desired behavior in terms of pseudo-periodic modes, they will have/not have imaginary part;
- having a reasonable time constant i.e.
    - sufficiently short to get satisfactory transients;
    - sufficiently large to avoid excessive inputs (torque applied).

(Recall that the time constant of an eigenvalue λ is the reciprocal of $-\mathrm{Re}(\lambda)$ and the response of the associate mode gets faster as the time constant decreases).

Example: servomechanism (cont.)

Take a time constant τ of half a time unit for both the eigenvalues:

$$\tau = \frac{1}{2}.$$

Thus, we have

$$\lambda_1 = \lambda_2 = -\frac{1}{\tau} = -2.$$

and, from the previous equations we get

$$\begin{aligned} k_1 &= -\frac{\lambda_1 \lambda_2}{\beta} = -\frac{4}{\beta} \\ k_2 &= \frac{\lambda_1 + \lambda_2}{\beta} = -\frac{4}{\beta} \end{aligned} \implies K = [k_1 \quad k_2] = [-\frac{4}{\beta} \quad -\frac{4}{\beta}]$$

Observation

Since $\lambda_1, \lambda_2 = -\frac{1}{\tau}$ we have, in general

$$k_1 = k_2 = -\frac{1}{\tau \beta}$$

showing that the control action increases as the time constant decreases (this is reasonable because to obtain faster transients we need stronger control action).

If the system is non-reachable, we have seen that, being K a feedback matrix, and T the similarity transformation to the Kalman reachability form, we have

$$
\begin{aligned}
T^{-1}(A+BK)T = T^{-1}AT + (T^{-1}B)\overbrace{(KT)}^{\begin{bmatrix}\hat{K}_1 & \hat{K}_2\end{bmatrix}} &= \\
= \begin{bmatrix} \hat{A}_{1,1} & \hat{A}_{1,2} \\ 0 & \hat{A}_{2,2} \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} \begin{bmatrix}\hat{K}_1 & \hat{K}_2\end{bmatrix} &= \\
= \begin{bmatrix} \hat{A}_{1,1} + \hat{B}_1\hat{K}_1 & \hat{A}_{1,2} + \hat{B}_1\hat{K}_2 \\ 0 & \hat{A}_{2,2} \end{bmatrix},
\end{aligned}
$$

thus, the eigenvalues of the reachable part can be assigned by solving the assignment problem:

$$\sigma(\hat{A}_{1,1} + \hat{B}_1\hat{K}_1) = \Lambda,$$

which is solvable because the pair $(\hat{A}_{1,1}, \hat{B}_1)$ is reachable. Having obtained $\hat{K}_1$ as the solution of the previous

Non-reachable systems (cont.)

problem, it is sufficient to take

$$\hat{K} = \begin{bmatrix} \hat{K}_1 & 0 \end{bmatrix}.$$

Finally, recalling that $\hat{K} = KT$, we have

$$K = \hat{K}T^{-1}.$$

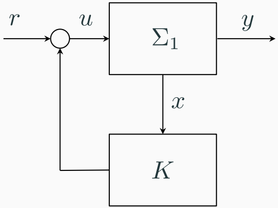
Clearly, provided that the assigned eigenvalues have strictly negative real part, the whole closed-loop system will be asymptotically stable if and only if the unreachable eigenvalues have strictly negative real part.

Discrete-time LTI systems

Consider the system:

$$\begin{cases} x(t+1) = & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{cases}$$

and take the input as $u = Kx + r$, where $K \in \mathbb{R}^{m \times n}$.



Static state feedback.

By substituting u is the state equation, we get:

$$\begin{cases} x(t+1) & = & (A + BK)x(t) + B\,r(t) \\ y(t) & = & (C + DK)x(t) + D\,r(t) \end{cases}$$

which is the the state-space representation of the closed-loop system. The closed-loop system has $A + BK$ as state matrix.

Discrete-time LTI systems (cont.)

The same theorem of the continuous-time case holds for the discrete-time because the eigenvalue assignment problem is a purely algebraic one ("find a matrix K such that $A + BK$ has given eigenvalues"). We report the theorem here for convenience.

Theorem

Given $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, the eigenvalues of the matrix $A + BK$ can be assigned to arbitrary (real or complex *conjugate*) locations, by a properly chosen $K \in \mathbb{R}^{m \times n}$, if and only if the pair (A, B) is reachable.

For the same reason, all the eigenvalue assignment methods shown for the continuous-time case can be employed for the discrete-time as well.

Needless to say, the asymptotic stability region is now the open unit circle.

State estimation

State estimation

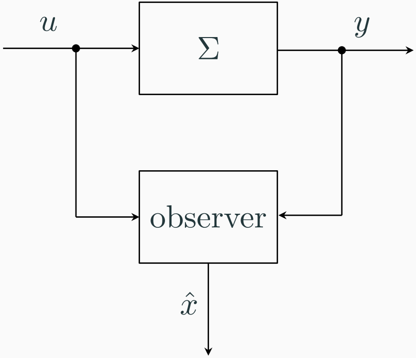
In some cases of practical interest, the whole state is not accessible, and only the output y can be measured. Possible reasons are:

1. some state variables are not measurable;
2. the sensor required for measuring a state variable is too expensive;
3. the noise is such that any possible measured value would be affected by a too large error;
4. ...

In those cases, under proper conditions, the state can be estimated from the available information (essentially, the input and the output in a given time interval).

Observer

The device that provides an estimate $\hat{x}$ of the state x of the system Σ based on the input u and the output y, in a given time interval, is called *state observer*.



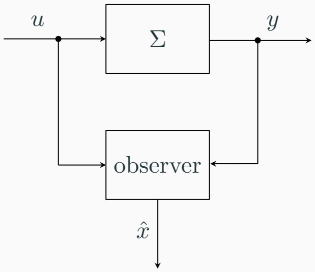
Luenberger observer

Given the system

$$\begin{cases} \dot{x} &= A\,x + B\,u \\ y &= C\,x + D\,u \end{cases}$$

the *Luenberger observer* is a defined as

$$\begin{cases} \dot{\hat{x}} &= A\,\hat{x} + B\,u + L(y - \hat{y}) \\ \hat{y} &= C\,\hat{x} + D\,u \end{cases}$$



where $L \in \mathbb{R}^{n \times p}$ is a suitably chosen matrix. Notice that:

- the Luenberger observer is a linear dynamic system;
- the order of such a system is n, the same of the system to be observed;
- the state $\hat{x}$ of the observer is the estimate of the true state x;
- the dynamics of the Luenberger observer is the same of the original system, except for a “correction term” $(y - \hat{y})$ that enters through the injection matrix L.

Luenberger observer (cont.)

Considering again the representation:

$$\begin{cases} \dot{\hat{x}} & = & A\,\hat{x} + B\,u + L(y - \hat{y}) \\ \hat{y} & = & C\,\hat{x} + D\,u \end{cases}$$

By substituting the second equation onto the first, and recalling that the output of the system is $y = Cx + Du$, we can write

$$\dot{\hat{x}} = (A - LC)\hat{x} + (B - LD)u + Ly$$

or, in compact form:

$$\dot{\hat{x}} = (A - LC)\hat{x} + [B - LD \quad L] \begin{bmatrix} u \\ y \end{bmatrix}$$

Thus:

- the dynamics of the observer is governed by $A - LC$;
- the input vector of the observer is composed by the input and the output of the observed system.

Error dynamics

The estimation error at time t is the difference between the actual state $x(t)$ and the estimated state $\hat{x}(t)$:

$$e(t) = x(t) - \hat{x}(t).$$

By taking the time derivative we get

$$\begin{aligned}\dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) = A\,x + B\,u - [A\,\hat{x} + B\,u + L(\overbrace{y - \hat{y}}^{Cx + Du - C\hat{x} - Du})] \\ &= A\,x + B\,u - [A\,\hat{x} + B\,u + LC(x - \hat{x})] = (A - LC)(x - \hat{x})\end{aligned}$$

thus:

$$\dot{e}(t) = (A - LC)e(t)$$

In other words, the error behaves as a linear system with no input, governed by the state matrix $A - LC$.

Error dynamics (cont.)

By integrating we get

$$e(t) = \exp\left[(A - LC)\,t\right] e(0).$$

If the eigenvalues of $A - LC$ lie in the open left half plane, then the system is asymptotically stable and

$$e(t) \longrightarrow 0 \qquad \text{as} \qquad t \longrightarrow \infty,$$

independent of the initial error $e(0) = x(0) - \hat{x}(0)$, i.e. independent of the observer's initialization. Moreover, the speed of convergence to zero is governed by the eigenvalues.

Eigenvalue assignment

Lemma

Given $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$, the eigenvalues of the matrix $A - LC$ can be assigned to arbitrary (real or complex *conjugate*) locations, by a properly chosen $L \in \mathbb{R}^{n\times p}$, if and only if the pair (A, C) is observable.

Proof.
The matrices

$$A - LC \qquad \text{and} \qquad (A - LC)^\top = A^\top - C^\top L^\top$$

have the same eigenvalues. By the eigenvalue assignment theorem, the eigenvalues of $A^\top - C^\top L^\top$ can be arbitrarily assigned by a proper choice of $L^\top$ if and only if the pair $(A^\top, C^\top)$ is reachable. By duality, the latter condition is equivalent to the pair (A, C) being observable. $\square$

Detectable systems

Definition

The system

$$\begin{cases} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{cases}$$

is said to be *detectable* (and likewise the pair (A, C)) if its unobservable eigenvalues (if any) have strictly negative real part.

Detectability is a necessary and sufficient condition for obtaining an asymptotically stable Luenberger observer.

Choice of the eigenvalues

When designing an observer, two opposite requirements must be taken into account:

- on the one hand, it is desirable that the estimation error converges to zero "fast enough";
- on the other hand, too fast eigenvalues lead to "too large" gain matrix L, rendering the estimate too sensitive to measurement noise.

Observe that the measurement noise is the only reason that prevents from using arbitrarily large observer eigenvalues (i.e. arbitrarily fast convergence), because the observer is typically implemented in a digital computer. On the contrary, when designing the feedback matrix K, besides the measurement noise of the state, there is the additional constraint of the maximum actuators' effort.

Choice of the eigenvalues (cont.)

The following is the Popov-Belevitch-Hautus (PBH) test for detectability.

Theorem

The continuous-time system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

is detectable if and only if

$$\operatorname{rank}\begin{bmatrix}\lambda I - A\\ C\end{bmatrix} = n, \quad \forall\lambda : \operatorname{Re}\lambda \geq 0.$$

The proof is analogous to that of PBH test for observability, except that here we restrict the attention to the unstable region of $\mathbb{C}$.

An analogous result holds for discrete-time systems (where the condition $\operatorname{Re}\lambda \geq 0$ is substituted with $|\lambda| \geq 1$).

Trivial observer

Is it possible to choose $L = 0$?

In that case, the correction term $y - \hat{y}$ is missing and the observer is, in practice, a replica of the system to be observed:

$$\dot{\hat{x}} = A\,\hat{x} + B\,u.$$

Such a choice has a number of disadvantages:

- since the state matrix of the observer is A, the estimation error converges to zero only if the system is asymptotically stable;
- there is no way of modifying the speed of convergence;
- suppose that $u(t) = 0,\ \ t \geq 0$, and the state of the observer is initialized as $\hat{x}(0) = 0$ (which is an obvious choice, in absence of specific information). We then have $e(0) = x(0) - 0 = x(0)$, thus

  $$\|e(t)\| = \|\exp\left(At\right)x(0)\| = \|x(t)\|\,.$$

  In other words, the estimation error is as big as the estimated variable (which is clearly a poor performance).

Example

Consider the observable pair

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Let $\alpha_d(\lambda) = \lambda^3 + d_2\lambda^2 + d_1\lambda + d_0$ be the characteristic polynomial to be assigned to $A - LC$.

We can formulate the dual problem of assigning the eigenvalues to $A_D + B_D K_D$, where

$$A_D = A^\top = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}, \qquad B_D = C^\top = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad K_D = -L^\top.$$

Example (cont.)

The similarity transform to the controller form is

$$A_{D_c} = T^{-1} A_D T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

where

$$T = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix}.$$

By applying

$$k_i = a_i - d_i, \quad i = 0, \ldots, n-1$$

we get immediately

$$K_{D_c} = \begin{bmatrix} -d_0 & -d_1 - 2 & -d_2 + 1 \end{bmatrix}.$$

Example (cont.)

Thus, the solution of the dual problem is

$$K_D = K_{D_c} T^{-1} = \begin{bmatrix} -d_0 & -d_1 - 2 & -d_2 + 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -d_2 + 1 & d_2 - d_1 - 3 & -d_0 + d_1 - 3d_2 + 5 \end{bmatrix}.$$

Finally:

$$L = -K_D^\top = \begin{bmatrix} d_2 - 1 \\ -d_2 + d_1 + 3 \\ d_0 - d_1 + 3d_2 - 5 \end{bmatrix}.$$

Ackermann's formula

The same result could be obtained by applying the *dual Ackermann's formula*:

$$L = \alpha_d(A)O^{-1}e_n$$

where O is the observability matrix of the pair (A, C):

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

The dual formula can be obtained from the primal, by applying the duality transform. More generally, **any procedure** for assigning the eigenvalues to $A + BK$ by a proper choice of K can be applied, by means of duality, to the problem of assigning the eigenvalues to $A - LC$ by a proper choice of L.

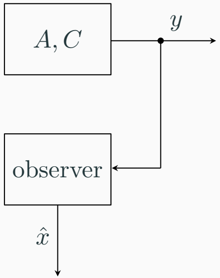
Example

Consider the system:

$$\left\{ \begin{array}{rcl} \dot{x} & = & A\,x \\ y & = & C\,x \end{array} \right. \quad \text{where} \quad A = \begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and the Luenberger observer:

$$\left\{ \begin{array}{rcl} \dot{\hat{x}} & = & A\,\hat{x} + L(y - \hat{y}) \\ \hat{y} & = & C\,\hat{x} \end{array} \right.$$



It is easy to check that the matrix

$$L = \begin{bmatrix} d_0 - 2 \\ d_1 - 2 \end{bmatrix}$$

assigns to $A - LC$ the following characteristic polynomial:

$$\alpha_d(\lambda) = \lambda^2 + d_1\lambda + d_0.$$

Example (cont.)

Indeed

$$\det(\lambda I - (A - LC)) = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & -d_0 \\ 1 & -d_1 \end{bmatrix}\right) = \lambda^2 + d_1\lambda + d_0.$$

The error $e(t) = x(t) - \hat{x}(t)$ is governed by the equation

$$\dot{e}(t) = (A - LC)e(t).$$

We now consider four different polynomials (i.e. different pairs of eigenvalues) and represent the error as a function of time, for $\hat{x}(0) \neq x(0)$, precisely

$$x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \hat{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Case A

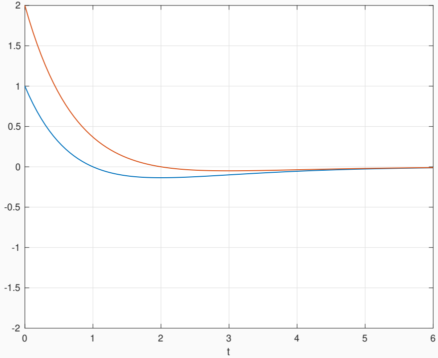
$$\lambda_1 = -1,\ \lambda_2 = -1 \quad \Longrightarrow \quad \alpha_d(\lambda) = (\lambda+1)^2 = \lambda^2 + 2\lambda + 1 \quad \Longrightarrow \quad L = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

- at $t = 0$ the error is equal to the state since the observer is initialized at zero:

$$e(0) = x(0) - \hat{x}(0) = x(0);$$

- then, it decreases, governed by the stable dynamics:

$$\dot{e}(t) = (A - LC)e(t)$$



Example (cont.)

Case B

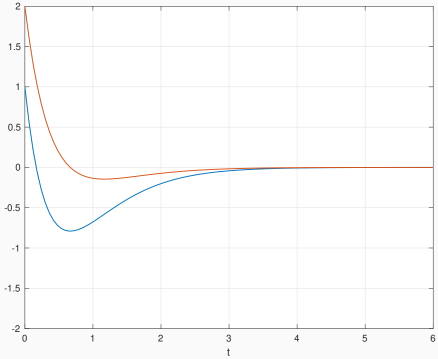
$$\lambda_1 = -2,\ \lambda_2 = -2 \quad \Longrightarrow \quad \alpha_d(\lambda) = (\lambda+2)^2 = \lambda^2 + 4\lambda + 4 \quad \Longrightarrow \quad L = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

- at $t = 0$ the error is equal to the state since the observer is initialized at zero:

$$e(0) = x(0) - \hat{x}(0) = x(0);$$

- then, it decreases, governed by the stable dynamics:

$$\dot{e}(t) = (A - LC)e(t)$$



The error transient is much shorter (about half) than that of case A. Indeed, the assigned eigenvalues are farther from the imaginary axis than in the previous case.

Example (cont.)

Case C

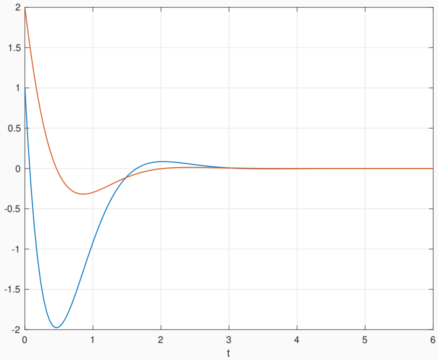
$$\begin{aligned} \lambda_1 &= -2-2j \\ \lambda_2 &= -2+2j \end{aligned} \implies \alpha_d(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2+4\lambda+8 \implies L = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

- at $t=0$ the error is equal to the state since the observer is initialized at zero:

$$e(0) = x(0) - \hat{x}(0) = x(0);$$

- then, it decreases, governed by the stable dynamics:

$$\dot{e}(t) = (A-LC)e(t)$$



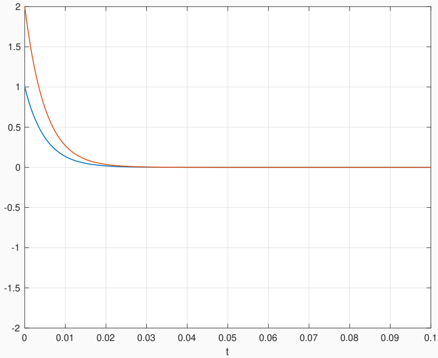
The duration of the transient is approximately the same as in case B, but pseudoperiodic modes appear because the eigenvalues have a non-zero imaginary part.

Example (cont.)

Case D

$$\begin{aligned} \lambda_1 &= -200 \\ \lambda_2 &= -0.5 \end{aligned} \implies \alpha_d(\lambda) = \lambda^2 + 200.5\lambda + 100 \implies L = \begin{bmatrix} 98 \\ 198.5 \end{bmatrix}$$

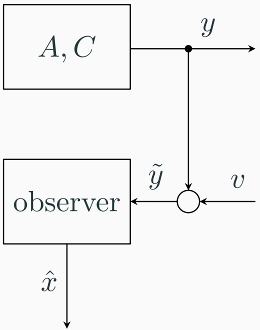
- the gain is extremely high;
- the transient is extremely short;
- however, problem may arise due to measurement noise, as shown next.



Example (cont.)

Let us take into account a measurement noise v:

$$\begin{cases} \dot{\hat{x}} & = & A\,\hat{x} + L(y + v - \hat{y}) \\ \hat{y} & = & C\,\hat{x} \end{cases}$$



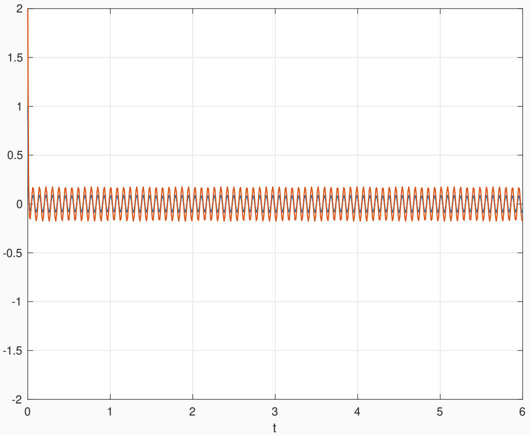
It is easy to check that the error obeys the following:

$$\dot{e}(t) = (A - LC)e(t) - Lv(t)$$

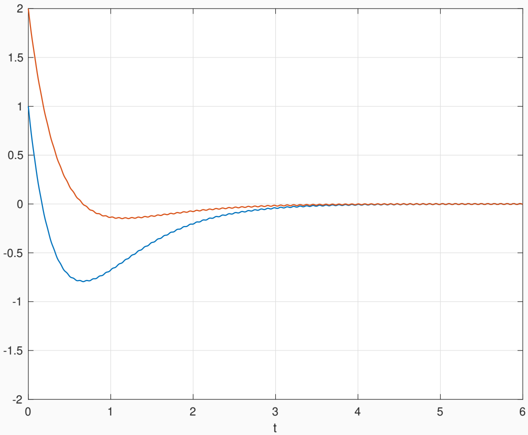
In particular, let

$$v(t) = 0.2\sin(80t)\,.$$

Example (cont.)



Observer D



Observer B

The slower observer B is less sensitive to the measurement noise than observer D.

Example (cont.)

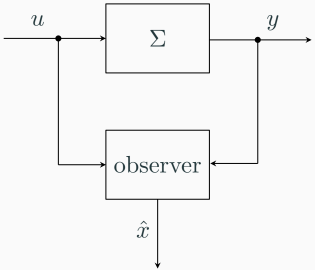
Homework:

1. compute the transfer functions from v to e_1 and e_2, in the cases B and D;
2. plot and compare the Bode diagrams.

Discrete-time LTI systems

There is a strict analogy with the continuous-time case. Given the discrete-time system

$$\begin{cases} x(t+1) & = & A\,x(t) + B\,u(t) \\ y(t) & = & C\,x(t) + D\,u(t) \end{cases}$$



the Luenberger observer is a defined as

$$\begin{cases} \hat{x}(t+1) & = & A\,\hat{x}(t) + B\,u(t) + L(y(t) - \hat{y}(t)) \\ \hat{y}(t) & = & C\,\hat{x}(t) + D\,u(t) \end{cases}$$

or, more compactly:

$$\hat{x}(t+1) = (A - LC)\hat{x}(t) + [B - LD \quad L] \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}.$$

Discrete-time LTI systems (cont.)

Analogously to the continuous-time case, the estimation error

$$e(t) = x(t) - \hat{x}(t)$$

is governed by the difference equation:

$$e(t+1) = (A - LC)e(t).$$

The eigenvalues of $(A - LC)$ can be assigned arbitrarily if and only if the pair (A, C) is observable.

The detectability also has an analogous definition:

Definition

The system

$$\begin{cases} x(t+1) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{cases}$$

is said to be *detectable* (and likewise the pair (A, C)) if its unobservable eigenvalues (if any) have modulus strictly less than 1.

Of course, the eigenvalues of $(A - LC)$ must belong to the open unit disk in order to guarantee the convergence to zero of the estimation error, independent of the initial condition.

Finally, as in the continuous-time case, the choice of the eigenvalues to be assigned must take into account two opposite needs:

- quick convergence of the estimation;
- insensitivity to measurement noise.

Deadbeat observer

The *deadbeat observer* is an observer designed in such a way that **the estimation error goes to zero in *finite* time**.

Contrary to the continuous-time systems, in the case of discrete-time systems it can be obtained by assigning suitable eigenvalues to the closed-loop matrix.

Indeed, if L is chosen to place all the eigenvalues of $A - LC$ in the origin:

$$\sigma(A - LC) = \{0, \dots, 0\},$$

then the characteristic polynomial of $A - LC$ becomes $p(z) = z^n$, where n is the order of the system. Thus, by the Cayley-Hamilton theorem, we have:

$$p(A - LC) = (A - LC)^n = 0.$$

As a consequence, we get

$$e(n) = x(n) - \hat{x}(n) = (A - LC)^n e(0) = 0.$$

Deadbeat observer (cont.)

It can be shown that the actual number of steps depends on the size of the largest Jordan block associated to the null eigenvalue (which is the sole eigenvalue).

Let J be the Jordan canonical form of $A - LC$. We have

$$(A - LC)^k = (TJT^{-1})^k = TJ^kT^{-1}$$

We know that J is a diagonal block matrix

$$J = \text{diag}\{J_1, \dots, J_m\}$$

where each block takes the form

$$J_i = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \\ & & & 1 \\ 0 & & \dots & 0 \end{bmatrix}. \tag{2}$$

Deadbeat observer (cont.)

By the properties of block matrices we have

$$J^k = \text{diag}\{J_1^k, \ldots, J_m^k\}$$

and, given the special form (2), the block J_i^k is null if and only if $k \geq n_i$, where n_i is the block dimension.

Denoting by

$$\bar{k} = \max_{i=1,\ldots,m} \{n_i\}$$

the dimension of the largest block, we have

$$(A - LC)^k = 0 \qquad \Longleftrightarrow \qquad k \geq \bar{k}.$$

Example

Consider the observable pair

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

To assign zero eigenvalues to $A - LC$ we need to assign the characteristic polynomial

$$\alpha_d(z) = z^3.$$

By applying the Ackermann's dual formula we get

$$L = \alpha_d(A)O^{-1}e_n = A^3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -5 \end{bmatrix}.$$

Deadbeat observer (cont.)

Thus, we have

$$A - LC = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \\ -5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -3 & 0 & 1 \\ 5 & 2 & -1 \end{bmatrix}$$

and

$$(A - LC)^2 = \begin{bmatrix} -2 & 1 & 1 \\ 2 & -1 & -1 \\ -6 & 3 & 3 \end{bmatrix}, \qquad (A - LC)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

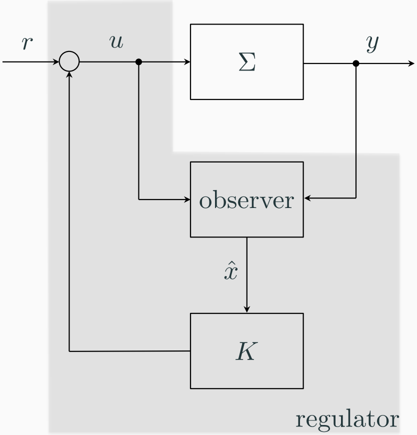
Thus, the error goes to zero in at most three steps.

Output feedback

Output feedback

When the state x cannot be measured, one could be tempted to use the state estimate $\hat{x}$ instead of the actual state x in the feedback law, thus obtaining

$$u = K\hat{x} + r.$$



In this case, the regulator is, overall, a dynamic system, composed of a dynamic part (the observer) and a static one (the feedback gain). In the following we analyze the properties of such a control scheme.

Consider the system

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

and the state observer

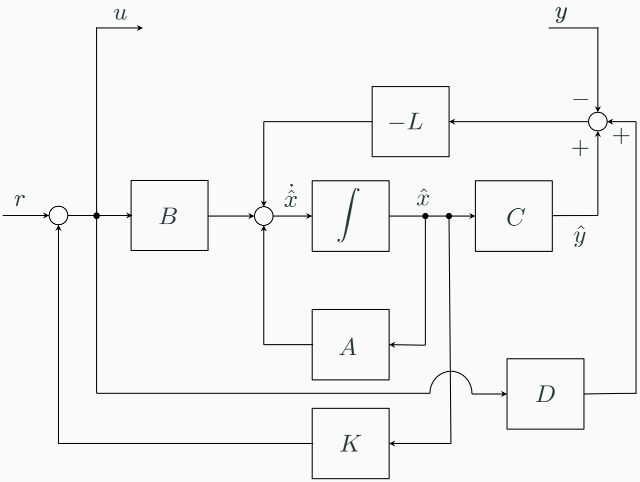
$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x} - Du).$$

We now apply the feedback law

$$u = K\hat{x} + r,$$

and analyze the properties of the obtained closed-loop system, whose more detailed block scheme is reported in the next slide.

Time-domain analysis (cont.)



The observer's equation can be written as

$$\dot{\hat{x}} = (A - LC)\hat{x} + \begin{bmatrix} B - LD & L \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}.$$

Recalling that $y = Cx + Du$, we get

$$\begin{aligned} \dot{\hat{x}} &= (A - LC)\hat{x} + LCx + Bu \\ &= (A - LC)\hat{x} + LCx + BK\hat{x} + Br \\ &= (A - LC + BK)\hat{x} + LCx + Br \end{aligned}$$

On the other hand, we have

$$\dot{x} = Ax + BK\hat{x} + Br$$

Thus, the whole closed-loop system is governed by:

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \overbrace{\begin{bmatrix} A & BK \\ LC & A - LC + BK \end{bmatrix}}^{A_{cl}} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \overbrace{\begin{bmatrix} B \\ B \end{bmatrix}}^{B_{cl}} r$$

$$y = \underbrace{[C \;\; DK]}_{C_{cl}} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + Dr$$

We now perform a change of basis, to analyze the stability of the system.

Let the new state vector be:

$$\begin{bmatrix} x \\ x - \hat{x} \end{bmatrix} = \begin{bmatrix} x \\ e \end{bmatrix}$$

It can be obtained by the change of basis

$$T \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} x \\ e \end{bmatrix}$$

Time-domain analysis (cont.)

It is immediate to check that $T = T^{-1}$, thus we get

$$T^{-1}A_{cl}T = \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix}, \qquad T^{-1}B_{cl} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$C_{cl}T = \begin{bmatrix} C+DK & -DK \end{bmatrix}$$

As a consequence, the new representation of the closed-loop system becomes

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} C+DK & -DK \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + Dr$$

Time-domain analysis (cont.)

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A + BK & -BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} C + DK & -DK \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + Dr$$

The block-triangular structure of the closed-loop state matrix has a fundamental consequence: the closed-loop eigenvalues are the union of the eigenvalues of $A + BK$ and $A - LC$:

$$\sigma(A_{cl}) = \sigma(A + BK) \cup \sigma(A - LC)$$

Such eigenvalues, under the reachability and observability hypotheses can be arbitrarily and independently assigned by a proper choice of K and L.

Separation theorem

We have thus proven the *separation theorem*:

Theorem

The closed-loop of the system

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

and the output feedback controller

$$\begin{cases} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - Cx - Du) \\ u &= K\hat{x} + r \end{cases}$$

results in a system whose eigenvalues are the union of the eigenvalues of the state feedback closed-loop matrix $A + BK$ with the eigenvalues of the observer matrix $A - LC$.

Separation theorem (cont.)

The theorem suggests a two-step procedure to design a controller when the state is not accessible:

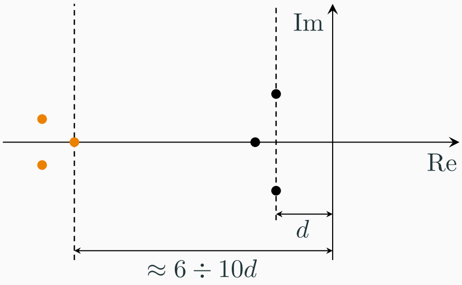
1. design the controller *as if* the state was accessible (thus, choose K in order to assign the desired eigenvalues to $A + BK$);
2. if the state is not accessible, design a state observer by assigning the desired eigenvalues to $A - LC$ by a proper choice of L.

Neither of the two choices influences the other and therefore the state and the error of estimation will evolve according to the respective assigned dynamics. This is called the *separation property*.

A rule of thumb

We have seen that the eigenvalues of $A - LC$ have to be chosen to guarantee a sufficient speed of convergence of the estimation error. What a sufficient speed is, depends on the dynamics of the system whose state is to be estimated.

A rule of thumb is the following: choose the observer's eigenvalues (orange, in the figure) from 6 to 10 times farther from the imaginary axis than the eigenvalues of $A + BK$ (black, in the figure).



Observation

Recall that the closed-loop system is governed by:

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

As it is clear, the closed-loop system is not reachable by the external input r. In particular, the estimation error e does not depend on r. Such a property is desirable, because the error

$$e(t) = x(t) - \hat{x}(t)$$

must converge to zero independent of the external input r.

Example

Consider the system:

$$\begin{cases} \dot{x} &= A\,x + B\,u \\ y &= C\,x \end{cases} \quad \text{where} \quad A = \begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

By letting

$$K = \begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix},$$

the eigenvalues of $A + BK$ are placed in correspondence to the roots of

$$\alpha(s) = (s+1)(s+1) = s^2 + 2s + 1.$$

Considering now the observer

$$\begin{cases} \dot{\hat{x}} &= A\,\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} &= C\,\hat{x} \end{cases} \quad \text{where} \quad L = \begin{bmatrix} d_0 - 2 \\ d_1 - 2 \end{bmatrix}$$

it can be easily checked that the eigenvalues of $A - LC$ are the roots of

$$\alpha_d(s) = s^2 + d_1 s + d_0.$$

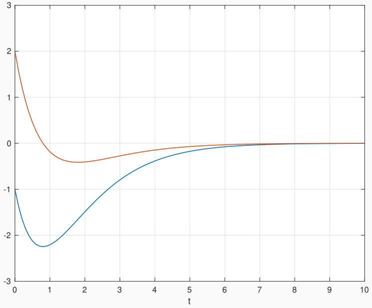
Example (cont.)

State feedback

By applying the state feedback $u = Kx$, from the initial state

$$x(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

we get the transient shown in the figure.



We now apply the feedback of the estimated state, using three different observers.

Case A

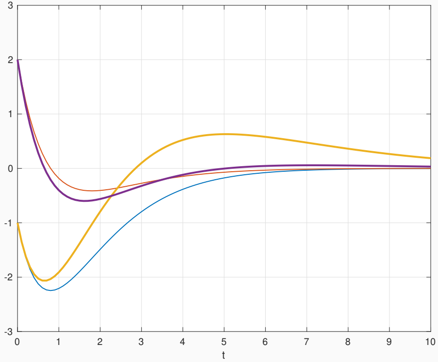
$$\alpha_d^{(1)}(s) = (s + \frac{1}{2})^2 \qquad \Longrightarrow \qquad L^{(1)} = \begin{bmatrix} -\frac{7}{4} \\ -1 \end{bmatrix}$$

- the control law is:

$$u(t) = K\hat{x}(t);$$

- the observer is initialized at zero:

$$\hat{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$



Case B

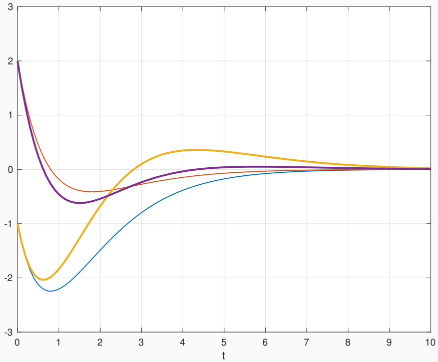
$$\alpha_d^{(2)}(s) = (s+5)^2 \qquad \Longrightarrow \qquad L^{(2)} = \begin{bmatrix} 23 \\ \\ 8 \end{bmatrix}$$

- the control law is:

$$u(t) = K\hat{x}(t);$$

- the observer is initialized at zero:

$$\hat{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$



Example (cont.)

Case C

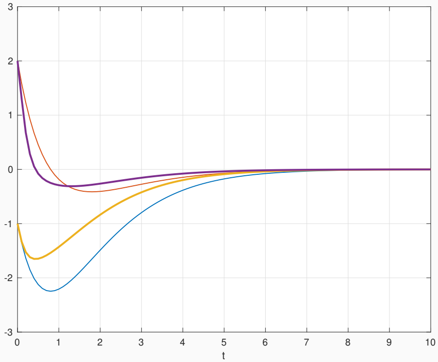
$$\alpha_d^{(3)}(s) = (s+10)^2 \qquad \Longrightarrow \qquad L^{(3)} = \begin{bmatrix} 98 \\ \\ 18 \end{bmatrix}$$

- the control law is:

$$u(t) = K\hat{x}(t);$$

- the observer is initialized at zero:

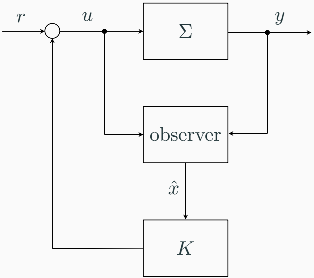
$$\hat{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$



Observations

The transients are clearly different.
Apparently, the control system of the Case C (which is based on the feedback of the estimated state) behaves better than the full state feedback system (the transient is shorter). However:

- the fact that a particular transient is shorter does not mean that this always occurs (it depends on the initialization of the observer);
- the performance should be evaluated also considering the moderation of the control action (strong controls may not be desirable or even admissible);
- always keep in mind that high observer gains expose you to the risk of amplifying measurement noise.



Consider the estimated state feedback:

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} &= \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r \\ y &= [C+DK \quad -DK] \begin{bmatrix} x \\ e \end{bmatrix} + Dr \end{aligned} \tag{3}$$

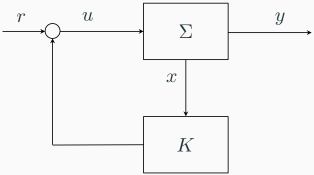
and compute the transfer function $T(s)$ from r to y.

Frequency-domain analysis (cont.)

By discarding the modes associated to e (which certainly do not appear in the transfer function, being unreachable), we get

$$T(s) = [(C + DK)[sI - (A + BK)]^{-1}B + D].$$

Such a transfer function coincides with the transfer function of the system in the following figure (that is, of the full state feedback system).



In other words, **from the input-output point of view, the control system behaves as if there was no observer**. However, recalling the meaning of the transfer function, it is clear that the above equivalence holds only when the transients have expired, or for null initial conditions (including, in particular $e(0) = 0$).

Frequency-domain analysis (cont.)

By assuming nonzero initial conditions, and applying the Laplace transform to equations (3), we get

$$\begin{aligned} Y(s) = {} & (C+DK)[sI-(A+BK)]^{-1}x(0) \\ & -\{(C+DK)[sI-(A+BK)]^{-1}BK[sI-(A-LC)]^{-1} \\ & +DK[sI-(A+BK)]^{-1}\}e(0) + T(s)R(s) \end{aligned}$$

that shows, in terms of transforms, how the output is affected by the initial conditions.

Notice that

- the colored term is due to the observer and is null if $\hat{x}(0) = x(0)$;
- the speed of convergence to zero of the contribution due to $e(0)$ depends on $A + BK$ as well as on $A - LC$.

Parameterization of all stabilizing controllers

Dynamic, output-feedback stabilization problem

Consider the system

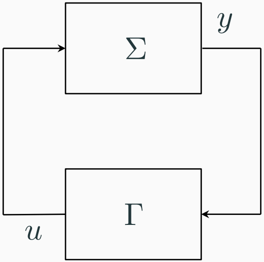
$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$ (the case $D \neq 0$ will be considered later on).

We want to find all the linear controllers (if any) of the form

$$\Gamma : \begin{cases} \dot{\eta}(t) &= \bar{A}\eta(t) + \bar{B}y(t) \\ u(t) &= \bar{C}\eta(t) + \bar{D}y(t) \end{cases}$$

such that the closed-loop system formed by Σ and Γ is (asymptotically) stable.



Dynamic, output-feedback stabilization problem (cont.)

By substituting, we get

$$\begin{cases} \dot{x} &= Ax + B(\bar{C}\eta + \bar{D}y) = Ax + B\bar{C}\eta + B\bar{D}Cx \\ \dot{\eta} &= \bar{A}\eta + \bar{B}Cx \end{cases}$$

or, in compact form:

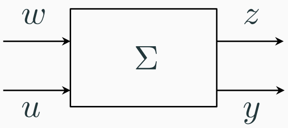
$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \underbrace{\begin{bmatrix} A + B\bar{D}C & B\bar{C} \\ \bar{B}C & \bar{A} \end{bmatrix}}_{\doteq A_{cl}} \begin{bmatrix} x \\ \eta \end{bmatrix}$$

where A_{cl} is the closed-loop matrix. Thus, the stability of the closed-loop depends on the eigenvalues of A_{cl}.

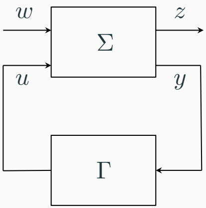
Dynamic, output-feedback stabilization problem (cont.)

We will actually consider a more general problem, in which the system to be controlled is

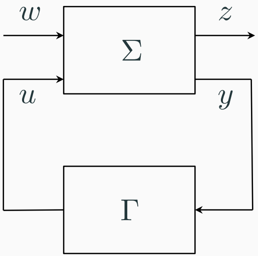
$$\Sigma : \begin{cases} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w \end{cases}$$



and search for the linear controllers Γ such that the closed-loop system in the figure below is stable.



Dynamic, output-feedback stabilization problem (cont.)



In this more general setting:

- u is the vector of the *control inputs*;
- y is the vector of *measured outputs*;
- w is the vector of *exogenous inputs* (containing, for instance, references and disturbances and measurement noise);
- z is the vector of *performance outputs*.

By substituting we get

$$\begin{cases} \dot{x} &= Ax + B_1 w + B_2\left(\bar{C}\eta + \bar{D}(C_2 x + D_{21} w)\right) \\ \dot{\eta} &= \bar{A}\eta + \bar{B}(C_2 x + D_{21} w) \\ z &= C_1 x + D_{11} w + D_{12}\left(\bar{C}\eta + \bar{D}(C_2 x + D_{21} w)\right) \end{cases}$$

or, in compact form:

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} &= \underbrace{\begin{bmatrix} A + B_2\bar{D}C_2 & B_2\bar{C} \\ \bar{B}C_2 & \bar{A} \end{bmatrix}}_{\doteq \tilde{A}_{cl}} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} B_1 + B_2\bar{D}D_{21} \\ \bar{B}D_{21} \end{bmatrix} w \\ z &= \begin{bmatrix} C_1 + D_{12}\bar{D}C_2 & D_{12}C \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \left(D_{11} + D_{12}\bar{D}D_{21}\right) w \end{aligned}$$

Dynamic, output-feedback stabilization problem (cont.)

Let r be the order of the controller (i.e. $\bar{A} \in \mathbb{R}^{r \times r}$). The closed-loop matrices A_{cl} and $\tilde{A}_{cl}$ can be written respectively as:

$$\begin{bmatrix} A + B\bar{D}C & B\bar{C} \\ \bar{B}C & \bar{A} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0_r \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I_r \end{bmatrix} \underbrace{\begin{bmatrix} \bar{D} & \bar{C} \\ \bar{B} & \bar{A} \end{bmatrix}}_{\doteq M} \begin{bmatrix} C & 0 \\ 0 & I_r \end{bmatrix}$$

and

$$\begin{bmatrix} A + B_2\bar{D}C_2 & B_2\bar{C} \\ \bar{B}C_2 & \bar{A} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0_r \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} \bar{D} & \bar{C} \\ \bar{B} & \bar{A} \end{bmatrix} \begin{bmatrix} C_2 & 0 \\ 0 & I_r \end{bmatrix}.$$

The above expressions are formally the same (B_2 and C_2 correspond to B and C, respectively). Thus, for simplicity of notation, we will refer to the first case.

Dynamic, output-feedback stabilization problem (cont.)

The dynamic output feedback stabilization problem can be stated as follows.

Problem statement: dynamic output feedback stabilization problem

Given the matrices A, B, C find, if they exist, an integer $r \geq 0$ and a matrix $M \in \mathbb{R}^{(m+r)\times(p+r)}$ such that

$$\begin{bmatrix} A & 0 \\ 0 & 0_r \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I_r \end{bmatrix} M \begin{bmatrix} C & 0 \\ 0 & I_r \end{bmatrix}$$

is Hurwitz (i.e. all its eigenvalues have strictly negative real part).

In the following, we provide necessary and sufficient conditions for the problem to admit a solution.

We begin with two intermediate results.

Lemma

Let $r \geq 0$ and define the extended matrices

$$A_r^e = \begin{bmatrix} A & 0 \\ 0 & 0_r \end{bmatrix} \qquad \text{and} \qquad B_r^e = \begin{bmatrix} B & 0 \\ 0 & I_r \end{bmatrix}.$$

Then, (A, B) is stabilizable if and only if (A_r^e, B_r^e) is stabilizable.

Proof.
It is an immediate consequence of the PBH test for stabilizability. Indeed, the matrix

$$\left[\begin{array}{c|c} \lambda I_{n+r} - A_r^e & B_r^e \end{array}\right] = \left[\begin{array}{cc|cc} \lambda I - A & 0 & B & 0 \\ 0 & \lambda I_r & 0 & I_r \end{array}\right]$$

has a rank drop for $\lambda : \mathrm{Re}(\lambda) \geq 0$ if and only if the matrix

$$\left[\begin{array}{c|c} \lambda I - A & B \end{array}\right]$$

has a rank drop for the same λ. $\square$

Dynamic, output-feedback stabilization problem (cont.)

Lemma

Let $r \geq 0$ and define the extended matrices

$$A_r^e = \begin{bmatrix} A & 0 \\ 0 & 0_r \end{bmatrix} \quad \text{and} \quad C_r^e = \begin{bmatrix} C & 0 \\ 0 & I_r \end{bmatrix}.$$

Then, (A, C) is detectable if and only if (A_r^e, C_r^e) is detectable.

Proof.
It is an immediate consequence of the PBH test for detectability. □

The following result provides necessary and sufficient conditions for the dynamic output feedback stabilization problem to admit a solution.

Theorem

Given the matrices $A^{n\times n}, B^{n\times m}, C^{p\times n}$, there exists an integer $r \geq 0$ and a matrix $M^{(m+r)\times(p+r)}$ such that

$$\begin{bmatrix} A & 0 \\ 0 & 0_r \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I_r \end{bmatrix} M \begin{bmatrix} C & 0 \\ 0 & I_r \end{bmatrix}$$

is Hurwitz if and only if

1. the pair (A, B) is stabilizable, and
2. the pair (A, C) is detectable.

Proof.
$(\Rightarrow)$

By hypothesis the matrix $A_r^e + B_r^e M C_r^e$ is Hurwitz. By viewing it as

$$A_r^e + \overbrace{B_r^e\, M}^{L^e}\!\!\!\!\underbrace{C_r^e}_{K^e}$$

it is clear that (A_r^e, B_r^e) is stabilizable, because there exists a matrix K^e such that $A_r^e + B_r^e K^e$ is Hurwitz. Therefore, by the first of the previous lemmas, (A, B) is stabilizable.
Similarly, (A_r^e, C_r^e) is detectable, because there exists a matrix L^e such that $A_r^e + L^e C_r^e$ is Hurwitz. Therefore, (A, C) is detectable.

($\Leftarrow$)

By hypothesis, there exist matrices K and L such that

$$A + BK \qquad \text{and} \qquad A + LC$$

are Hurwitz. Take M as follows:

$$M = \begin{bmatrix} 0 & K \\ -L & A + BK + LC \end{bmatrix}$$

(which implies that $r = n$). It is easy to check that the closed-loop matrix becomes

$$A_{cl} = A_r^e + B_r^e M C_r^e = \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix}.$$

Such a matrix can be proven to be Hurwitz by a proper change of basis.

Dynamic, output-feedback stabilization problem (cont.)

Indeed, take

$$T = \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \qquad \text{whose inverse is} \qquad T^{-1} = \begin{bmatrix} I_n & I_n \\ 0 & I_n \end{bmatrix}.$$

We have:

$$TA_{cl}T^{-1} = \begin{bmatrix} A+LC & 0 \\ -LC & A+BK \end{bmatrix}$$

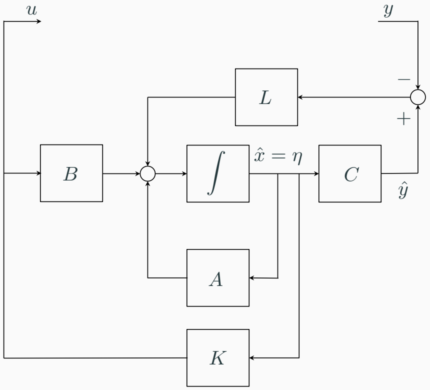
which is Hurwitz. □

Notice that, in view of the separation theorem, we already knew that the stabilizability and detectability are sufficient conditions for stabilizing a system by means of output feedback: indeed they allow for designing an observer-based stabilizing controller. We now have proven that they are also necessary.

Observe also that the controller employed in the second part of the proof, i.e.

$$M = \begin{bmatrix} 0 & K \\ -L & A + BK + LC \end{bmatrix}$$

is nothing more than an observer-based controller, whose block scheme is reported next.



The following theorem (Q-parameterization) is a fundamental result in Control Theory.

Theorem

Given the matrices $A^{n\times n}, B^{n\times m}, C^{p\times n}$. Let (A, B) be stabilizable and (A, C) be detectable. Let $K^{m\times n}$ and $L^{n\times p}$ be such that $A+BK$ and $A+LC$ are Hurwitz. Then:

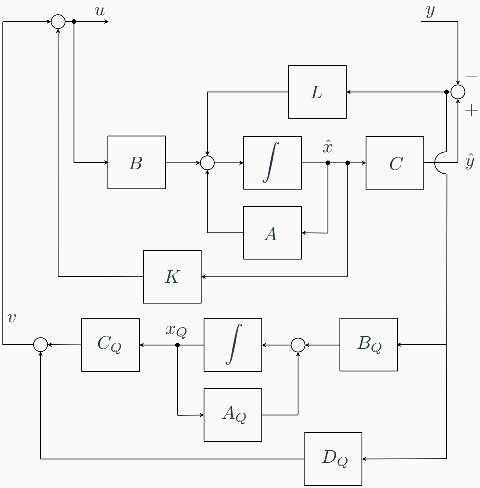
- for every stabilizing controller $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ there exists a stable system (A_Q, B_Q, C_Q, D_Q) such that a realization of the controller, possibly with stable, yet uncontrollable and/or unobservable modes is:

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}} \\ \dot{x}_Q \end{bmatrix} &= \begin{bmatrix} A+BK+LC+BD_QC & BC_Q \\ B_QC & A_Q \end{bmatrix} \begin{bmatrix} \hat{x} \\ x_Q \end{bmatrix} + \begin{bmatrix} -L-BD_Q \\ -B_Q \end{bmatrix} y \\ u &= \begin{bmatrix} K+D_QC & C_Q \end{bmatrix} \begin{bmatrix} \hat{x} \\ x_Q \end{bmatrix} - D_Q y \end{aligned} \tag{4}$$

- for every stable system (A_Q, B_Q, C_Q, D_Q) the controller (4) is stabilizing.

Before proving the theorem, some observations are in order.

The block diagram of the controller is the following:



Q-parameterization (cont.)

Thus, the controller is composed of an observer-based controller (whose state is $\hat{x}$) and a stable linear system (A_Q, B_Q, C_Q, D_Q). The stable linear system is driven by the output estimation error $\hat{y} - y$ and its output v is added to the feedback:

$$u = K\hat{x} + v.$$

The theorem states that **all the linear stabilizing controllers** can be obtained by fixing K and L and varying (A_Q, B_Q, C_Q, D_Q) among the set of the stable linear systems (of any order).

In other words, the family of all the stabilizing controllers for the system (A, B, C) can be viewed as a parametric family, depending on the “parameter” (A_Q, B_Q, C_Q, D_Q). Such a result is known as the *Q-parameterization* or *Youla-Kučera parameterization*.

The result is important because it allows to cast the controller design problem in terms of a search in the parameter space:

- the stability of the closed-loop is guaranteed by the parameterization;
- the parameter can be chosen in order to satisfy some prescribed performance specifications.

Proof.
($\Leftarrow$)

We want to show that for all choices of (A_Q, B_Q, C_Q, D_Q), the controller given by (4) is stabilizing. Let A_Q be Hurwitz and B_Q, C_Q, D_Q be matrices of appropriate dimensions. The closed-loop system obtained by employing the controller (4) is governed by the following equations:

$$
\begin{aligned}
\dot{x} &= Ax + Bu = Ax + B\left[(K + D_Q C)\hat{x} + C_Q x_Q - D_Q y\right] = \\
&\quad Ax + B\left[(K + D_Q C)\hat{x} + C_Q x_Q - D_Q C x\right] \\
\dot{\hat{x}} &= \left(A + BK + LC + BD_Q C\right)\hat{x} + BC_Q x_Q - (L + BD_Q)Cx \\
\dot{x}_Q &= B_Q C\hat{x} + A_Q x_Q - B_Q C x
\end{aligned}
$$

Hence, the closed-loop matrix is

$$
A_{cl} = \begin{bmatrix} A - BD_Q C & B(K + D_Q C) & BC_Q \\ -(L + BD_Q)C & A + BK + LC + BD_Q C & BC_Q \\ -B_Q C & B_Q C & A_Q \end{bmatrix}
$$

Q-parameterization (cont.)

By performing the change of variables:

$$\begin{bmatrix} x \\ \hat{x} \\ x_Q \end{bmatrix} \longrightarrow \begin{bmatrix} x - \hat{x} \\ x_Q \\ \hat{x} \end{bmatrix}$$

we can recognize that A_{cl} is Hurwitz. Indeed, denoting by n_Q the order of the matrix A_Q, the change of variable can be obtained by

$$T = \begin{bmatrix} I_n & -I_n & 0 \\ 0 & 0 & I_{n_Q} \\ 0 & I_n & 0 \end{bmatrix} \quad \text{whose inverse is} \quad T^{-1} = \begin{bmatrix} I_n & 0 & I_n \\ 0 & 0 & I_n \\ 0 & I_{n_Q} & 0 \end{bmatrix}$$

leading to

$$\hat{A}_{cl} = TA_{cl}T^{-1} = \begin{bmatrix} A + LC & 0 & 0 \\ -B_Q C & A_Q & 0 \\ -(L + BD_Q) & BC_Q & A + BK \end{bmatrix}$$

which is clearly Hurwitz.

($\Rightarrow$)

Suppose that the controller $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is stabilizing and let r be its order. Then, the closed-loop matrix

$$A_Q \doteq A_r^e + B_r^e \begin{bmatrix} \bar{D} & \bar{C} \\ \bar{B} & \bar{A} \end{bmatrix} C_r^e$$

is Hurwitz. Define now

$$B_Q \doteq \begin{bmatrix} L - B\bar{D} \\ -\bar{B} \end{bmatrix}, \quad C_Q \doteq \begin{bmatrix} -K + \bar{D}C & \bar{C} \end{bmatrix}, \quad D_Q \doteq -\bar{D}.$$

We want to show that the controller (4), with A_Q, B_Q, C_Q, D_Q defined as above, has the same transfer function (i.e. is a realization) of the controller $\bar{A}, \bar{B}, \bar{C}, \bar{D}$. By substituting in (4) and partitioning x_Q in a way consistent to B_Q:

$$x_Q = \begin{bmatrix} x_{Q_1} \\ x_{Q_2} \end{bmatrix}$$

we find the following state-space equations for the controller.

Q-parameterization (cont.)

$$\begin{aligned}
\dot{\hat{x}} &= (A + BK + LC - B\bar{D}C)\hat{x} + B(-K + \bar{D}C)x_{Q_1} + B\bar{C}x_{Q_2} + (-L + B\bar{D})y \\
\dot{x}_{Q_1} &= (L - B\bar{D})C\hat{x} + (A + B\bar{D}C)x_{Q_1} + B\bar{C}x_{Q_2} + (-L + B\bar{D})y \\
\dot{x}_{Q_2} &= -\bar{B}C\hat{x} + \bar{B}Cx_{Q_1} + \bar{A}x_{Q_2} - \bar{B}y \\
u &= (K - \bar{D}C)\hat{x} + (-K + \bar{D}C)x_{Q_1} + \bar{C}x_{Q_2} + \bar{D}y
\end{aligned}$$

or, in compact form:

$$\left[\begin{array}{c} \dot{\hat{x}} \\ \dot{x}_{Q_1} \\ \dot{x}_{Q_2} \\ \hline u \end{array}\right] = \left[\begin{array}{ccc|c} A + BK + LC - B\bar{D}C & B(-K + \bar{D}C) & B\bar{C} & -L + B\bar{D} \\ (L - B\bar{D})C & A + B\bar{D}C & B\bar{C} & -L + B\bar{D} \\ -\bar{B}C & \bar{B}C & \bar{A} & \bar{B} \\ \hline K - \bar{D}C & -K + \bar{D}C & \bar{C} & \bar{D} \end{array}\right] \left[\begin{array}{c} \hat{x} \\ x_{Q_1} \\ x_{Q_2} \\ \hline y \end{array}\right].$$

We now perform the change of variables:

$$\begin{bmatrix} \hat{x} \\ x_{Q_1} \\ x_{Q_2} \end{bmatrix} \longrightarrow \begin{bmatrix} \hat{x} - x_{Q_1} \\ x_{Q_1} \\ x_{Q_2} \end{bmatrix}$$

corresponding to the change of base matrix:

$$T = \begin{bmatrix} I_n & -I_n & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_r \end{bmatrix},$$

obtaining

$$\left[\begin{array}{c} \dot{\hat{x}} - \dot{x}_{Q_1} \\ \dot{x}_{Q_1} \\ \dot{x}_{Q_2} \\ \hline u \end{array}\right] = \left[\begin{array}{ccc|c} A + BK & 0 & 0 & 0 \\ (L - B\bar{D})C & A + LC & B\bar{C} & -L + B\bar{D} \\ -\bar{B}C & 0 & \bar{A} & \bar{B} \\ \hline K - \bar{D}C & 0 & \bar{C} & \bar{D} \end{array}\right] \left[\begin{array}{c} \hat{x} - x_{Q_1} \\ x_{Q_1} \\ x_{Q_2} \\ \hline y \end{array}\right].$$

The first states (which are stable) are unreachable, thus can be removed without affecting the transfer function.

We obtain:

$$\begin{bmatrix} \dot{x}_{Q_1} \\ \dot{x}_{Q_2} \\ \hline u \end{bmatrix} = \left[\begin{array}{cc|c} A+LC & B\bar{C} & -L+B\bar{D} \\ 0 & \bar{A} & \bar{B} \\ \hline 0 & \bar{C} & \bar{D} \end{array}\right] \begin{bmatrix} x_{Q_1} \\ x_{Q_2} \\ \hline y \end{bmatrix}.$$

The first states (which are stable) are unobservable, thus can be removed without affecting the transfer function. We get:

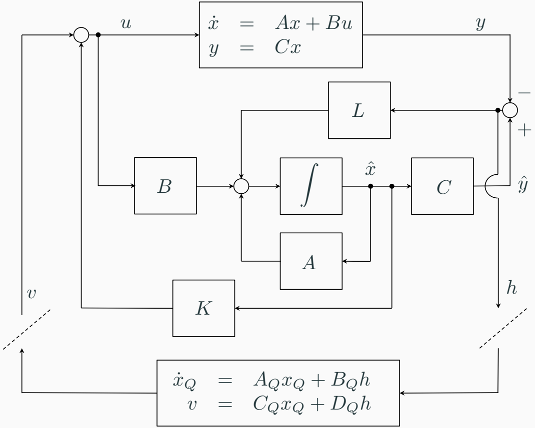
$$\begin{bmatrix} \dot{x}_{Q_2} \\ \hline u \end{bmatrix} = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] \begin{bmatrix} x_{Q_2} \\ \hline y \end{bmatrix},$$

i.e., the stabilizing controller $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$. Thus we have proven that the controller (4), with A_Q, B_Q, C_Q, D_Q defined as above, is a realization of the controller $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$. $\square$

Notice that the controller (4), which is often called the *Q-augmented controller*, is not necessarily minimal, but is certainly stabilizable and detectable (indeed, unobservable and/or unreachable modes may exists but, as shown in the proof, they are stable).

Observation 1

It is worth computing the transfer function "seen by the parameter": it is sufficient to disconnect the system and compute the transfer function from v to $h \doteq \hat{y} - y$, as in the figure below.



Observations (cont.)

We have:

$$\begin{aligned} \dot{x} &= Ax + Bv + BK\hat{x} \\ \dot{\hat{x}} &= A\hat{x} + Bv + BK\hat{x} + L(C\hat{x} - Cx) \\ h &= C\hat{x} - Cx \end{aligned}$$

i.e., in compact form

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} v \; .$$

$$h = \begin{bmatrix} -C & C \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

We now perform the change of variables:

$$\begin{bmatrix} x \\ \hat{x} \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ x - \hat{x} \end{bmatrix}$$

Observations (cont.)

corresponding to the matrix

$$T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}.$$

We thus have (notice that $T = T^{-1}$):

$$T^{-1}AT = \begin{bmatrix} A+BK & -BK \\ 0 & A+LC \end{bmatrix}, \quad TB = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad CT^{-1} = \begin{bmatrix} 0 & -C \end{bmatrix}.$$

Hence:

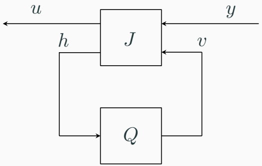
$$\begin{bmatrix} \dot{x} \\ \dot{x} - \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A+BK & -BK \\ 0 & A+LC \end{bmatrix} \begin{bmatrix} x \\ x - \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v$$

$$h = \begin{bmatrix} 0 & -C \end{bmatrix} \begin{bmatrix} x \\ x - \hat{x} \end{bmatrix}.$$

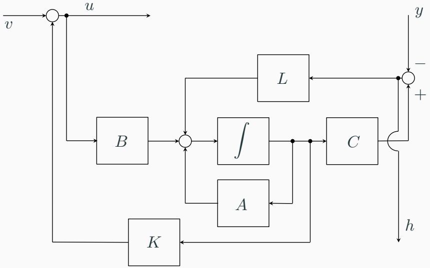
It is immediate to check that all the modes are either unobservable or unreachable: as a consequence, the transfer function from v to h is equal to zero.

Observation 2

The generic stabilizing controller can be represented as in the figure:



where J is the transfer matrix from (y, v) to (u, h), corresponding to the scheme



Observations (cont.)

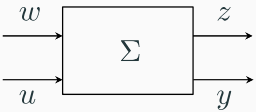
having the following state-space description:

$$\begin{cases} \dot{\hat{x}} = (A + BK + LC)\hat{x} - Ly - Bv \\ u = K\hat{x} + v \\ h = C\hat{x} - y \end{cases}$$

Observations (cont.)

Observation 3

If the system is of type



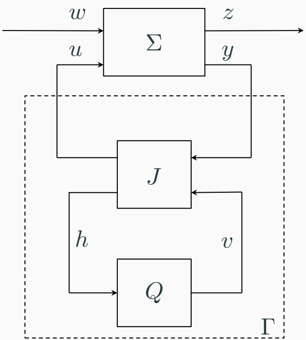
i.e.

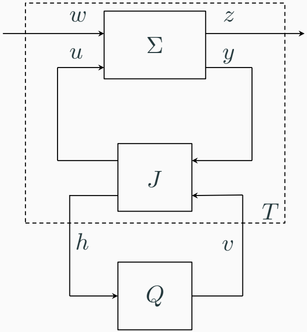
$$\begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w \end{cases}$$

it is sufficient to employ, for the parameterization, the triplet (A, B_2, C_2). Of course, the pairs (A, B_2) and (A, C_2) need to be stabilizable and detectable, respectively. The transfer matrix J will have the following realization

$$\begin{cases} \dot{\hat{x}} = (A + B_2 K + L C_2)\hat{x} - Ly - B_2 v \\ u = K\hat{x} + v \\ h = C_2 \hat{x} - y \end{cases}$$

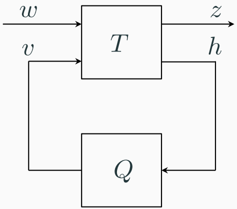
By applying the Q-augmented controller we get the scheme on the left, which is equivalent to that on the right, obtained by grouping in a different way.





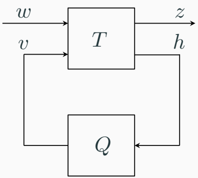
Closed-loop transfer matrix (cont.)

Thus, we have



where $T = \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & 0 \end{bmatrix}$ (we have already proven that $T_{22} = 0$). We can write

$$\begin{cases} z = T_{11}(s)w + T_{12}(s)v \\ h = T_{21}(s)w \\ v = Q(s)h \end{cases} \implies z = \underbrace{\left(T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)\right)}_{T_{cl}(s)} w.$$



Thus, we have the following important property.

Property

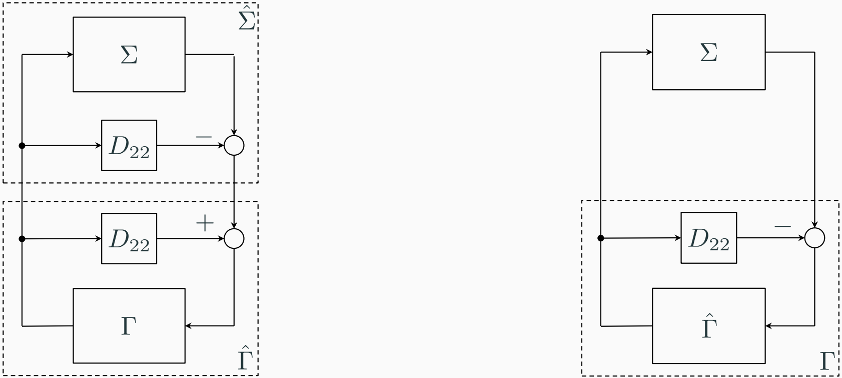
The Q-parameterization results in a closed-loop transfer function $T_{cl}(s)$ between the exogenous inputs w and the performance outputs z of the form

$$T_{cl}(s) = T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)$$

where $Q(s) = C_Q(sI - A_Q)^{-1}B_Q + D_Q$ and $T_{11}(s), T_{12}(s), T_{21}(s)$ are stable transfer matrices. In other words, the closed-loop transfer matrix is **affine in** $Q(s)$.

The property is important because affine dependence considerably simplifies the search in the parameter space, leading to a design technique known as Q-*design*.

To conclude, we show how to remove the hypothesis $D_{22} = 0$. It is sufficient to add and subtract D_{22} in the loop, as in figure, obtaining an auxiliary system $\hat{\Sigma}$ such that $\hat{D}_{22} = 0$.



Then, the family of all controllers $\hat{\Gamma}$ that stabilize $\hat{\Sigma}$ is found via Q-parameterization. The final controller Γ is obtained from $\hat{\Gamma}$ by feeding back the signal $-D_{22}u$.

Applying feedback to nonlinear systems

Applying feedback to nonlinear systems

Consider the nonlinear system (which is strictly proper, for ease of calculation):

$$\begin{cases} \dot{x}(t) & = & f(x(t), u(t)) \\ y(t) & = & g(x(t)), \end{cases}$$

and let $(\bar{x}, \bar{u}) = (0, 0)$ be an equilibrium pair. By applying the Taylor's series expansion, the previous can be written as

$$\begin{cases} \dot{x}(t) & = & Ax(t) + Bu(t) + \zeta(x(t), u(t)) \\ y(t) & = & Cx(t) + \xi(x(t)), \end{cases} \tag{5}$$

where the residuals $\zeta(x, u)$ and $\xi(x)$, are infinitesimals of order greater than 1 with respect to their arguments.

By neglecting the residuals, we obtain a linear system $\Sigma(A, B, C)$ (i.e., the linearized system around the considered equilibrium):

$$\Sigma = \begin{cases} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t). \end{cases}$$

Applying feedback to nonlinear systems (cont.)

In the previous sections we have seen some techniques for designing regulators for linear systems.

One may ask whether a stabilizing regulator designed based on Σ (i.e., capable of stabilizing Σ) is able to stabilize the equilibrium of the nonlinear system as well. Remarkably, the answer is positive.

Indeed, let Γ be a stabilizing regulator for the linear system Σ (no matter how it has been obtained):

$$\Gamma : \begin{cases} \dot{\eta}(t) & = & \bar{A}\eta(t) + \bar{B}y(t) \\ u(t) & = & \bar{C}\eta(t) + \bar{D}y(t). \end{cases} \tag{6}$$

Applying the regulator to the nonlinear system amounts to joining (5) and (6), thus obtaining:

$$\begin{aligned} \dot{x} &= Ax + B(\bar{C}\eta + \bar{D}y) + \zeta(x,u) = \\ &= Ax + B\bar{C}\eta + B\bar{D}y + \zeta(x,u) = \\ &= Ax + B\bar{C}\eta + B\bar{D}(Cx + \xi(x)) + \zeta(x,u) = \\ &= (A + B\bar{D}C)x + B\bar{C}\eta + B\bar{D}\xi(x) + \zeta(x,u) \\ \dot{\eta} &= \bar{A}\eta + \bar{B}(Cx + \xi(x)) = \\ &= \bar{A}\eta + \bar{B}Cx + \bar{B}\xi(x). \end{aligned}$$

Applying feedback to nonlinear systems (cont.)

The previous can be written in compact form as:

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \underbrace{\begin{bmatrix} A + B\bar{D}C & B\bar{C} \\ \bar{B}C & \bar{A} \end{bmatrix}}_{\doteq A_{cl}} \begin{bmatrix} x \\ \eta \end{bmatrix} + \underbrace{\begin{bmatrix} \zeta(x,u) + B\bar{D}\xi(x) \\ \bar{B}\xi(x) \end{bmatrix}}_{\text{infinitesimal of order greater than } 1},$$

where the last term is easily proven to be an infinitesimal of order greater than 1 with respect to x and η.

The previous is the state equation of the overall nonlinear control system, obtained by applying the linear regulator Γ to the nonlinear system. Clearly, such a control system has the origin as an equilibrium state. The stability of the equilibrium can be studied by linearizing the system, which amounts to neglecting the residual. In other words, recalling the Lyapunov's indirect method, the stability depends on the eigenvalues of A_{cl}.

On the other hand, A_{cl} is precisely the closed-loop matrix obtained when applying Γ to the linear system Σ. Since Γ stabilizes Σ, the eigenvalues of A_{cl} have strictly negative real part, implying the asymptotic stability of the equilibrium.

References

References

Hautus, M. (1977). A simple proof of Heymann's Lemma. *IEEE Transactions on Automatic Control*, 22:885 – 886.

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Lecture 6

State feedback and output feedback

END