Control Theory

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Lecture 7: Optimal control

Introduction

Roughly speaking, by optimal control we mean the formulation of a control problem in terms of an optimization problem.

Optimal control problem

Consider the system

$$\dot{x}(t) = f(x(t), u(t)) \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

where $f(\cdot, \cdot) \in \mathcal{C}^1$ (i.e., is continuously differentiable).
Given the initial time t_0, the initial state $x(t_0)$ and the "horizon" T, the optimal control problem is: find the input

$$u^o(t), \quad t \in [t_0, T]$$

such that the following cost is minimized:

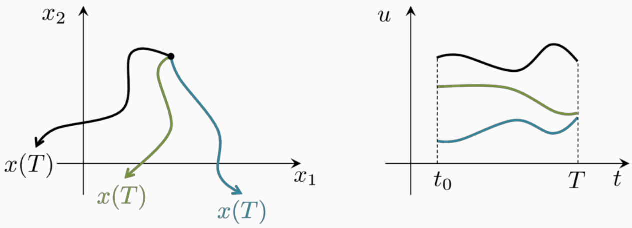
$$J(x(t_0), u(\cdot), t_0) = \int_{t_0}^{T} l(x(\tau), u(\tau))d\tau + m(x(T)),$$

where $l : \mathbb{R}^{n+m} \to \mathbb{R}^+$ and $m : \mathbb{R}^n \to \mathbb{R}^+$ are both continuously differentiable.

The meaning of the previous formulation is the following.

At time t_0 the state is $x(t_0)$. To each control action $u(\cdot)$ corresponds a movement (of the state and the input) and thus a value of the cost J.

The goal is to choose the control action resulting in the minimum of that cost.



Introduction (cont.)

Notice that the cost

$$J(x(t_0), u(\cdot), t_0) = \int_{t_0}^{T} l(x(\tau), u(\tau))d\tau + m(x(T))$$

depends on some design parameters, in particular

- the horizon T;
- the *running cost function* $l(\cdot, \cdot)$;
- the *terminal cost function* $m(\cdot)$.

Thus, the optimality of $u^o(\cdot)$ is relative to those parameters.

A typical choice for the running cost and terminal cost is:

$$l(x, u) = x^\top Qx + u^\top Ru$$
$$m(x) = x^\top Sx$$

where $Q, S \succeq 0$ and $R \succ 0$.

Introduction (cont.)

We thus obtain a *quadratic* cost:

$$J(x(t_0), u(\cdot), t_0) = \int_{t_0}^{T} x^\top(\tau)Qx(\tau) + u^\top(\tau)Ru(\tau)d\tau + x^\top(T)Sx(T).$$

Such a choice corresponds to the following goals:

- steer x close to the origin (S);
- keep x close to the origin during the transient (Q);
- achieve the previous goals using a “small” control (R).

In the scalar case ($n = 1, m = 1$) we have:

$$\begin{aligned} l(x,u) &= qx^2 + ru^2 \\ m(x) &= sx^2 \end{aligned} \qquad q, s \geq 0, \quad r > 0$$

and the choice of q, r, s reflects the relative importance of the three objectives (that are different and conflicting).

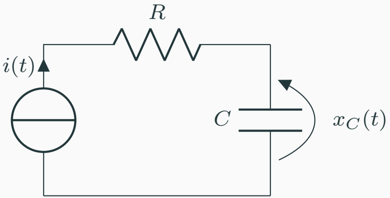
Example

Consider the circuit in figure and let $q(t)$ be the charge of the capacitor at time t. We have

$$i(t) = \frac{dq(t)}{dt} = \frac{dCx_C(t)}{dt} = C\dot{x}_C(t)$$

Thus, by letting $u(t) = i(t)$ and $x(t) = x_C(t)$:

$$\dot{x}(t) = \frac{u(t)}{C}$$



We formulate the following control problem: for a given $x_C(t_0) = \bar{x}_C \neq 0$, minimize both

- the residual charge $q(T)$;
- the energy dissipated by the resistor during the interval $[t_0, T]$.

The two objectives can be expressed through the cost

$$J = s x_C^2(T) + \int_{t_0}^{T} R i^2(\tau) d\tau, \qquad s \geq 0$$

Different values of $s \geq 0$ correspond to different relative importance of the two objectives. For instance, in the extreme case when $s = 0$, the optimal solution is

$$u(t) = 0, \quad \forall t \in [t_0, T]$$

achieving $J = 0$. Notice that, by definition, the cost J is positive, thus any input achieving $J = 0$ is certainly optimal.

Two approaches

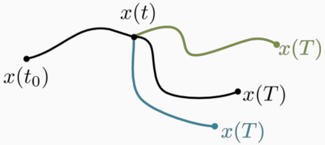
There exist two different approaches to the optimal control:

- Pontryagin's *maximum principle* (Pontryagin (1987));
- Bellman's *dynamic programming* (Bellman (1957)).

The former is based on the calculus of variations and provides an open-loop solution. The latter provides a closed-loop solution, but is often intractable both analytically and numerically (with the notable exception of the linear-quadratic problem).

In the following we present the dynamic programming approach. Such approach is based on the *principle of optimality*.

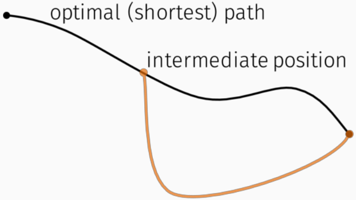
The principle of optimality



Consider an optimal control problem over the time interval $[0, T]$ and suppose the solution is known. Suppose to follow the optimal trajectory to a time t, arriving at state $x(t)$. Now, consider a new problem, initiated at time t and state $x(t)$. Under broad assumptions, **the solution to the new problem is the remainder of the original solution**.

The Principle of Optimality: From any point on an optimal trajectory, the remaining trajectory is optimal for the corresponding problem initiated at that point.

The principle of optimality (cont.)



Example (Shortest path to home) From a given location in the city, find the minimum distance path to home. In this case, the objective function is the path length (to be minimized). Suppose to solve the problem and start traveling along the optimal path. The state is the current position. From any intermediate position, the optimal path is clearly the remaining part of the original optimal path.

Optimal cost-to-go

Let $u_{[a,b]}$ denote the set of bounded continuous functions $u(\cdot)$ defined in the interval $[a, b]$.

Definition

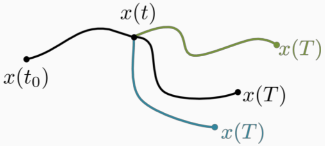
With reference to the control problem introduced before, the function

$$
\begin{aligned}
J^o(x,t) &= \min_{u[t,T]} J(x, u(\cdot), t) \\
&= \min_{u[t,T]} \left\{ \int_t^T l(x(\tau), u(\tau))d\tau + m(x(T)) \right\}
\end{aligned}
$$

where $t \in [t_0, T]$ is said the *optimal cost-to-go*.

In other words, the optimal cost-to-go is the optimal value of the cost for the subproblem starting at time $t \in [t_0, T]$ and state x, and ending at time T.

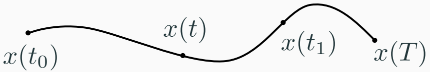
Optimal cost-to-go (cont.)



Clearly, $J^o(x, t)$ depends on the initial state of subproblem (i.e., x) and on the time t at which the subproblem starts (since as t changes, the overall optimization interval changes and the optimal cost may change as well).

In the following, we will focus on how the optimal cost-to-go changes as the initial state x of the subproblem changes in time. For that reason, we will write $J^o(x(t), t)$.

Optimal cost-to-go (cont.)



In view of the principle of optimality, the optimal cost-to-go has the following property:

$$J^o(x(t),t) = \min_{u[t,t_1]} \left\{ \int_t^{t_1} l(x(\tau),u(\tau))d\tau + J^o(x(t_1),t_1) \right\}$$

where $x(t_1)$ is an intermediate state of the optimal trajectory.

From a teaching perspective, it is convenient to introduce the dynamic programming approach for discrete-time systems, and subsequently move back to continuous-time systems.

Hamilton-Jacobi-Bellman equation for discrete-time systems

Consider the discrete-time system:

$$x(k+1) = f(x(k), u(k))$$

and the cost

$$J(x(k_0), u(\cdot), k_0) = \sum_{k=k_0}^{N-1} l(x(k), u(k)) + m(x(N))$$

and let $k_0 \leq k \leq N$.

By applying the principle of optimality, the optimal cost-to-go can be written as:

$$J^o(x,k) = \overbrace{l(x, u^o(x,k))}^{\text{cost of the first optimal step}} + \underbrace{J^o\left(f(x, u^o(x,k)), k+1\right)}_{\text{optimal cost-to-go from the arrival state}}$$

But $u^o(x,k))$, being the optimal input at time k if the state is x, is indeed the control value u that minimizes

$$l(x,u) + J^o\left(f(x,u), k+1\right).$$

Hamilton-Jacobi-Bellman equation for discrete-time systems (cont.)

Thus, we can write

$$J^o(x,k) = \min_u \left\{ l(x,u) + J^o\left(f(x,u), k+1\right) \right\}$$

which is the *Hamilton-Jacobi-Bellman equation* for discrete-time systems. The companion boundary condition is

$$J^o(x,N) = m(x).$$

The optimal control is thus

$$u^o(x,k) = \operatorname*{argmin}_u \left\{ l(x,u) + J^o\left(f(x,u), k+1\right) \right\}. \tag{1}$$

In other words, for each state x and time k, the optimal control $u^o(x,k)$ is the one that minimizes the sum of

1. the running cost of the first step $l(x,u)$, and
2. the optimal cost-to-go from the state reached, at time $k+1$, by applying the control u.

The HJB equation admits a “backward in time” solution, starting from the boundary condition at time N:

$$
\begin{aligned}
J^o(x, N) &= m(x) \\
J^o(x, N-1) &= \min_u \{ l(x,u) + J^o(f(x,u), N) \} \\
J^o(x, N-2) &= \min_u \{ l(x,u) + J^o(f(x,u), N-1) \} \\
&\vdots \\
J^o(x, k_0) &= \min_u \{ l(x,u) + J^o(f(x,u), k_0+1) \}
\end{aligned}
$$

The functions on the left side of the previous equations represent the solution of the HJB equation. Solving the HJB equation allows to compute the optimal control by the (1).

Hamilton-Jacobi-Bellman equation for discrete-time systems (cont.)

Algorithm Backward solution of the HJB equation

Input: $f(\cdot,\cdot), m(\cdot), l(\cdot,\cdot), N, k_0$

Output: $J^o(x,k)$, $k = k_0, \dots, N$

```
J^o(x, N) ← m(x)
k ← (N − 1)
while k ≥ k_0 do
    J^o(x, k) ← min_u {l(x, u) + J^o(f(x, u), k + 1)}
    k ← (k − 1)
end while
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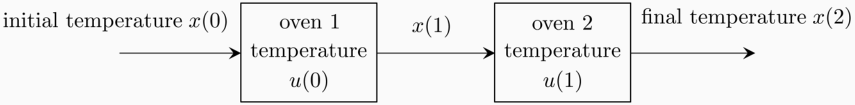
Example (Two ovens)



A certain material is passed through two ovens in sequence. The goal is to reach a final temperature $\bar{x}$ with minimum energy consumption.

- $x(0)$: initial temperature of the material;
- $x(k)$: temperature of the material at the exit of oven k;
- $u(k)$: average temperature of oven $k+1$.

The dynamics and the cost are thus

$$\begin{aligned} x(k+1) &= (1-a)x(k) + au(k) \\ J &= u^2(0) + u^2(1) + \underbrace{r(x(2)-\bar{x})^2}_{\text{terminal cost}}, \quad r > 0 \end{aligned}$$

The final optimal cost-to-go is

$$J^o(x,2) = r(x-\bar{x})^2.$$

The first iteration leads to

$$J^o(x,1) = \min_u \left\{u^2 + J^o(f(x,u),2)\right\} = \min_u \left\{u^2 + r((1-a)x + au - \bar{x})^2\right\},$$

and the minumum is attained by

$$u = \frac{ra(\bar{x}-(1-a)x)}{1+ra^2}.$$

By substitution, we get

$$J^o(x,1) = \frac{r((1-a)x-\bar{x})^2}{1+ra^2}.$$

The second iteration leads to:

$$J^o(x,0) = \min_u \left\{u^2 + J^o(f(x,u),1)\right\} = \min_u \left\{u^2 + \frac{r((1-a)((1-a)x+au)-\bar{x})^2}{1+ra^2}\right\},$$

and the minimum is attained by

$$u = \frac{r(1-a)a(\bar{x}-(1-a)^2x)}{1+ra^2(1+(1-a)^2)}.$$

Finally, by substitution, we obtain:

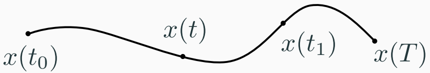
$$J^o(x,0) = \frac{r((1-a)^2x - \bar{x})^2}{1 + ra^2(1 + (1-a)^2)}.$$

Thus, the optimal input sequence is:

$$u(0) = \frac{r(1-a)a\left(\bar{x} - (1-a)^2x(0)\right)}{1 + ra^2(1 + (1-a)^2)}, \qquad u(1) = \frac{ra\left(\bar{x} - (1-a)x(1)\right)}{1 + ra^2}.$$

Notice that the optimal input sequence takes a form of a feedback control law (the input $u(k)$ is a function of the state $x(k)$).

The idea of solving the optimal control problem "backward in time" applies to the continuous-time systems as well, albeit the maths are more involved.



Recall that, in view of the principle of optimality, the optimal cost-to-go has the following property:

$$J^o(x(t), t) = \min_{u[t,t_1]} \left\{ \int_t^{t_1} l(x(\tau), u(\tau))d\tau + J^o(x(t_1), t_1) \right\}$$

where $x(t_1)$ is an intermediate state of the optimal trajectory.

HJB equation for continuous-time systems (cont.)

Let $t_1 = t + dt$. By the mean value theorem there exists $\alpha \in]0,1[$ such that:

$$J^o(x(t),t) = \min_{u[t,t+dt]} \{l(x(t+\alpha dt), u(t+\alpha dt))dt + J^o(x(t+dt), t+dt)\}$$

On the other hand we can write:

$$J^o(x(t+dt), t+dt) = J^o(x(t),t) + \frac{\partial J^o(x(t),t)}{\partial x}\frac{dx}{dt}dt + \frac{\partial J^o(x(t),t)}{\partial t}dt + o(dt).$$

By substituting, simplifying and dividing by dt we get

$$0 = \min_{u[t,t+dt]} \left\{ l(x(t+\alpha dt), u(t+\alpha dt)) + \frac{\partial J^o(x(t),t)}{\partial x} f(x(t),u(t)) + \frac{\partial J^o(x(t),t)}{\partial t} + \frac{o(dt)}{dt} \right\}$$

In the limit as $dt \to 0$ the higher order term $o(dt)/dt$ disappears and the minimum is taken over the instantaneous value of u at time t.

Pulling $\dfrac{\partial J^o(x(t),t)}{\partial t}$ outside the minimum, as it does not depend on u, we obtain

$$\frac{\partial J^o(x,t)}{\partial t} = -\min_u \left\{ l(x,u) + \frac{\partial J^o(x,t)}{\partial x} f(x,u) \right\}. \tag{2}$$

Notice that, as t is fixed, x and u are vectors (and not functions of time), thus the dependency on t has been dropped.

Equation (2) is a partial differential equation known as the *Hamilton-Jacobi-Bellman equation* (HJB equation). The HJB equation must hold for any $t \in [t_0, T[$, and is associated to the boundary condition:

$$J^o(x,T) = m(x). \tag{3}$$

Based on the HJB equation, the solution to the optimal control problem can be found in two steps:

1. find u^o that minimizes
$$\left\{ l(x,u) + \frac{\partial J^o(x,t)}{\partial x} f(x,u) \right\},$$
having the form:
$$u^o = k\left(x, \frac{\partial J^o(x,t)}{\partial x}\right);$$
2. solve the HJB equation:
$$\frac{\partial J^o(x,t)}{\partial t} = -\left[l(x,u^o) + \frac{\partial J^o(x,t)}{\partial x} f(x,u^o) \right]$$
with the boundary condition
$$J^o(x,T) = m(x).$$

Unfortunately, step 2 is typically difficult for nonlinear systems and/or non-quadratic cost.

Example

As an example, consider the scalar integrator:

$$\dot{x}(t) = u(t), \qquad x, u \in \mathbb{R}$$
$$t_0 = 0$$

and the cost

$$J(x(0), u(\cdot), 0) = \int_0^T \left(u^2(\tau) + x^2(\tau) + \frac{1}{2}x^4(\tau) \right) d\tau.$$

Clearly, we have $l(x, u) = u^2 + x^2 + \frac{1}{2}x^4$ and $f(x, u) = u$ thus the first step is minimizing

$$u^2 + x^2 + \frac{1}{2}x^4 + \frac{\partial J^o(x,t)}{\partial x}u$$

w.r.t. u. Taking the derivative w.r.t. u and equating to zero, we get that the minimizing u, i.e., u^o, must satisfy

$$2u^o + \frac{\partial J^o(x,t)}{\partial x} = 0,$$

thus:

$$u^o = -\frac{1}{2}\frac{J^o(x,t)}{\partial x}.$$

Example (cont.)

We now have to solve

$$\frac{\partial J^o(x,t)}{\partial t} = -\left(\frac{1}{4}\left(\frac{\partial J^o(x,t)}{\partial x}\right)^2 + x^2 + \frac{x^4}{2} - \frac{1}{2}\left(\frac{\partial J^o(x,t)}{\partial x}\right)^2\right)$$
$$= \frac{1}{4}\left(\frac{\partial J^o(x,t)}{\partial x}\right)^2 - x^2 - \frac{x^4}{2}$$

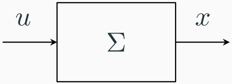
with the boundary condition

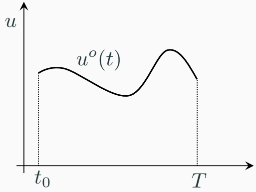
$$J^o(x,T) = 0.$$

Despite the apparent simplicity of the optimal control problem, the solution of the above equation is, quoting from Anderson and Moore (2007), “extraordinarily complex”.

Observation

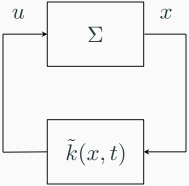
Observe that, despite the optimal control problem has been formulated in an **open-loop** fashion ("find the optimal control in the form $u^o(t), \quad t \in [t_0, T]$")





the actual solution takes a **closed-loop** form:

$$u^o = k\left(x, \frac{\partial J^o(x,t)}{\partial x}\right) = \tilde{k}(x,t)$$



i.e., is a (time-varying) state-feedback.

Finite-horizon linear-quadratic optimal control

Finite-horizon linear-quadratic optimal control

The *linear-quadratic optimal control problem* (LQ problem) is a particular case of the problem defined earlier, in which:

- the system is linear;
- the cost is quadratic (in both x and u).

Let the system be

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t = 0, \quad x(0) = x_0$$

and the cost be

$$J(x_0, u(\cdot), 0) = \int_0^T x^\top(\tau)Qx(\tau) + u^\top(\tau)Ru(\tau)d\tau + x^\top(T)Sx(T)$$

where $Q = Q^\top \succeq 0$, $S = S^\top \succeq 0$ and $R = R^\top \succ 0$.

By applying the general method shown before, we have that

$$\min_u \left\{ l(x,u) + \frac{\partial J^o(x,t)}{\partial x} f(x,u) \right\}$$

becomes

$$\min_u \underbrace{\left\{ x^\top Q x + u^\top R u + \frac{\partial J^o(x,t)}{\partial x} (Ax + Bu) \right\}}_{\text{convex function of } u}.$$

The unique absolute minimum can be found by taking the gradient with respect to u and equating to zero. We get

$$2u^\top R + \frac{\partial J^o(x,t)}{\partial x} B = 0,$$

which is equivalent to

$$Ru = -\frac{1}{2} B^\top \left[\frac{\partial J^o(x,t)}{\partial x} \right]^\top.$$

Finally, recalling that R is positive definite, hence invertible, we get

$$u^o = -\frac{1}{2} R^{-1} B^\top \left[\frac{\partial J^o(x,t)}{\partial x} \right]^\top.$$

Finite-horizon linear-quadratic optimal control (cont.)

The next step is substituting the minimizer in the HJB equation. We get:

$$\frac{\partial J^o(x,t)}{\partial t} = -x^\top Qx - \frac{1}{4}\frac{\partial J^o(x,t)}{\partial x}BR^{-1}B^\top\left[\frac{\partial J^o(x,t)}{\partial x}\right]^\top - \\ -\frac{\partial J^o(x,t)}{\partial x}\left(Ax - \frac{1}{2}BR^{-1}B^\top\left[\frac{\partial J^o(x,t)}{\partial x}\right]^\top\right) \tag{4}$$

and the boundary condition

$$J^o(x,T) = x^\top Sx. \tag{5}$$

It can be shown (see for instance Anderson and Moore (2007), page 22) that the solution takes the form

$$J^o(x,t) = x^\top P(t)x$$

with $P(t)$ symmetric. In other words, the optimal cost-to-go is quadratic in the state. By the boundary condition (5) we have

$$x^\top P(T)x = x^\top Sx, \quad \forall x \qquad \Longrightarrow \qquad P(T) = S.$$

Finite-horizon linear-quadratic optimal control (cont.)

Moreover:

$$\frac{\partial J^o(x,t)}{\partial t} = \frac{\partial x^\top P(t) x}{\partial t} = x^\top \dot{P}(t) x$$
$$\frac{\partial J^o(x,t)}{\partial x} = \frac{\partial x^\top P(t) x}{\partial x} = 2x^\top P(t)$$

By substituting in (4) we obtain

$$\begin{aligned} -x^\top \dot{P}(t) x &= x^\top Q x - x^\top P(t) B R^{-1} B^\top P(t) x + 2x^\top P(t) A x \\ &= x^\top Q x - x^\top P(t) B R^{-1} B^\top P(t) x + 2x^\top \left(\frac{P(t)A + A^\top P(t)}{2} \right) x \end{aligned}$$

Since the equation above must hold true for all x, we have

$$\dot{P}(t) + Q - P(t) B R^{-1} B^\top P(t) + P(t) A + A^\top P(t) = 0,$$

which is called the *Riccati differential equation*.

If $P(t)$ is a solution of the Riccati differential equation, with boundary condition $P(T) = S$, then

$$J^o(x,t) = x^\top P(t)x \qquad t \in [0,T].$$

Moreover, the optimal control law is

$$u^o(x,t) = -R^{-1}B^\top P(t)x = -K(t)x.$$

Observe that:

- since $R \succ 0$, the control law is well-defined (R^{-1} does exist);
- since, by definition, $J^o(x,t) \geq 0$ it follows that $P(t) \succeq 0$, $\forall t \in [0,T]$;
- the control law is time-varying, although the system is not;
- $K(t)$ can be pre-computed off-line (by solving the Riccati equation);
- there exists an alternative formulation of the Riccati equation (prove it as exercise):

$$\dot{P}(t) + Q + P(t)\left(A - BK(t)\right) + \left(A^\top - K^\top(t)B^\top\right)P(t) + K^\top(t)RK(t) = 0.$$

Infinite-horizon linear-quadratic optimal control

Infinite-horizon linear-quadratic optimal control

From a practical point of view, a time-varying control law, defined in $[0, T]$ is not very useful. Indeed:

- if some disturbances occur such that the state departs form the optimal trajectory, what control should we apply?
- what control should be applied for $t > T$?

By formulating an *infinite horizon* control problem, we can get a time-invariant control law which is (under suitable assumptions) stabilizing.

Infinite-horizon linear-quadratic optimal control (cont.)

Consider the finite-horizon cost:

$$J(x_0, u(\cdot), 0) = \int_0^T x^\top(\tau) Q x(\tau) + u^\top(\tau) R u(\tau) d\tau + x^\top(T) S x(T).$$

For $S = 0$ and $T \to \infty$ we get

$$J(x_0, u(\cdot), 0) = \int_0^\infty x^\top(\tau) Q x(\tau) + u^\top(\tau) R u(\tau) d\tau.$$

Infinite-horizon linear-quadratic optimal control (cont.)

The following theorem is a useful intermediate result.

Theorem

Given the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and finite-horizon cost

$$J(x_0, u(\cdot), 0) = \int_0^T x^\top(\tau)Qx(\tau) + u^\top(\tau)Ru(\tau)d\tau,$$

if the pair (A, B) is reachable, then

1. The solution of the Riccati differential equation, with the boundary condition $P(T) = 0$, converges, for $T \to \infty$, to a constant matrix $\bar{P} \succeq 0$ that satisfies the *algebraic Riccati equation*:

$$0 = A^\top P + PA + Q - PBR^{-1}B^\top P;$$

2. the optimal control law converges to

$$u(x(t)) = -R^{-1}B^\top \bar{P}x(t) = -\bar{K}x(t).$$

Infinite-horizon linear-quadratic optimal control (cont.)

Proof.
We begin by considering a family of finite-horizon control problems, depending on the horizon T. The optimal cost-to-go will therefore depend on the horizon:

$$J^o(x(t), t, T) = \min_{u[t,T]} \int_t^T x^\top(\tau)Qx(\tau) + u^\top(\tau)Ru(\tau)d\tau.$$

The Riccati differential equation will depend on T as well:

$$\begin{cases} -\dot{P}(t, T) = A^\top P(t, T) + P(t, T)A + Q - P(t, T)BR^{-1}B^\top P(t, T) \\ P(T, T) = 0 \end{cases}$$

We already know that the optimal cost-to-go is a quadratic function of the current state:

$$J^o(x(t), t, T) = x^\top(t)P(t, T)x(t).$$

Take a generic t and $\bar{T} \in [t, T]$. Since (A, B) is reachable, there exists a control

$$\bar{u}(\cdot) : [t, \bar{T}] \longrightarrow \mathbb{R}^m$$

such that $x(\bar{T}) = 0$.

Infinite-horizon linear-quadratic optimal control (cont.)

Now, suppose to apply the following control:

$$u(\tau) = \begin{cases} \bar{u}(\tau) & \tau \in [t, \bar{T}] \\ 0 & \tau > \bar{T} \end{cases} \tag{6}$$

As a consequence, the system reaches the origin at time $\bar{T}$ and remains there. The cost-to-go for this particular choice of input is

$$\begin{aligned} J(x(t), t, T) &= \int_t^{\bar{T}} x^\top(\tau) Q x(\tau) + u^\top(\tau) R u(\tau) d\tau + \underbrace{\int_{\bar{T}}^{T} x^\top(\tau) Q x(\tau) + u^\top(\tau) R u(\tau) d\tau}_{=0} \\ &\doteq \bar{J} \end{aligned}$$

Clearly, since the control law (6) is not optimal, in general, we will have

$$J^o(x(t), t, T) \leq \bar{J}.$$

We have thus shown that $J^o(x(t), t, T)$ is bounded from above.

Infinite-horizon linear-quadratic optimal control (cont.)

We can also show that $J^o(x(t), t, T)$ is monotonically non decreasing (as a function of T). Indeed, suppose that

$$J^o(x(t), t, \tilde{T}) < J^o(x(t), t, T) \qquad \text{for} \quad \tilde{T} > T \tag{7}$$

We could then apply, in the interval $[t, T]$, the optimal control associated to the horizon $\tilde{T}$. Let J be the corresponding cost. We then would have:

$$\begin{cases} J \geq J^o(x(t), t, T) & \text{(because the control may not be optimal for the horizon } T\text{)} \\ J \leq J^o(x(t), t, \tilde{T}) & \text{(because the integrand terms are positive)} \end{cases}$$

Thus we get

$$J^o(x(t), t, \tilde{T}) \geq J^o(x(t), t, T)$$

which contradicts (7).

Infinite-horizon linear-quadratic optimal control (cont.)

We can conclude that $J^o(x(t), t, T)$, being bounded from above and monotonically non decreasing, admits a limit for $T \to \infty$. Since $J^o(x(t), t, T) = x^\top(t)P(t,T)x(t)$, it follows that $P(t,T)$ admits a limit for $T \to \infty$:

$$\lim_{T\to\infty} P(t,T) = \bar{P}(t).$$

The matrix $\bar{P}(t)$ is symmetric and positive semidefinite, because it is a limit of symmetric and positive semidefinite matrices (the solutions af Riccati differential equations).

It remains to prove that $\bar{P}(t)$ does not depend on t and is thus the solution of the algebraic Riccati equation.

Consider two infinite horizon control problems (for the same A, B, Q, R), defined in $[t_1, \infty]$ and $[t_2, \infty]$, where $t_1 \neq t_2$. If the initial state x_0 is the same, the optimal cost-to-go must necessarily be the same. Indeed, being the system time-invariant, the actual optimal control problem is exactly the same in the two cases, i.e., minimize a cost over an infinite horizon, starting from x_0. As a consequence:

$$x_0^\top \bar{P}(t_1) x_0 = x_0^\top \bar{P}(t_2) x_0 \qquad \forall x_0, \quad \forall t_1, t_2,$$

which implies that

$$\bar{P}(t_1) = \bar{P}(t_2) = \bar{P}.$$

The fact that the control law becomes

$$u(x(t)) = -R^{-1} B^\top \bar{P} x(t) = -\bar{K} x(t).$$

can be easily verified by substitution. □

Notice that:

- the control law is a linear time-invariant state feedback;
- the closed-loop system is governed by:

$$\dot{x}(t) = (A - B\bar{K})x(t);$$

- so far, there is no guarantee that the closed-loop system is asymptotically stable.

Infinite-horizon linear-quadratic optimal control (cont.)

To obtain an asymptotically stabilizing control law, all the components of the state must affect the cost. Here, a fundamental role is played by the matrix Q.

The matrix Q can be factorized (possibly, in a non-unique way) as:

$$Q = C_q^\top C_q.$$

Indeed, any real symmetric matrix is orthogonally similar to a diagonal matrix, thus a possible factorization is:

$$Q = U^\top \Lambda U = U^\top \sqrt{\Lambda}\sqrt{\Lambda} U = \left(\sqrt{\Lambda} U\right)^\top \underbrace{\sqrt{\Lambda} U}_{C_q}.$$

The cost can thus be rewritten as

$$\begin{aligned} J(x_0, u(\cdot), 0) &= \int_0^\infty x^\top(\tau) C_q^\top C_q x(\tau) + u^\top(\tau) R u(\tau) d\tau \\ &= \int_0^\infty \tilde{y}^\top(\tau) \tilde{y}(\tau) + u^\top(\tau) R u(\tau) d\tau, \end{aligned}$$

where we have introduced a pseudo-output $\tilde{y}(t) = C_q x(t)$.

It is intuitive that the need of adequately weighting the state can be expressed in terms of the observability of the pair (A, Cq). To formalize such intuition, we need some intermediate results.

Lemma

Let

$$Q = C_{q_1}^\top C_{q_1} = C_{q_2}^\top C_{q_2}.$$

Then:

$$(A, C_{q_1}) \quad \text{is observable} \qquad \iff \qquad (A, C_{q_2}) \quad \text{is observable.}$$

Proof.

Recall that the observability is equivalent to the observability Gramian $W_o(\bar{t})$ being invertible for some $\bar{t} > 0$.

Take $\bar{t} > 0$ and let $W_o^{(1)}(\bar{t})$ and $W_o^{(2)}(\bar{t})$ be the observability Gramians of the pair (A, C_{q_1}) and (A, C_{q_2}), respectively. We have:

$$W_o^{(1)}(\bar{t}) = \int_0^{\bar{t}} e^{A^\top \tau} C_{q_1}^\top C_{q_1} e^{A\tau} d\tau = \int_0^{\bar{t}} e^{A^\top \tau} C_{q_2}^\top C_{q_2} e^{A\tau} d\tau = W_o^{(2)}(\bar{t})$$

□

The previous lemma allows, when the observability is concerned, to consider the generic factorization $C_q^\top C_q$ of Q.

Lemma

$$\bar{P} \quad \text{is positive definite} \qquad \Longleftrightarrow \qquad (A, C_q) \quad \text{is observable.}$$

Proof.
$(\Leftarrow)$

By construction $\bar{P}$ is symmetric and positive semidefinite. We show that if $x_0^\top \bar{P} x_0 = 0$ for some $x_0 \neq 0$, the pair (A, C_q) is not observable. If $x_0^\top \bar{P} x_0 = 0$ holds true, then the optimal cost $J^o(x_0)$ from x_0 is zero. As a consequence, the optimal control must be identically zero for the whole optimal transient from x_0. Thus the optimal transient from x_0 is a natural movement:

$$\dot{x}(t) = Ax(t) \qquad \text{which implies} \qquad x(t) = e^{At} x_0.$$

Infinite-horizon linear-quadratic optimal control (cont.)

The optimal cost can thus be written as

$$J^o(x_0) = \int_0^\infty \underbrace{x_0^\top e^{A^\top \tau} C_q^\top C_q e^{A\tau} x_0}_{\|C_q e^{A\tau} x_0\|^2} \, d\tau = 0.$$

Since the integrand is positive, it must be zero for any $\tau \geq 0$. Thus, for any $\bar{t} > 0$, we have:

$$\int_0^{\bar{t}} x_0^\top e^{A^\top \tau} C_q^\top C_q e^{A\tau} x_0 d\tau = x_0^\top \underbrace{\int_0^{\bar{t}} e^{A^\top \tau} C_q^\top C_q e^{A\tau} d\tau}_{W_o(\bar{t})} \, x_0 = 0.$$

As a consequence, the observability Gramian is not positive definite for all $t \geq 0$, which implies that the pair (A, C_q) is not observable.

$(\Rightarrow)$

Let $\bar{P} \succ 0$ and suppose by contradiction that (A, C_q) is not observable. Then, there exists $x_0 \neq 0$ such that:

$$C_q e^{At} x_0 = 0, \qquad \forall t \geq 0$$

(indeed, it is sufficient to pick any nonzero vector belonging to the unobservable subspace). Thus, the cost achieved by applying a null input from x_0 is:

$$J(x_0) = \int_0^\infty x_0^\top e^{A^\top \tau} C_q^\top C_q e^{A\tau} x_0 d\tau = 0.$$

Such a cost, being zero, is necessarily optimal. As a consequence we have:

$$x_0^\top \bar{P} x_0 = 0, \qquad x_0 \neq 0$$

which contradicts the condition $\bar{P} \succ 0$. □

Observation

Observe that if $Q \succ 0$, then the observability of (A, C_q) is guaranteed.

Indeed, if $C_q^\top C_q \succ 0$, the rank of C_q must be n. As a consequence:

$$\text{rank} \begin{bmatrix} C_q \\ C_q A \\ \vdots \\ C_q A^{n-1} \end{bmatrix} = n,$$

thus the pair is observable.

Fundamental theorem of linear quadratic control

Theorem

Given the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and the cost

$$J(x(0), u(\cdot), 0) = \int_0^\infty x^\top(\tau)Qx(\tau) + u^\top(\tau)Ru(\tau)d\tau,$$

where $Q = Q^\top = C_q^\top C_q \succeq 0$ and $R = R^\top \succ 0$.

If the pair (A, B) is reachable and the pair (A, C_q) is observable, then

1. the optimal control law is

   $$u(x) = -R^{-1}B^\top \bar{P}x = -\bar{K}x,$$

   where $\bar{P}$ is the unique positive definite solution of the algebraic Riccati equation

   $$0 = A^\top P + PA + Q - PBR^{-1}B^\top P;$$

2. the closed-loop system $\dot{x}(t) = (A - B\bar{K})x(t)$ is asymptotically stable.

Proof.
As for the point 1, it only remains to prove that the Riccati equation admits a unique positive definite solution. We will prove this statement after proving point 2, i.e., the stability of the closed-loop.

Consider the Lyapunov function $V(x) = x^\top \bar{P} x$, which is positive definite thanks to the previous lemma. Then:

$$\begin{aligned}
\dot{V}(x) &= \dot{x}^\top \bar{P} x + x^\top \bar{P} \dot{x} \\
&= x^\top \left(A^\top - \bar{K}^\top B^\top\right) \bar{P} x + x^\top \bar{P} \left(A - B\bar{K}\right) x \\
&= x^\top \left(A^\top \bar{P} - \bar{K}^\top B^\top \bar{P} + \bar{P} A - \bar{P} B \bar{K}\right) x \\
&= x^\top \left(\underbrace{A^\top \bar{P} - \bar{P} B R^{-1} B^\top \bar{P} + \bar{P} A}_{-Q} - \bar{K}^\top R \bar{K}\right) x \\
&= -x^\top \left(Q + \bar{K}^\top R \bar{K}\right) x
\end{aligned}$$

Notice that, being $R \succ 0$ we have $\bar{K}^\top R \bar{K} \succeq 0$. If $Q \succ 0$ we have $\dot{V} \prec 0$ and the stability is proven.

Fundamental theorem of linear quadratic control (cont.)

Otherwise, we have only $\dot{V} \preceq 0$ and we need to prove (according to the Krasowskii criterion), that there are no perturbed initial conditions such that $\dot{V}(x(t)) = 0 \; \forall t$. Notice that $V(x) = J^o(x(t))$ and suppose that there exists an initial state $\bar{x}_0 \neq 0$ such that, for the corresponding state movement $\bar{x}(t)$, we have

$$\frac{dJ^o(\bar{x}(t))}{dt} = 0, \quad \forall t.$$

Since $Q \succeq 0$ and $\bar{K}^\top R \bar{K} \succeq 0$, the integrand must be identically zero, i.e.

$$\bar{x}^\top(t) Q \bar{x}(t) = 0 \quad \forall t \qquad \text{and} \qquad \bar{x}^\top(t) \bar{K}^\top R \bar{K} \bar{x}(t) = 0 \quad \forall t.$$

From the latter, and from the fact that $R \succ 0$, it follows that $\bar{K}\bar{x}(t) = 0 \; \forall t$, thus $u(\bar{x}(t)) = 0 \; \forall t$. Hence the state movement $\bar{x}(t)$ can only be a natural movement:

$$\bar{x}(t) = e^{At}\bar{x}_0.$$

By substituting in $\bar{x}^\top(t) Q \bar{x}(t) = 0$, we obtain:

$$\left(e^{At}\bar{x}_0\right)^\top Q e^{At}\bar{x}_0 = \bar{x}_0^\top e^{A^\top t} C_q^\top C_q e^{At}\bar{x}_0 = 0, \quad \forall t.$$

Fundamental theorem of linear quadratic control (cont.)

The previous equation implies that the observability Gramian $W_o(t)$ of the pair (A, C_q) is not positive definite, which contradicts the observability of (A, C_q). Thus, the sought state movement does not exist and, by the Krasowskii criterion, the closed-loop system is asymptotically stable.

To prove that the Riccati equation admits a unique positive definite solution, suppose that two positive definite solutions $\bar{P}_1 \succ 0$ and $\bar{P}_2 \succ 0$ exist. We can write

$$\begin{aligned}
\left(\bar{P}_1 - \bar{P}_2\right) A_{cl_1} + A_{cl_2}^\top \left(\bar{P}_1 - \bar{P}_2\right) = \left(\bar{P}_1 - \bar{P}_2\right) \left(A - B\bar{K}_1\right) + \\
+ \left(A^\top - \bar{K}_2^\top B^\top\right) \left(\bar{P}_1 - \bar{P}_2\right) = \left(\bar{P}_1 - \bar{P}_2\right) \left(A - BR^{-1}B^\top \bar{P}_1\right) + \\
+ \left(A^\top - \bar{P}_2 BR^{-1}B^\top\right) \left(\bar{P}_1 - \bar{P}_2\right) = \left(\bar{P}_1 A + A^\top \bar{P}_1 - \bar{P}_1 BR^{-1}B^\top \bar{P}_1\right) - \\
- \left(\bar{P}_2 A + A^\top \bar{P}_2 - \bar{P}_2 BR^{-1}B^\top \bar{P}_2\right) = -Q + Q = 0.
\end{aligned}$$

Thus:

$$A_{cl_2}^{\top}\left(\bar{P}_1-\bar{P}_2\right)+\left(\bar{P}_1-\bar{P}_2\right)A_{cl_1}=0,$$

which is of the form:

$$AX+XB=C,$$

known as the *Sylvester equation*. Such an equation admits a unique solution if and only if A and $-B$ have no common eigenvalues, i.e.:

$$\lambda_i(A)+\lambda_j(B)\neq 0 \quad \forall i,j.$$

Clearly, the condition is true, since all the eigenvalues of both A_{cl_1} and A_{cl_2} have strictly negative real part. Thus, we have

$$\bar{P}_1-\bar{P}_2=0.$$

□

Observations

Notice that:

- the algebraic Riccati equation can be solved by means of suitable numerical algorithms; in Matlab, the whole linear quadratic regulator problem can be solved by

```
K = lqr(A, B, Q, R);
```

- the sublevel sets of $x^\top \bar{P} x$ are positively invariant under the control law $u = -\bar{K}x$.

Example 1

Consider the scalar system

$$\dot{x}(t) = x(t) + u(t), \qquad x \in \mathbb{R}, \quad (A = 1, B = 1)$$

and the cost

$$J = \int_0^\infty x^\top(\tau) Q x(\tau) + u^\top(\tau) R u(\tau) d\tau.$$

The corresponding algebraic Riccati equation is

$$A^\top \bar{P} + \bar{P} A + Q - \bar{P} B R^{-1} B^\top \bar{P} = 0,$$

where $\bar{P}$ is a scalar. We get:

$$\bar{P} + \bar{P} + Q - \frac{\bar{P}^2}{R} = 0 \quad \Longrightarrow \quad \frac{\bar{P}^2}{R} - 2\bar{P} - Q = 0,$$

thus, solving for $\bar{P}$:

$$\bar{P} = \frac{2 \pm \sqrt{4 + 4\dfrac{Q}{R}}}{\dfrac{2}{R}} = R\left(1 \pm \sqrt{1 + \frac{Q}{R}}\right).$$

Examples (cont.)

Observe that if $R > 0$ and $Q > 0$ there exists a unique positive solution (in the scalar case, "positive (semi)definite" is equivalent to "greater than (or equal to) zero").

We now consider three cases:

1. $Q = 3$, $R = 1$

   In this case we have:
   $$\bar{P} = 1 + \sqrt{1+3} = 3 \quad \Longrightarrow \quad \bar{K} = R^{-1}B^\top \bar{P} = 3$$
   thus the closed loop eigenvalue is
   $$A_{cl} = A - B\bar{K} = 1 - 1 \cdot 3 = -2 < 0.$$

2. $Q = 8$, $R = 1$

   In this case we have:
   $$\bar{P} = 1 + \sqrt{1+8} = 4 \quad \Longrightarrow \quad \bar{K} = R^{-1}B^\top \bar{P} = 4$$
   thus the closed loop eigenvalue is
   $$A_{cl} = A - B\bar{K} = 1 - 4 = -3 < 0.$$

Examples (cont.)

3. $Q = 15$, $R = 1$

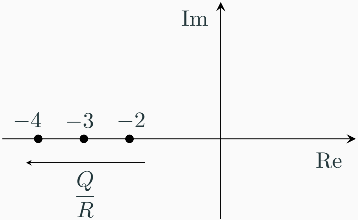
   In this case we have:

$$\bar{P} = 1 + \sqrt{1+15} = 5 \quad \Longrightarrow \quad \bar{K} = R^{-1}B^\top \bar{P} = 5$$

   thus the closed loop eigenvalue is

$$A_{cl} = A - B\bar{K} = 1 - 5 = -4 < 0.$$

Observe that the time constant decreases as Q/R increases, thus larger Q/R means shorter transients.



Question: for $Q = 0$ we get $\bar{P} = 2R$ and $\bar{K} = 2$, but clearly, the optimal control is identically zero. Thus the obtained solution leads to a non-optimal control law. Why?

Example 2

Consider the system

$$\dot{x}(t) = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \qquad Q = q > 0, \quad R^{-1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}.$$

The algebraic Riccati equation

$$A^\top \bar{P} + \bar{P} A + Q - \bar{P} B R^{-1} B^\top \bar{P} = 0$$

becomes

$$q - \bar{P} \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \bar{P} = 0$$

or

$$q - \bar{P}^2 \left(b_1^2 \rho_1 + b_2^2 \rho_2 \right) = 0,$$

whose positive solution is

$$\bar{P} = \sqrt{\frac{q}{b_1^2 \rho_1 + b_2^2 \rho_2}},$$

Examples (cont.)

leading to the gain matrix

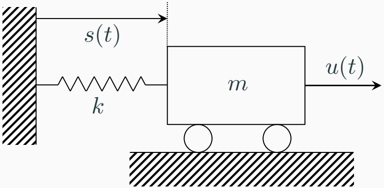
$$\bar{K} = R^{-1}B^\top \bar{P} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \sqrt{\frac{q}{b_1^2\rho_1 + b_2^2\rho_2}} = \begin{bmatrix} \rho_1 b_1 \sqrt{\dfrac{q}{b_1^2\rho_1 + b_2^2\rho_2}} \\ \rho_2 b_2 \sqrt{\dfrac{q}{b_1^2\rho_1 + b_2^2\rho_2}} \end{bmatrix}.$$

Notice how the magnitude of the components of the gain matrix depends on ρ_1 and ρ_2.

Example 3

Consider a mass-spring system as in figure.
The equation of motion is

$$m\ddot{s}(t) + ks(t) = u(t).$$



$$\begin{cases} x_1(t) = s(t) \\ x_2(t) = \dot{s}(t) \end{cases} \implies \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\frac{k}{m}x_1(t) + \frac{u(t)}{m} \end{cases}$$

Letting, for simplicity, $k = 1$, $\omega^2 = \frac{1}{m}$ we get

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \omega^2 \end{bmatrix} u(t).$$

By letting the cost be

$$J = \int_0^\infty x_1^2(t) + ru^2(t)dt,$$

the matrices that define the LQ problem are:

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \omega^2 \end{bmatrix}, \quad Q = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}}, \quad R = r.$$

The pair (A, B) is reachable:

$$\text{rank} \begin{bmatrix} B & AB \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & \omega^2 \\ \omega^2 & 0 \end{bmatrix} = 2.$$

The pair (A, C_q) is observable:

$$\text{rank} \begin{bmatrix} C_q \\ C_q A \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2.$$

The Riccati equation

$$A^\top P + PA + Q - PBR^{-1}B^\top P = 0$$

becomes

$$\begin{bmatrix} 0 & -\omega^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} -$$

$$-\frac{1}{r} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ \omega^2 \end{bmatrix} \begin{bmatrix} 0 & \omega^2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, we get

$$\begin{bmatrix} -2\omega^2 p_{12} + 1 & p_{11} - \omega^2 p_{22} \\ p_{11} - \omega^2 p_{22} & 2p_{12} \end{bmatrix} - \frac{\omega^4}{r} \begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

corresponding to the three independent equations:

Examples (cont.)

$$1 - 2\omega^2 p_{12} - \frac{\omega^4}{r} p_{12}^2 = 0 \quad (8)$$

$$p_{11} - \omega^2 p_{22} - \frac{\omega^4}{r} p_{12} p_{22} = 0 \quad (9)$$

$$2p_{12} - \frac{\omega^4}{r} p_{22}^2 = 0 \quad (10)$$

From (8) we get:

$$p_{12} = \frac{2\omega^2 \pm \sqrt{4\omega^4 + \frac{r\omega^4}{r}}}{-2\frac{\omega^4}{r}} = \frac{-r \pm r\sqrt{1 + \frac{1}{r}}}{\omega^2},$$

and from (10) we see that $p_{12} \geq 0$, thus we choose the solution

$$p_{12} = \frac{-r + r\sqrt{1 + \frac{1}{r}}}{\omega^2}.$$

Now, observe that, for P to be positive definite, it is necessarily $p_{22} > 0$. Hence, from (10) we obtain

$$p_{22} = \sqrt{2p_{12}\frac{r}{\omega^4}} = \cdots = \frac{r}{\omega^3}\sqrt{2\left(\sqrt{1+\frac{1}{r}}-1\right)}.$$

Finally, from (9) :

$$p_{11} = \omega^2 p_{22} + \frac{\omega^4}{r}p_{12}p_{22} = \cdots = \frac{r}{\omega}\sqrt{2\left(1+\frac{1}{r}\right)\left(\sqrt{1+\frac{1}{r}}-1\right)}.$$

The gain matrix is thus:

$$K = R^{-1}B^\top P = \frac{1}{r}\begin{bmatrix} 0 & \omega^2 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

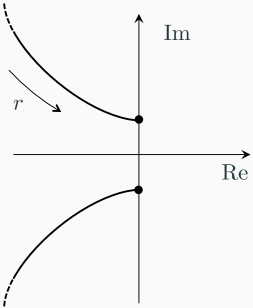
where

$$k_1 = \sqrt{1+\frac{1}{r}} - 1$$

$$k_2 = \frac{1}{\omega}\sqrt{2\left(\sqrt{1+\frac{1}{r}}-1\right)}$$

The locus of the eigenvalues of $A - BK$, as r varies, is reported in figure. Notice that:

- the eigenvalues are always complex conjugate;
- as $r \to \infty$, the two eigenvalues converge to the purely imaginary eigenvalues of the open-loop system.



Normalization

Often the matrices Q and R are taken to be diagonal, i.e.

$$Q = \begin{bmatrix} q_1 & 0 & \cdots & \\ 0 & q_2 & & \\ \vdots & & \ddots & \\ & & & q_n \end{bmatrix}, \; q_i \geq 0 \quad \text{and} \quad R = \begin{bmatrix} r_1 & 0 & \cdots & \\ 0 & r_2 & & \\ \vdots & & \ddots & \\ & & & r_m \end{bmatrix}, \; r_i > 0$$

thus the cost is:

$$J = \int_0^{\infty} \left[q_1 x_1^2(\tau) + q_2 x_2^2(\tau) + \cdots + q_n x_n^2(\tau) + r_1 u_1^2(\tau) + \cdots + r_m u_m^2(\tau) \right] d\tau.$$

However, it is often the case that the individual components of the state and the input take on values in different intervals, possibly differing of several orders of magnitude. As a consequence, is important to **normalize** the contributions of the individual components.

Choice of the design parameters (cont.)

Assume that

$$|x_i| \leq x_{i,\text{max}} \qquad i = 1, \dots, n$$
$$|u_i| \leq u_{i,\text{max}} \qquad i = 1, \dots, m$$

Letting

$$q_i = \frac{\tilde{q}_i}{x_{i,\text{max}}^2} \qquad \text{and} \qquad r_i = \frac{\tilde{r}_i}{u_{i,\text{max}}^2}$$

the cost becomes

$$J = \int_0^\infty \tilde{q}_1 \underbrace{\frac{x_1^2(\tau)}{x_{1,\text{max}}^2}}_{0 \div 1} + \dots + \tilde{r}_1 \underbrace{\frac{u_1^2(\tau)}{u_{1,\text{max}}^2}}_{0 \div 1} + \dots d\tau$$

thus $\tilde{q}_i$ and $\tilde{r}_i$ do weight contributions in the range $[0, 1]$.

As a matter of fact, the LQ control is frequently employed in an iterative fashion (thus refining the design parameters Q and R until a satisfactory control system is obtained). In that case, the *Bryson's rule* provides a choice for the first iteration. It consist of choosing $\tilde{q}_i = 1$ and $\tilde{r}_i = 1$, thus obtaining

$$Q = \begin{bmatrix} \frac{1}{x_{1,\text{max}}^2} & & & \\ & \frac{1}{x_{2,\text{max}}^2} & & \\ & & \ddots & \\ & & & \frac{1}{x_{n,\text{max}}^2} \end{bmatrix}, \quad R = \begin{bmatrix} \frac{1}{u_{1,\text{max}}^2} & & & \\ & \frac{1}{u_{2,\text{max}}^2} & & \\ & & \ddots & \\ & & & \frac{1}{u_{m,\text{max}}^2} \end{bmatrix}.$$

Choice of the design parameters (cont.)

Output weighting

Let the system have an output transform:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases} \qquad y \in \mathbb{R}^p$$

and suppose we want the output y to appear in the cost. We can simply choose:

$$Q = C^\top \bar{Q} C, \qquad \bar{Q} \succeq 0, \bar{Q} \in \mathbb{R}^{p \times p}.$$

It it easy to check that the cost becomes

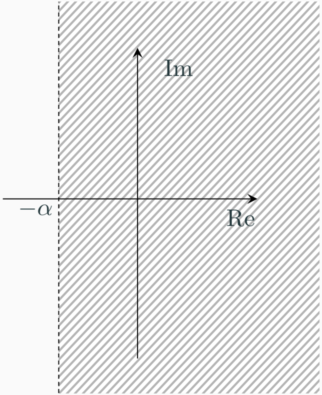
$$J = \int_0^\infty y^\top(\tau)\bar{Q}y(\tau) + u^\top(\tau)Ru(\tau)d\tau.$$

Notice that:

- such a choice guarantees that $Q \succeq 0$;
- the pair A, C_q where $C_q^\top C_q = Q = C^\top \bar{Q} C$ must be observable;
- the state must be accessible for performing the feedback.

Linear quadratic control with prescribed degree of stability

It is possible to formulate a variant of the linear quadratic control problem such that the eigenvalues of $A - B\bar{K}$ lie to the left of a given $-\alpha$, for a given $\alpha > 0$.



$$\dot{x} = Ax + Bu$$

$$Q = C_q^\top C_q \succeq 0$$

$$R \succ 0$$

(A, B) reachable

(A, C_q) observable

$\alpha > 0$ assigned

Although the stability is a Boolean property (a system is either stable or unstable), historically the wording "degree of stability" is employed (Anderson and Moore (1969)).

Lemma

If the matrix $M \in \mathbb{R}^{n\times n}$ has eigenvalues λ_i, $i = 1, \ldots, n$ then the matrix $M + \alpha I$ has eigenvalues $\lambda_i + \alpha$, $i = 1, \ldots, n$.

Proof.

Let λ_i and v_i be an eigenvalue and the associated eigenvector of M, respectively. Then:

$$Mv_i = \lambda_i v_i.$$

On the other hand:

$$(M + \alpha I)\, v_i = Mv_i + \alpha I v_i = \lambda_i v_i + \alpha v_i = (\lambda_i + \alpha)\, v_i,$$

meaning that $\lambda_i + \alpha$ is an eigenvalue of $M + \alpha I$. □

Linear quadratic control with prescribed degree of stability (cont.)

Consider the auxiliary cost:

$$J(x_0, u(\cdot), 0) = \int_0^{\infty} e^{2\alpha\tau} \left[x^\top(\tau) Q x(\tau) + u^\top(\tau) R u(\tau) \right] d\tau$$

and define:

$$\tilde{x}(t) = e^{\alpha t} x(t) \qquad \text{auxiliary state}$$
$$\tilde{u}(t) = e^{\alpha t} u(t) \qquad \text{auxiliary input}$$

We have

$$\begin{aligned}\dot{\tilde{x}}(t) &= \alpha e^{\alpha t} x(t) + e^{\alpha t} \dot{x}(t) = e^{\alpha t} [\overbrace{(A + \alpha I)}^{\doteq \tilde{A}} x(t) + B u(t)] = \\ &= \tilde{A} \tilde{x}(t) + B \tilde{u}(t)\end{aligned}$$

It follows that the cost can be written as

$$J(x_0, u(\cdot), 0) = \int_0^{\infty} \tilde{x}^\top(\tau) Q \tilde{x}(\tau) + \tilde{u}^\top(\tau) R \tilde{u}(\tau) d\tau \qquad (11)$$

We now consider an LQ problem for the auxiliary system

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + B\tilde{u}(t)$$

and the cost (11). Observe that

$$(A, B) \quad \text{reachable} \qquad \Longrightarrow \qquad (\tilde{A}, B) \quad \text{reachable}$$

as it follows immediately by the PBH test, since

$$\left[\begin{array}{c|c} A + \alpha I - \lambda I & B \end{array}\right] = \left[\begin{array}{c|c} A - (\lambda - \alpha) I & B \end{array}\right].$$

Analogously, it can be shown that

$$(A, C_q) \quad \text{observable} \qquad \Longrightarrow \qquad (\tilde{A}, C_q) \quad \text{observable}$$

Thus, the auxiliary problem satisfies the hypotheses of the fundamental theorem of the LQ control.

Its solution is

$$\tilde{u}^o(t) = -R^{-1}B^\top \bar{P}_\alpha \tilde{x}(t) = -\bar{K}_\alpha \tilde{x}(t),$$

where $\bar{P}_\alpha$ is the positive definite solution of the algebraic Riccati equation:

$$\tilde{A}^\top \bar{P}_\alpha + \bar{P}_\alpha \tilde{A} + Q - \bar{P}_\alpha B R^{-1} B^\top \bar{P}_\alpha = 0.$$

Moreover, the eigenvalues of $\left(\tilde{A} - B\bar{K}_\alpha\right)$ have strictly negative real part (we denote by $\lambda_i(M)$ the ith eigenvalue of M):

$$\mathrm{Re}\left(\lambda_i\left(\tilde{A} - B\bar{K}_\alpha\right)\right) < 0 \quad \forall i,$$

which implies, for the previous lemma:

$$\mathrm{Re}\left(\lambda_i\left(A - B\bar{K}_\alpha\right)\right) < -\alpha \quad \forall i.$$

Linear quadratic control with prescribed degree of stability (cont.)

In practice, the procedure is the following.

Given $A, B, Q = C_q^\top C_q \succeq 0, R \succ 0$, with (A, B) reachable and (A, C_q) observable, perform the steps:

1. choose $\alpha > 0$;
2. compute $\tilde{A} = A + \alpha I$;
3. solve the Riccati equation
$$\tilde{A}^\top \bar{P}_\alpha + \bar{P}_\alpha \tilde{A} + Q - \bar{P}_\alpha B R^{-1} B^\top \bar{P}_\alpha = 0;$$
4. implement the control law
$$u(t) = -R^{-1} B^\top \bar{P}_\alpha x(t) = -\bar{K}_\alpha x(t).$$

Example

$$\dot{x}(t) = x(t) + u(t), \qquad x \in \mathbb{R}$$

The Riccati equation for $\tilde{A} = A + \alpha = 1 + \alpha$ is

$$(1+\alpha)\bar{P}_\alpha + \bar{P}_\alpha(1+\alpha) + Q - \bar{P}_\alpha R^{-1} \bar{P}_\alpha = 0,$$

whose positive solution is

$$\bar{P}_\alpha = R\left[(1+\alpha) + \sqrt{(1+\alpha)^2 + \frac{Q}{R}}\right].$$

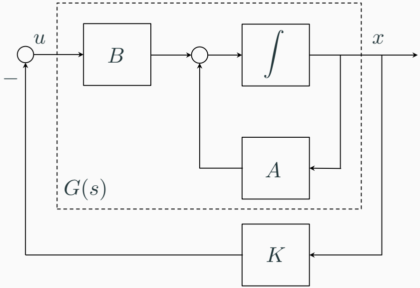
Thus we have

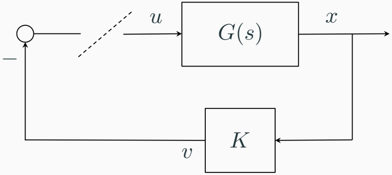
$$\bar{K}_\alpha = R^{-1} B^\top \bar{P}_\alpha = (1+\alpha) + \sqrt{(1+\alpha)^2 + \frac{Q}{R}}$$

and, finally

$$A - B\bar{K}_\alpha = 1 - (1+\alpha) - \sqrt{(1+\alpha)^2 + \frac{Q}{R}} = -\alpha - \sqrt{(1+\alpha)^2 + \frac{Q}{R}} < -\alpha.$$

Consider the following schemes, where K has been obtained by solving an LQR problem.





$$X(s) = \underbrace{(sI - A)^{-1} B}_{\doteq G(s)} U(s)$$

i.e., $G(s)$ is the transfer matrix from the input u to the state x.

Thus

$$L(s) = K (sI - A)^{-1} B = KG(s)$$

is the open-loop transfer matrix (obtained by opening the loop and computing the transfer matrix from u to v).

Frequency domain properties of the linear quadratic regulator (cont.)

The transfer function

$$J(s) = I + L(s)$$

is called the *return difference*. Consider now the algebraic Riccati equation:

$$Q = -A^\top P - PA + K^\top RK.$$

By adding and subtracting sP to the right member we get:

$$\begin{aligned} Q &= sP - sP - A^\top P - PA + K^\top RK \\ &= P(sI - A) + (-sI - A^\top)P + K^\top RK. \end{aligned}$$

Multiplying each member by $G^\top(-s) = B^\top(-sI - A^\top)^{-1}$ (to the left) and by $G(s) = (sI - A)^{-1}B$ (to the right) we obtain

$$\begin{aligned} G^\top(-s)QG(s) = G^\top(-s)\, & \overbrace{P(sI - A)(sI - A)^{-1}B}^{K^\top R} + \\ & + \underbrace{B^\top(-sI - A^\top)^{-1}(-sI - A^\top)P}_{RK}(sI - A)^{-1}B + \\ & + G^\top(-s)K^\top RKG(s). \end{aligned}$$

Hence we have

$$G^\top(-s)QG(s) = L^\top(-s)R + RL(s) + L^\top(-s)RL(s),$$

and, by adding R to both members

$$\begin{aligned} R + G^\top(-s)QG(s) &= R(I + L(s)) + L^\top(-s)R(I + L(s)) \\ &= \underbrace{(I + L^\top(-s))}_{J^\top(-s)} R \underbrace{(I + L(s))}_{J(s)} \end{aligned}$$

and, finally we obtain the so-called *Kalman's equality*:

$$J^\top(-s)RJ(s) = R + G^\top(-s)QG(s).$$

The Kalman's equality allows to establish the stability margins guaranteed by the LQR.

Frequency domain properties of the linear quadratic regulator (cont.)

We consider, in the following, the single-input case ($m = 1$), thus the Kalman's equality is scalar. By letting $s = j\omega$ we obtain

$$J^\top(-j\omega)RJ(j\omega) = R + G^\top(-j\omega)Q(j\omega)$$

and, since $J(s)$ and $G(s)$ are rational:

$$J^*(j\omega)RJ(j\omega) = R + G^*(j\omega)QG(j\omega), \tag{12}$$

where we denote by M^* the conjugate transpose of M. Recalling that $Q = C_p^\top C_p$, we have

$$G^*(j\omega)QG(j\omega) = \|C_pG(j\omega)\|^2 \geq 0.$$

On the other hand, $J^*(j\omega)RJ(j\omega) = R\,\|J(j\omega)\|^2$. Thus, since $R > 0$, we get, from (12):

$$R\,\|J(j\omega)\|^2 \geq R.$$

Then, we obtain

$$\|1 + L(j\omega)\|^2 \geq 1$$

and, finally:

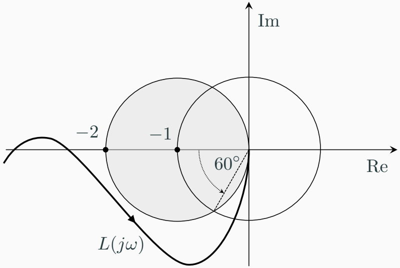
$$|1 + L(j\omega)| \geq 1, \qquad \forall\omega \in \mathbb{R}.$$

Frequency domain properties of the linear quadratic regulator (cont.)

The condition $|1 + L(j\omega)| \geq 1$ implies that the Nyquist plot of $L(s)$ does not enter the disk of radius 1 centered in -1.

As a consequence, the LQR guarantees:

- a **vector margin of stability of at least** 1;
- a **positive gain margin of** $+\infty$;
- a **negative gain margin of at least** $20\log_{10}(0.5) \approx -6\,\mathrm{dB}$;
- a **phase margin of at least** $\pm 60°$.



In the multi-input case ($m > 1$), it can be shown that the given margins are guaranteed for each input channel separately.

Frequency domain properties of the linear quadratic regulator (cont.)

We conclude by showing (for the single-input case) that the magnitude of the Bode plot of the complementary sensitivity function $T(s) = \dfrac{L(s)}{1+L(s)}$ cannot fall faster than 20dB/decade. Indeed, by the *matrix inversion lemma*[1] we can write:

$$\begin{aligned} T(j\omega) &= 1 - \frac{1}{1+L(j\omega)} \\ &= 1 - \left(1 + K(j\omega I - A)^{-1}B\right)^{-1} \\ &= K\left(j\omega I - A + BK\right)^{-1} B \end{aligned}$$

Thus, we have

$$\lim_{\omega\to\infty} \frac{T(j\omega)}{\frac{1}{j\omega}} = \lim_{\omega\to\infty} j\omega T(j\omega) = KB = -RB^\top PB,$$

meaning that $|T(j\omega)|$ decreases as fast as $\left|\dfrac{1}{j\omega}\right|$ i.e., at a rate of 20dB/decade.

The (slow) -20dB/decade magnitude decrease is the main drawback of the state feedback LQR controllers. Indeed, it may not be sufficient to reject high-frequency disturbances and/or to guarantee robustness against unmodeled dynamics.

[1] also known as Woodbury identity: $(A+BCD)^{-1} = A^{-1} - A^{-1}B\left(C^{-1} + DA^{-1}B\right)^{-1}DA^{-1}$.

Finite horizon

Consider the system

$$x(k+1) = Ax(k) + Bu(k)$$

and the cost

$$J(x(0), u(\cdot), 0) = \sum_{0}^{N-1} \left(x^\top(k) Q x(k) + u^\top(k) R u(k) \right) \; + \; x^\top(N) S x(N)$$

where $Q \succeq 0$, $R \succ 0$, $S \succeq 0$ and N is an assigned horizon.

It can be shown that:

1. the optimal cost-to-go is

   $$J^o(x,k) = x^\top P(k) x,$$

   where $P(k)$ is the solution of the *Riccati difference equation*:

$$\begin{cases} P(k) = Q + A^\top P(k+1) A - A^\top P(k+1) B \left(R + B^\top P(k+1) B \right)^{-1} B^\top P(k+1) A \\ P(N) = S \end{cases}$$

2. the optimal control law is:

$$u^o(k) = -K(k)x(k)$$

where

$$K(k) = \left(R + B^\top P(k+1)B\right)^{-1} B^\top P(k+1)A.$$

Observe that the Riccati equation can be easily solved backwards from time N.

Infinite horizon

As for the infinite horizon case, the results are analogous to the continuous-time case, as summarized by the following theorem.

Theorem

Given the system

$$x(k+1) = Ax(k) + Bu(k)$$

and the cost

$$J = \sum_{0}^{\infty} \left(x^\top(k) Q x(k) + u^\top(k) R u(k) \right), \quad Q = C_q^\top C_q \succeq 0,\ R = R^\top \succ 0,$$

if the pair (A, B) is reachable and the pair (A, C_q) is observable, then

1. the optimal control law is
$$u(x) = \left(R + B^\top \bar{P} B\right)^{-1} B^\top \bar{P} A = -\bar{K}x,$$
where $\bar{P}$ is the unique positive definite solution of the algebraic Riccati equation
$$\bar{P} = Q + A^\top \bar{P} A - A^\top \bar{P} B \left(R + B^\top \bar{P} B\right)^{-1} B^\top \bar{P} A;$$
2. the closed-loop system $x(k+1) = (A - B\bar{K})x(k)$ is asymptotically stable;
3. the optimal cost is $J^o(x) = x^\top \bar{P} x$.

Optimal set-point control

Optimal set-point control via state feedback

Consider the system (strictly proper, for simplicity):

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ z(t) &= Gx(t) \end{cases}$$

So far, we have considered *regulation* problems, i.e., the goal was to regulate the state (or a controlled output) to zero.

Now we consider a different problem: we want the controlled output z to converge optimally to a given nonzero constant *set-point* r, corresponding to an equilibrium pair $(x_{\text{eq}}, u_{\text{eq}})$, for which $z = r$.

Let the cost be

$$J = \int_0^\infty \tilde{z}^\top(t)\bar{Q}\tilde{z}(t) + \tilde{u}^\top(t)R\tilde{u}(t)dt \tag{13}$$

where $\tilde{z} \doteq z - r$ and $\tilde{u} = u - u_{\text{eq}}$.

The equilibrium pair $(x_{\text{eq}}, u_{\text{eq}})$ must satisfy the equation

$$\begin{cases} Ax_{\text{eq}} + Bu_{\text{eq}} &= 0 \\ Gx_{\text{eq}} &= r \end{cases} \iff \underbrace{\begin{bmatrix} A & B \\ G & 0 \end{bmatrix}}_{\doteq M} \begin{bmatrix} x_{\text{eq}} \\ u_{\text{eq}} \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix} \tag{14}$$

Optimal set-point control via state feedback (cont.)

Regarding the existence of solutions, we distinguish three cases:

1. when the number of inputs m is strictly smaller than the number p of controlled outputs, the system is said to be *underactuated*, and the previous equation may not have a solution (there are more equations than unknowns);
2. when the number of inputs is equal to the number of outputs, equation (14) admits a solution for any $r \neq 0$ only if M is nonsingular; in that case the solution is:

$$\begin{bmatrix} x_{\text{eq}} \\ u_{\text{eq}} \end{bmatrix} = M^{-1} \begin{bmatrix} 0 \\ r \end{bmatrix};$$

3. when the number of inputs is strictly larger than the number of outputs we have an *overactuated system* and equation (14) admits generally multiple solutions. In that case, if M is full-row rank, a solution can be found via the pseudoinverse:

$$\begin{bmatrix} x_{\text{eq}} \\ u_{\text{eq}} \end{bmatrix} = M^\top \left(MM^\top\right)^{-1} \begin{bmatrix} 0 \\ r \end{bmatrix}.$$

Optimal set-point control via state feedback (cont.)

To reduce the optimal set-point problem to a regulation problem, we consider an auxiliary system with state:

$$\tilde{x} \doteq x - x_{\text{eq}}$$

whose dynamics are

$$\begin{cases} \dot{\tilde{x}} = Ax + Bu = A(x - x_{\text{eq}}) + B(u - u_{eq}) + \underbrace{(Ax_{\text{eq}} + Bu_{\text{eq}})}_{=0} \\ \tilde{z} = Gx - r = G(x - x_{\text{eq}}) + \overbrace{(Gx_{\text{eq}} - r)}^{=0} \end{cases}$$

Thus we obtain

$$\begin{cases} \dot{\tilde{x}} = A\tilde{x} + B\tilde{u} \\ \tilde{z} = G\tilde{x} \end{cases}$$

which, along with the cost (13), defines an optimal regulation problem. Let K be the corresponding optimal gain. We have:

$$\tilde{u}(t) = -K\tilde{x}(t).$$

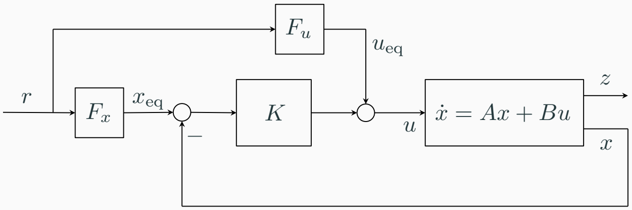
Back to the original input and state variables u and x, we have

$$u(t) = -K(x(t) - x_{\text{eq}}) + u_{\text{eq}}.$$

Observing that the solution of (14) can be written in the form

$$x_{\text{eq}} = F_x r, \quad u_{\text{eq}} = F_u r,$$

for suitably defined matrices F_x and F_u, we conclude that the control architecture can be represented as in figure.



It can be shown that, for a single controlled output, if A has an eigenvalue at the origin, and the mode is observable through z, then $u_{\text{eq}} = 0$, thus $F_u = 0$ and the feed-forward term $F_u r$ is absent.

Optimal set-point control via output feedback

When the state is not accessible, we can employ the control law:

$$u(t) = -K(\hat{x}(t) - x_{\text{eq}}) + u_{\text{eq}}.$$

where $\hat{x}$ is the state estimate produced by an observer:

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly = (A - LC - BK)\hat{x} + BKx_{\text{eq}} + Bu_{\text{eq}} + Ly.$$

$$\dot{\bar{x}} = -(A - LC - BK)\hat{x} + (A - BK)x_{\text{eq}} - Ly = (A - LC - BK)\bar{x} - L(y - Cx_{\text{eq}})$$

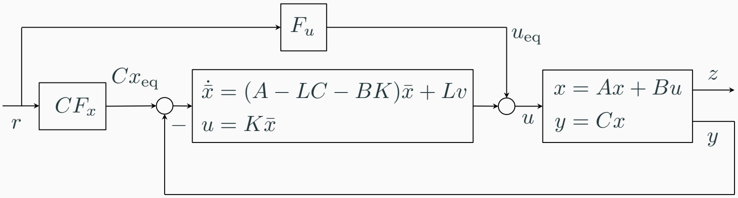
$$\dot{\bar{x}} = (A - LC - BK)\bar{x} - L(y - Cx_{\text{eq}}) \quad (15)$$

$$u = K\bar{x} + u_{\text{eq}} \quad (16)$$

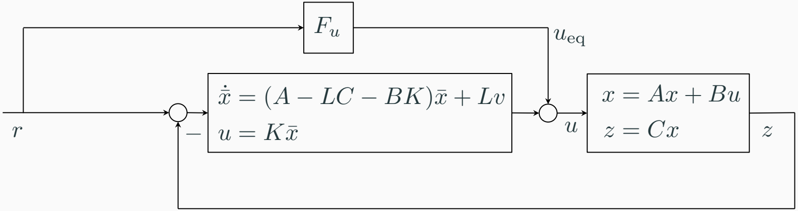
The closed-loop system is governed by

$$\begin{bmatrix} \dot{x} \\ \dot{\bar{x}} \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & A - LC - BK \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \end{bmatrix} + \begin{bmatrix} F_u \\ LCF_x \end{bmatrix} r \tag{17}$$

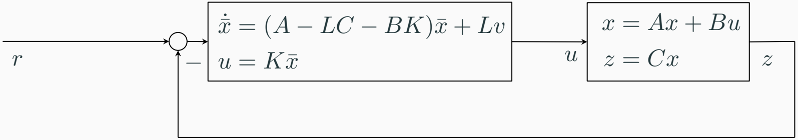
corresponding to the following control scheme:



If $z = y$ (i.e., the controlled output and the measured output coincide) we have $G = C$, thus $Cx_{\text{eq}} = r$. As a consequence, the scheme becomes:



If, in addition, $F_u = 0$, we obtain the usual feedback configuration:



Optimal control and linear programming

Minimum-time

Consider the single-input system

$$x(k+1) = \Phi x(k) + \Gamma u(k), \qquad \Phi \in \mathbb{R}^{n\times n}, \;\; \Gamma \in \mathbb{R}^{n\times 1}$$

and let the initial state $x(0) = \alpha$ be assigned. We want to find the minimum number of steps N and an input sequence $\{u(0), u(1), \dots, u(N-1)\}$ such that:

$$x(N) = \beta \quad \text{assigned}$$
$$|u(k)| \leq 1 \quad \forall k = 0 \dots N-1$$

A possible approach is:

1. fix N (for instance $N = 1$);
2. minimize the cost

$$J(N) = \max_{0\leq k \leq N-1} |u(k)|$$

over $\{u(0), u(1), \dots, u(N-1)\}$, subject to the constraint

$$\Phi^N \alpha + \sum_{k=0}^{N-1} \Phi^{N-k-1} \Gamma u(k) = \beta; \tag{18}$$

3. increase N until we get $J(N) \leq 1$.

The problem of step 2 can be formulated as a linear programming problem in the following way. Let M be an auxiliary variable such that

$$\begin{cases} M \geq u(k), & k = 0 \dots N-1 \\ M \geq -u(k), & k = 0 \dots N-1 \end{cases} \tag{19}$$

Then the problem becomes

$$\min M \qquad \text{subject to} \quad (18) \quad \text{and} \quad (19),$$

and must be solved iterating over N until we obtain a cost $M \leq 1$.

Minimum-time (cont.)

The Matlab function `linprog` can be employed for solving linear programming problems. Precisely,

```
linprog(f,A,b,Aeq,Beq)
```

solves the problem

$$\min_x f^\top x \qquad \text{s.t.} \quad Ax \leq b, \;\; A_{eq}x = b_{eq}.$$

Thus, the vector of the decision variables will be:

$$x = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \\ M \end{bmatrix}.$$

Since $M = f^\top x$, we set

$$f = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}^\top.$$

By the inequality constraints (19) we have

$$A = \left[\begin{array}{ccccc|c} 1 & 0 & \dots & & & -1 \\ 0 & 1 & 0 & \dots & & -1 \\ \vdots & & \ddots & & & \vdots \\ & & & 1 & & -1 \\ \hline -1 & 0 & \dots & & & -1 \\ 0 & -1 & 0 & \dots & & -1 \\ \vdots & & \ddots & & & \vdots \\ & & & -1 & & -1 \end{array}\right], \qquad b = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{array}\right].$$

Finally, by the equality constraint (18), we have

$$A_{eq} = \left[\begin{array}{cccccc} \Phi^{N-1}\Gamma & \Phi^{N-2}\Gamma & \dots & \Phi\Gamma & \Gamma & 0 \end{array}\right]$$

$$b_{eq} = \beta - \Phi^N \alpha$$

Another possibility to solve the minimum-time problem is that of finding the minimum N such that the following problem is feasible:

$$\min_{u(0),\ldots,u(N-1)} 0 \qquad \text{(dummy objective function)}$$

s.t.

$$-1 \leq u(k) \leq 1 \qquad k = 0, \ldots, N-1$$

$$\Phi^N \alpha + \sum_{k=0}^{N-1} \Phi^{N-k-1} \Gamma u(k) = \beta$$

Suggested reading: Zadeh and Whalen (1962).

In the previous setting, linear *state constraints* can be introduced, for instance the box constraints (figure below, left):

$$-\gamma \leq x(k) \leq \gamma, \qquad k = 0, \ldots, N$$

where the $\leq$ denotes a component-wise inequality.

Or more general constraints (figure below, right) such as:

$$c_i^\top x(k) \leq c_{i0}, \qquad k = 0, \ldots, N, \quad i = 1, \ldots, R$$

Minimum-time (cont.)

To add those constraints to the linear programming problem, observe that:

$$x(k) = \Phi^k x(0) + \sum_{i=0}^{k-1} \Phi^{k-1-i} \Gamma u(i).$$

In other words:

$$\begin{aligned} x(0) &= Ix(0) \\ x(1) &= \Phi x(0) + \Gamma u(0) \\ &\vdots \end{aligned}$$

or, in compact form:

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} = \begin{bmatrix} I \\ \Phi \\ \Phi^2 \\ \vdots \\ \Phi^N \end{bmatrix} x(0) + \begin{bmatrix} 0 & \dots & & \\ \Gamma & 0 & \dots & \\ \Phi\Gamma & \Gamma & 0 & \dots \\ & \ddots & \ddots & \\ \Phi^{N-1}\Gamma & \dots & \Phi\Gamma & \Gamma \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix}$$

Thus, the generic constraint

$$c_i^\top x(k) \leq c_{i0}, \qquad k = 0 \dots N$$

can be written as

$$\underbrace{\begin{bmatrix} 0 & \dots & & \\ c_i^\top \Gamma & 0 & \dots & \\ c_i^\top \Phi\Gamma & c_i^\top \Gamma & 0 & \dots \\ & \ddots & \ddots & \\ c_i^\top \Phi^{N-1}\Gamma & \dots & c_i^\top \Phi\Gamma & c_i^\top \Gamma \end{bmatrix}}_{\mathcal{A}_i} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix} \leq \underbrace{\begin{bmatrix} -c_i^\top x(0) \\ -c_i^\top \Phi x(0) \\ -c_i^\top \Phi^2 x(0) \\ \vdots \\ -c_i^\top \Phi^N x(0) \end{bmatrix} + \begin{bmatrix} c_{i0} \\ c_{i0} \\ c_{i0} \\ \vdots \\ c_{i0} \end{bmatrix}}_{\mathcal{B}_i}.$$

Minimum-time (cont.)

If R is the number of linear constraints, we get

$$A = \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_R \end{bmatrix}, \qquad b = \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \\ \vdots \\ \mathcal{B}_R \end{bmatrix}.$$

The linear constraints on $u(k)$ can be easily introduced, by adding further row-blocks to the matrices A and b.

Minimum-fuel

The minimum-fuel problem consists of minimizing, for fixed N, the cost

$$J = \sum_{k=0}^{N-1} c_k |u(k)|$$

subject to

$$\Phi^N \alpha + \sum_{k=0}^{N-1} \Phi^{N-k-1} \Gamma u(k) = \beta.$$

In other words, we want to steer the state from $x(0) = \alpha$ to $x(N) = \beta$ in N steps while minimizing the "fuel", assumed to be the weighted sum of the absolute values of the control sequence.

Minimum-fuel (cont.)

To reduce the problem to a linear programming one, we introduce the auxiliary variables $g_0, g_1, \dots, g_{N-1}$ and solve:

$$\min_{\substack{g_k, u(k) \\ 0 \le k \le N-1}} \quad \sum_{k=0}^{N-1} c_k g_k$$

s.t.

$$-g_k - u(k) \le 0 \qquad k = 0, \dots, N-1$$

$$-g_k + u(k) \le 0 \qquad k = 0, \dots, N-1$$

$$\Phi^N \alpha + \sum_{k=0}^{N-1} \Phi^{N-k-1} \Gamma u(k) = \beta$$

Kalman filter (outline)

Kalman filter

A fundamental result in Control Theory is the **duality between optimal control and optimal estimation**.

The dual counterpart of the linear quadratic regulator is the *Kalman filter*, which is basically an optimal observer.

In the following, we sketch the basic results, exploiting the duality of linear systems. For a thorough treatment of the subject, refer for instance to Magni and Scattolini (2014).

Kalman filter (cont.)

We begin by stating a particular LQ problem: given the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

we consider the auxiliary output $z(t) = Hx(t)$ and the cost:

$$J = \int_0^\infty z^\top(t)z(t) + u^\top(t)u(t)dt.$$

This is a standard LQ problem where $Q = H^\top H \succeq 0$ and $R = I \succ 0$. If (A, B) is reachable and (A, H) observable, we can find the optimal gain matrix K by solving the Riccati equation

$$A^\top P + PA + H^\top H - PBB^\top P = 0, \tag{20}$$

and letting

$$K = B^\top P.$$

We now consider the closed-loop system:

$$\begin{aligned} \dot{x}(t) &= (A - BK)\, x(t) + I\Delta(t) \qquad (21) \\ \begin{bmatrix} u(t) \\ z(t) \end{bmatrix} &= \begin{bmatrix} -K \\ H \end{bmatrix} x(t) \qquad (22) \end{aligned}$$

having $\Delta(t)$ as input and the compound vector $[u^\top(t) \quad z^\top(t)]^\top$ as output. The impulse response of the system is clearly:

$$W(t) = \begin{bmatrix} -K \\ H \end{bmatrix} e^{(A-BK)t} I = \begin{bmatrix} -Ke^{(A-BK)t} \\ He^{(A-BK)t} \end{bmatrix}.$$

It can be shown that the optimal gain is the one that minimizes

$$\int_0^\infty \mathrm{tr}\left[W^\top(t) W(t)\right] dt = \int_0^\infty \sum_{i,j} W_{ij}^2(t) dt$$

which is the energy of the impulse response.

Indeed, the cost is:

$$J = \int_0^\infty x^\top(t)H^\top Hx(t) + u^\top(t)u(t)dt,$$

and the optimal gain K is such that it is minimized (for any initial state $x(0) = x_0$) by the control $u(t) = -Kx(t)$. Since we have

$$x(t) = e^{(A-BK)t}x_0,$$

the optimal cost can be written as:

$$\begin{aligned} J^o(x_0) &= \int_0^\infty x_0^\top \overbrace{e^{(A-BK)^\top t}\left(H^\top H + K^\top K\right)e^{(A-BK)t}}^{W^\top(t)W(t)} x_0 dt \\ &= \int_0^\infty x_0^\top W^\top(t)W(t)x_0 dt \end{aligned}$$

Now, by taking

$$x_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

Kalman filter (cont.)

we observe that the optimal gain is such that the quantity:

$$J^o(x_0) = \int_0^\infty x_0^\top W^\top(t) W(t) x_0 dt = \int_0^\infty \sum_{i=1}^n W_{i1}^2(t) dt$$

is minimized. By taking successively

$$x_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \quad \dots \quad x_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

we prove that

$$\int_0^\infty \sum_{i=1}^n W_{i2}^2(t) dt, \quad \dots \quad , \int_0^\infty \sum_{i=1}^n W_{in}^2(t) dt$$

are minimized and thus

$$\sum_{j=1}^n \int_0^\infty \sum_{i=1}^n W_{ij}^2(t) dt = \int_0^\infty \sum_{i,j} W_{ij}^2(t) dt = \int_0^\infty \operatorname{tr}\left[W^\top(t) W(t)\right] dt$$

is minimized.

Kalman filter (cont.)

We now consider a state estimation problem, in the presence of

- a *process disturbance* $v(t)$ and
- a *measurement noise* $w(t)$.

Let the system be

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ev(t) \\ y(t) &= Cx(t) + w(t) \end{aligned}$$

and consider the standard observer

$$\dot{\hat{x}}(t) = (A - LC)\hat{x}(t) + Bu(t) + Ly(t). \tag{23}$$

By defining the estimation error as $e(t) = \hat{x}(t) - x(t)$ we get

$$\dot{e}(t) = (A - LC)e(t) + Lw(t) - Ev(t) \tag{24}$$

$$\eta(t) = Ie(t) \tag{25}$$

where $\eta(t)$ is the error itself.

Kalman filter (cont.)

The aim is to choose L optimally, in terms of rejection of disturbance and noise. A possible choice is minimizing the cost:

$$J = \int_0^\infty \operatorname{tr}\left[W_\eta^\top(t) W_\eta(t)\right] dt$$

where $W_\eta(t)$ is the impulse response matrix of the system (24)-(25). Intuitively, we are trying to render small the effect of the disturbance and the noise on the estimation error.

By taking the dual of (24)-(25) we get:

$$\left[\begin{array}{c|cc} A - LC & L & -E \\ \hline I & 0 & 0 \end{array}\right] \quad \overset{\text{dual}}{\longleftrightarrow} \quad \left[\begin{array}{c|c} A^\top - C^\top L^\top & I \\ \hline \begin{bmatrix} L^\top \\ -E^\top \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right]$$

which is the same as (21)-(22) if we apply the substitutions shown on the right side.

$$\left[\begin{array}{c|c} A - BK & I \\ \hline \begin{bmatrix} -K \\ H \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right] \qquad \begin{array}{ccccccc} A & \leftarrow & A^\top & & B & \leftarrow & C^\top \\ K & \leftarrow & -L^\top & & H & \leftarrow & -E^\top \end{array}$$

Kalman filter (cont.)

Thus, we can exploit duality for solving the problem.

Since:

- $\mathrm{tr}\left[MM^\top\right] = \mathrm{tr}\left[M^\top M\right]$, and
- the impulse response matrix of the dual system is the transpose impulse response matrix of the primal,

we can simply apply the substitutions to the primal Riccati equation (20), obtaining the dual Riccati equation:

$$AP + PA^\top + EE^\top - PC^\top CP = 0, \tag{26}$$

solve for the positive definite $\bar{P}$, and let the optimal filter gain be

$$L = \bar{P}C^\top. \tag{27}$$

The Kalman filter is most frequently presented in a stochastic framework. Consider the system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ev(t) \\ y(t) &= Cx(t) + w(t) \end{aligned}$$

and assume that the disturbance v and the measurement noise w are uncorrelated zero-mean Gaussian white-noise stochastic processes with unit covariance:

$$\mathbb{E}\left[v(t)v^\top(\tau)\right] = \delta(t-\tau)I, \qquad \mathbb{E}\left[w(t)w^\top(\tau)\right] = \delta(t-\tau)I,$$

where $\mathbb{E}$ denotes expectation.

Then, if (A, E) is reachable and (A, C) observable, the estimator (23), with L taken according to (26) and (27), is

- asymptotically stable;
- optimal, in the sense that it minimizes the asymptotic expectation of the estimation error:

$$J = \lim_{t\to\infty} \mathbb{E}\left[\|x(t) - \hat{x}(t)\|^2\right].$$

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Lecture 7
Optimal control

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