Correspondence

On Optimal Control and Linear Programming*

L. A. Zadeh

Many of the problems of optimal control of discrete-time systems can be formulated as, or readily reduced to, problems in nonlinear programming. Some of these nonlinear programming problems can in turn be reduced in well-known ways to linear programming problems. As a result, some of the problems of optimal control of discrete-time systems can be reduced to linear programming problems. The purpose of this communication is to exhibit a fairly obvious way in which this can be done in the case of linear, discrete-time systems.

Consider the following standard problem as an illustration. Let S be a discrete-time system characterized by the state equation

$$\mathbf{x}_{t+1} = A\mathbf{x}_t + Bu_t, \tag{1}$$

where $\mathbf{x}_t$ is the state vector at time t ($t=0, 1, 2, \cdots$), u_t is the scalar input at time t, A is an $n\times n$ matrix and B is an n vector. Assume 1) that S is initially (at $t=0$) in a specified state α, 2) that it is desired to take S to a specified state β in minimal time, and 3) that u_t is constrained by $|u_t|\leq 1$ for all t. Let T be the instant at which $x_t=\beta$ for the first time. Then the problem is to find a sequence $\{u_t\}$, $0\leq t \leq T-1$, which minimizes T.

From the state equation (1) it is a simple matter to deduce an explicit relation for the state of S at any time $t>0$ in terms of the state of S at time 0 and the input sequence $u_0, \cdots, u_{t-1}$. It reads

$$\mathbf{x}_t = A^t\mathbf{x}_0 + \sum_{\tau=1}^{t} \boldsymbol{h}_\tau u_{t-\tau}, \tag{2}$$

where $\boldsymbol{h}_t$, the state impulsive response, is the inverse z transform of the matrix product $(zI-A)^{-1}B$. Thus, in order that an input sequence $u_0, \cdots, u_{T-1}$ take S from $\boldsymbol{\alpha}$ (at $t=0$) to $\boldsymbol{\beta}$ (at $t=T$) it is necessary and sufficient that

$$\sum_{\tau=1}^{T} \boldsymbol{h}_\tau u_{t-\tau} = \boldsymbol{\beta} - A^T\boldsymbol{\alpha}. \tag{3}$$

In terms of this relation, the optimal control problem can be stated as follows. Find a sequence $u_0, \cdots, u_{T-1}$ which satisfies (3) with the smallest possible value of T, subject to the constraint $|u_t|\leq 1$, $t=0, \cdots, T-1$.

Now as was pointed by Krasovskii,[1] this problem can be reduced to the solution of Problem A stated below.

Problem A: Starting with a fixed T, minimize the quantity

$$Q(T) = \max_{0\leq t\leq T-1} |u_t| \tag{4}$$

over the u_t subject to the constraint

$$\sum_{\tau=1}^{T} \boldsymbol{h}_\tau u_{t-\tau} = \boldsymbol{\beta} - A^T\boldsymbol{\alpha}. \tag{5}$$

Krasovskii also pointed out that Problem A is equivalent to Problem B stated below.

Problem B: Starting with a fixed T, minimize the quantity

$$Q(T) = \sum_{t=1}^{T} |\boldsymbol{\xi}\cdot\boldsymbol{h}_t|,$$

$$\boldsymbol{\xi}\cdot\boldsymbol{h}_t = \text{scalar product of } \boldsymbol{\xi} \text{ and } \boldsymbol{h}_t \tag{6}$$

over real n vectors $\boldsymbol{\xi}$ subject to the constraint

$$(\boldsymbol{\beta} - A^T\boldsymbol{\alpha})\cdot\boldsymbol{\xi} = 1. \tag{7}$$

In both these problems, if the minimum value of $Q(T)$ exceeds one, start with a larger value of T. If $Q(T)<1$, reduce T until $Q(T)$ equals one. For this value of T, say $T=T_m$, the corresponding minimizing sequence $u_0, \cdots, u_{T-1}$ is a solution to the original problem, and T_m is the minimal value of T.

Now both Problems A and B can readily be put into a linear programming form by using known techniques.[2] Specifically, in the case of Problem B, introduce a set of auxiliary variables $z_1, \cdots, z_T$ satisfying the $2T$ inequalities

$$z_t\geq(\boldsymbol{\xi}\cdot\boldsymbol{h}_t),\quad z_t\geq-(\boldsymbol{\xi}\cdot\boldsymbol{h}_t),\quad t=1, \cdots, T, \tag{8}$$

which together imply

$$z_t \geq |\boldsymbol{\xi}\cdot\boldsymbol{h}_t|. \tag{9}$$

Next, form the objective function

$$\tilde{Q} = \sum_{t=1}^{T} z_t. \tag{10}$$

Now, for any $\boldsymbol{\xi}$ satisfying (7) the minimum value of $\tilde{Q}$ is attained when $z_t=|\boldsymbol{\xi}\cdot\boldsymbol{h}_t|$, $t=1, \cdots, T$. Consequently, the minimization of $\tilde{Q}$ is equivalent to the minimization of Q. In this way, the solution of Problem B is reduced to the minimization of the linear form $\tilde{Q}$ subject to the inequality constraints (8) and equality constraints (7). Another way of attaining the reduction is to introduce $2T$ non-negative variables v_t and w_t such that

$$\tilde{Q} = \sum_{t=1}^{T} (v_t + w_t) \tag{11}$$

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[1] N. N. Krasovskii, "On the theory of optimal regulation," *Automatika i Telemekhanika*, vol. 18, pp. 960–970; November, 1957. An exposition of Krasovskii's method is given in the author's chapter, "Optimal control problems in discrete-time systems," in "Computer Control Systems Technology," C. T. Leondes, Ed., McGraw-Hill Book Co., Inc., New York, N. Y.; 1961.

[2] C. E. Lemke and A. Charnes, "Extremal problems in linear inequalities," Carnegie Institute of Technology, Pittsburgh, Pa., Tech Rept. No. 36; May, 1953.

and

$$v_t - \omega_t = \xi \cdot h_t, \quad t = 1, \cdots, T. \quad (12)$$

Here for any ξ satisfying (7), the minimum value of $\tilde{Q}$ is attained by setting $w_t = 0$ if $\xi \cdot h_t$ is non-negative and $v_t = 0$ if $\xi \cdot h_t$ is negative. Then, clearly, $Q = \tilde{Q}$.

Problem A is closely related to the so called Tchebycheff problem,[3] which was originally formulated and solved by Laplace in 1799,[4] and which can be reduced to a linear programming problem in many ways. For example, let M be an auxiliary variable satisfying the inequalities

$$M \geq u_t, \; M \geq -u_t, \; t = 0, \cdots, T-1. \quad (13)$$

Then Problem A reduces to: Minimize M subject to the inequality constraints (13) and equality constraints (5).

There are many other problems of optimal control of linear discrete-time systems which can be reduced in various ways to linear programming problems. Some of these are discussed in the accompanying communication by B. Whalen.

The author is indebted to Prof. G. Dantzig for the Vallée-Poussin reference.[4]

L. A. Zadeh
Dept. of Elec. Engrg.
University of California
Berkeley, Calif.

B. H. Whalen[5]

As Prof. Zadeh points out above, Krasovskii's formulation of the minimal-time problem for linear discrete systems can be reduced to problems in linear programming. As shown below, a different formulation leads to a different linear program. A minimal-fuel problem is also discussed.

The Minimal-Time Problem

The definitions and the notation in Prof. Zadeh's communication are assumed. The problem is then to choose $T, u_0, \cdots, u_{T-1}$ such that T is minimized, and such that

$$\beta = A^T x_0 + \sum_{t=1}^{T} h_t u_{T-t}$$

$$|u_0| \leq 1, \; t = 0, 1, \cdots, T-1. \quad (1)$$

Krasovskii's idea was to fix T and then minimize over u_t the function

$$\max \{ |u_t|, t = 0, 1, \cdots, T-1 \} \quad (2)$$

subject to the constraints

$$\beta = A^T x_0 + \sum_{t=1}^{T} h_t u_{T-t}. \quad (3)$$

The alternative suggested here is to fix T and then minimize over u_t the function

$$\max \{ |e_i|, i = 1, \cdots n \}$$

where

$$e = (e_1, \cdots, e_n) = \beta - A^T x_0 - \sum_{t=1}^{T} h_t u_{T-t} \quad (4)$$

subject to the constraints

$$|u_t| \leq 1. \quad (5)$$

To reduce this problem to a linear program, introduce a new variable g and minimize g, subject to the $2n$ constraints

$$\left.\begin{aligned} g + e_i \geq 0 \\ g - e_i \geq 0 \end{aligned}\right\} \quad i = 1, \cdots, n, \quad (6)$$

and the $2T$ constraints

$$\left.\begin{aligned} u_t \geq -1 \\ -u_t \geq -1 \end{aligned}\right\} \quad t = 0, 1, \cdots, T-1. \quad (7)$$

Since the e_i are linear functions of u_t, and since g is a linear functional of the variables $g, u_0, \cdots, u_{T-1}$, the above problem is a linear program in these $T+1$ variables.[6] If the minimal g is nonzero, T is adjusted and the process repeated. For the least T such that the minimal g is zero, the solution of the program is the solution of the original minimal-time problem.

For the regulator problem ($\beta = 0$), T should always be adjusted to a larger value, although this is not necessarily true for the control problem. Obviously, a technique which determines only the value of the program, *i.e.*, the minimal g, is satisfactory for all the iterations but the last one.

A Minimal-Fuel Problem

Suppose it is desired to minimize the cost function

$$\sum_{t=0}^{T-1} c_t |u_t| \quad (8)$$

subject to the constraints

$$\beta = A^T x_0 + \sum_{t=1}^{T} h_t u_{T-t}. \quad (9)$$

To reduce this problem to a linear program, introduce the new variables $g_0, \cdots, g_{T-1}$. The equivalent linear program is then

$$\min_{g_t, u_t} \sum_{t=0}^{T-1} c_t g_t \quad (10)$$

subject to the constraints

$$\left.\begin{aligned} g_t + u_t \geq 0 \\ g_t - u_t \geq 0 \\ u_t \geq -1 \\ -u_t \geq -1 \end{aligned}\right| \quad t = 0, 1, \cdots, T-1$$

$$\beta = A^T x_0 + \sum_{t=1}^{T} h_t u_{T-t}. \quad (11)$$

Although this program is more complex than the program for the minimum-time problem, no iterations are involved, since T is specified. If one does not know in advance that the first set of constraints can be satisfied, a more appropriate cost function would include both the fuel and the final error. If the cost is linear in the $|e_i|$, this more general problem can be reduced to a linear program by the same method used above.

Another alternative which results in a linear program is to minimize fuel subject to an inequality constraint on the $|e_i|$.

Conclusion

It should be noted that the dimensions of the linear programs obtained here, as well as those in Prof. Zadeh's communication are proportional to T for large T. However, the special character of the constraints makes it possible to employ specialized algorithms which greatly reduce the computational labor for large values of T.

Aside from practical considerations, there are some theoretical advantages to this approach. For example, from the elementary fact that a solution of the linear program lies on a vertex of the constraint set, one can see that for both the minimal-time and the minimal-fuel problems, the optimal control is "bang-bang" for at least $T-n$ values of t. This is true whether the system is oscillatory or not, and one can thus obtain bounds on the error caused by using an input which can only be plus or minus one.

Another aspect of the linear programming formulation is that every linear program has an equivalent "dual" program. In fact, Krasovskii's observation that Problems "A" and "B" are equivalent is easily proved by demonstrating that the corresponding linear programs are dual to each other.

B. H. Whalen
University of California
Berkeley, Calif.

[3] E. Stiefel, "Note on Jordan elimination, linear programming and Tchebycheff approximation," *Numerische Mathematik*, vol. 2, pp. 1–17; 1960.

[4] C. J. de la Vallee-Poussin, "Sur la methode de l'approximation minimum," *Ann. Soc. Sci. (Bruxelles)*, vol. 35, pt. 2, pp. 1–16; 1911.

[5] Received March 5, 1962. The research reported here was supported by the National Science Foundation.

[6] D. Gale, "The Theory of Linear Economic Models," McGraw-Hill Book Co., Inc., New York, N. Y., chs. 1, 3, 4; 1960.

Plant-Adaptive Optimal Systems*

It is interesting to note that often in the synthesis of optimal control systems one obtains part of the solution to the adaptive problem [1]. Consider, for example, a plant with dynamics, in state space, of the form

$$\dot{x} = f(x, u, w) \quad (1)$$

where x represents the plant output, u the control input, and w a constant plant-parameter vector. The optimal control system then takes the form of Fig. 1.

The value of the parameters in the controller depend, of course, on the plant-parameter vector w. If a relationship between the parameters of the controller and w can be established, then an adaptive path may be included in the optimal system to maintain optimality. See Fig. 2.

The adaptive path must both identify the vector w and actuate the optimal controller. The purpose of this note is to indicate how the relationship between controller parameters and plant parameters may be obtained. The results are best illustrated by a simple example.

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