Control Theory

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Lecture 8: Model predictive control

Model predictive control

Consider the discrete-time system

$$x(k+1) = f(x(k), u(k)) \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f(0,0) = 0$ and $f \in \mathcal{C}^1$. The following constraints are defined:

$$u(k) \in \mathcal{U}, \quad x(k) \in \mathcal{X}, \quad \forall k \in \mathbb{Z} = \{0, 1, 2, \dots\}\,. \tag{2}$$

Given the horizon N, let a cost be defined as

$$J(x(k), u(\cdot), k) = \sum_{i=0}^{N-1} \left(\|x(k+i)\|_Q^2 + \|u(k+i)\|_R^2 \right) + V_f(x(k+N)), \tag{3}$$

where $\|z\|_M^2$ is a shorthand for $z^\top M z$, $Q = Q^\top \succ 0$, $R = R^\top \succ 0$ and $V_f : \mathbb{R}^n \longrightarrow \mathbb{R}^+$.

Model predictive control (cont.)

For each time k, we can formulate the following optimization problem:

Find the optimal input sequence

$$\underline{u}^o = \{u^o(k), u^o(k+1), \dots, u^o(k+N-1)\}$$

such that the cost (3) is minimized and the constraints (2) are satisfied, in accordance to the dynamics (1).

The receding-horizon principle

The *model predictive control* (MPC) is based on the so-called *receding-horizon principle*:

At time k, solve the optimization problem over the finite horizon $[k, k+N]$ and apply only the first input $u^o(k)$ of the obtained optimal sequence $\underline{u}^o$. At time $k+1$, solve a new optimization problem over the interval $[k+1, k+N+1]$, apply the first input of the optimal sequence and so on.

Notice that in doing so, we are implicitly defining a time-invariant state-feedback control law:

$$u(k) = K_{RH}(x(k)). \tag{4}$$

Indeed, since the horizon is shifted forward at each time step, the optimization problem depends only on the current state $x(k)$, irrespective of k.

The receding-horizon principle (cont.)

It should be observed that the control law (4) does not guarantee the stability of the equilibrium, unless some conditions are met.

In the following, we report a formulation of the problem (in terms of the choice of cost and constraints) that guarantees the stability; more precisely, that renders the origin $x = 0$ an asymptotically stable equilibrium.

Stability of the MPC

Consider the system

$$x(k+1) = f(x(k), u(k))$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f(0,0) = 0$ and $f \in \mathcal{C}^1$.

Let the constraints be:

$$u(k) \in \mathcal{U}, \quad x(k) \in \mathcal{X}, \quad \forall k \in \mathbb{Z} = \{0, 1, 2, \dots\}.$$

where the sets $\mathcal{U} \subset \mathbb{R}^m$ and $\mathcal{X} \subset \mathbb{R}^n$ are bounded and closed sets, containing the origin in their interior.

Suppose that an *auxiliary control law* $u = K_a(x)$ is given that renders positively invariant the set $\mathcal{X}_f \subset \mathcal{X}$ containing the origin, guaranteeing the satisfaction of the constraints. In other words, the system

$$x(k+1) = f(x(k), K_a(x(k)))$$

is such that , if $x(\bar{k}) \in \mathcal{X}_f$ then:

1. $x(k) \in \mathcal{X}_f \quad \forall k \geq \bar{k}$, and
2. $u(k) = K_a(x(k)) \in \mathcal{U} \quad \forall k \geq \bar{k}$.

Stability of the MPC (cont.)

Consider now the following optimization problem (where $Q = Q^\top \succ 0$ and $R = R^\top \succ 0$):

$$
\begin{aligned}
\min_{u(\cdot)} J(x(k), u(\cdot), k) \quad &= \quad \sum_{i=0}^{N-1} \left(\|x(k+i)\|_Q^2 + \|u(k+i)\|_R^2 \right) + V_f(x(k+N)) \\
&\text{s.t.} \\
x(k+1) \quad &= \quad f(x(k), u(k)) \\
x(k+i) \quad &\in \quad \mathcal{X} \qquad i = 0, \dots, N-1 \\
u(k+i) \quad &\in \quad \mathcal{U} \qquad i = 0, \dots, N-1 \\
x(k+N) \quad &\in \quad \mathcal{X}_f
\end{aligned}
$$

Notice, in particular, the presence of the terminal constraint.

Stability of the MPC (cont.)

The following theorem provides a sufficient condition for the stability.

Theorem

If for all $x \in \mathcal{X}_f$ the following inequality is satisfied:

$$V_f\left(f\left(x(k), K_a(x(k))\right)\right) - V_f(x(k)) + \left(\|x(k)\|_Q^2 + \|K_a(x(k))\|_R^2\right) \leq 0 \tag{5}$$

then the origin $x = 0$ is an asymptotically stable equilibrium for the closed-loop system obtained by applying the control law (4).

Stability of the MPC (cont.)

Proof.

Let the optimal input sequence at time k be

$$\underline{u}_k^o = \{u_k^o(k), u_k^o(k+1), \ldots, u_k^o(k+N-1)\}\,.$$

If, at time k, we apply $u_k^o(k)$, the next state will be

$$x^o(k+1) = f(x(k), u_k^o(k)).$$

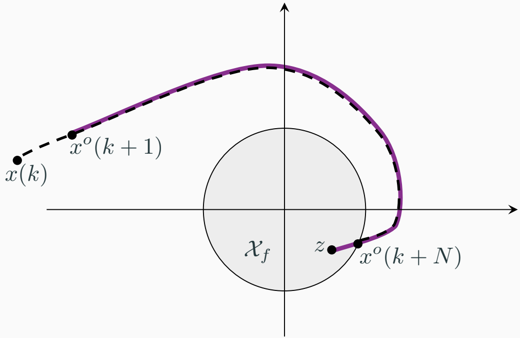
Consider now the sequence

$$\underline{u}_{k+1} = \{u_k^o(k+1), \ldots, u_k^o(k+N-1), K_a(x^o(k+N))\}$$

where $x^o(k+N)$ is the last state of the optimal state sequence at time k. The sequence is certainly admissible (it satisfies the constraints) for the optimization problem formulated at time $k+1$ from the state $x^o(k+1)$.

Stability of the MPC (cont.)

Indeed, it results in a trajectory which overlaps the optimal trajectory of the previous step, except for the last step (from $x^o(k+N)$ to z, in the figure). But the last step does not violates the constraints because the auxiliary control law $K_a(x)$ is admissible in $\mathcal{X}_f$.



However, the sequence $\underline{u}_{k+1}$ is not optimal, in general.

Let $\tilde{J}(x^o(k+1), k+1)$ be the cost obtained by applying $\underline{u}_{k+1}$ from the state $x^o(k+1)$. We have:

$$J^o(x^o(k+1), k+1) \leq \tilde{J}(x^o(k+1), k+1),$$

and subtracting $J^o(x(k), k)$ we get

$$\begin{aligned} J^o(x^o(k+1), k+1) - J^o(x(k), k) &\leq \tilde{J}(x^o(k+1), k+1) - J^o(x(k), k) = \\ &= V_f\left(f\left(x^o(k+N), K_a(x^o(k+N))\right)\right) - V_f(x^o(k+N)) + \\ &\quad + \left(\|x^o(k+N)\|_Q^2 + \|K_a(x^o(k+N))\|_R^2\right) - \left(\|x(k)\|_Q^2 + \|K_{RH}(x(k))\|_R^2\right). \end{aligned}$$

Indeed, the contributes to $\tilde{J}$ and J^o from $k+1$ to $k+N-1$ cancel each other because the trajectories overlap. Notice that the colored term is ≤ 0 by (5), since $x^o(k+N) \in \mathcal{X}_f$.

As a consequence, we have

$$J^o(x^o(k+1), k+1) - J^o(x(k), k) \leq -\left(\|x(k)\|_Q^2 + \|K_{RH}(x(k))\|_R^2\right)$$

and the right-hand side of the inequality is zero only if $x(k) = 0$, because Q is positive definite. Thus, the function

$$\Delta J^o(x) \doteq J^o(x^+) - J^o(x),$$

where $x^+ = f(x, K_{RH}(x))$ is negative definite. Since $J^o(x)$ is positive definite, we conclude by the Lyapunov Theorem that the origin is asymptotically stable. $\square$

Zero terminal constraint

A possible way to enforce the condition

$$V_f\left(f\left(x(k), K_a(x(k))\right)\right) - V_f(x(k)) + \left(\|x(k)\|_Q^2 + \|K_a(x(k))\|_R^2\right) \leq 0 \quad \forall x \in \mathcal{X}_f$$

is employing a *zero terminal constraint*, i.e., letting

$$\begin{aligned} K_a(x) &= 0 \\ \mathcal{X}_f &= \{0\} \\ V_f(x) &= 0 \end{aligned}$$

It is immediate to check that the required condition is satisfied by the sole state $(x = 0)$ belonging to $\mathcal{X}_f$. Moreover, the null control law $K_a(x) = 0$ renders positively invariant the set $\mathcal{X}_f$ and is admissible, since $0 \in \mathcal{U}$.

The zero terminal constraint is a conceptually simple approach, but has a significant disadvantage: the constraint

$$x^o(k+N) \in \mathcal{X}_f = \{0\}$$

can be difficult to satisfy, both in the linear, control-constrained, case and in the nonlinear (even unconstrained) case.

With reference to linear systems, the *quasi-infinite horizon MPC* is formulated as follows:

$$\begin{aligned} K_a(x) &= -K_{LQ}x \\ \mathcal{X}_f &= \{x : x^\top P x \le \alpha\} \subset \mathcal{X} \\ V_f(x) &= x^\top P x \end{aligned}$$

where α is a sufficiently small positive scalar and the matrices K_{LQ} and P are obtained by solving an infinite horizon optimal linear quadratic control problem.

More precisely, given the system

$$x(k+1) = Ax(k) + Bu(k),$$

we solve the infinite horizon LQ problem using the same matrices Q and R appearing in the cost (3). Thus we get the auxiliary control law:

$$K_a(x) = -K_{LQ}x,$$

and the matrix P, as the positive definite solution of the algebraic Riccati equation

$$P = A^\top PA + Q - A^\top PB \underbrace{\left(R + B^\top PB\right)^{-1} B^\top PA}_{K_{LQ}}$$

Quasi-infinite horizon MPC (cont.)

and of the equivalent equation:

$$(A - BK_{LQ})^\top P (A - BK_{LQ}) - P = -\left(Q + K_{LQ}^\top R K_{LQ}\right).$$

Since $\mathcal{X}_f$ is a sublevel set of the LQ cost, it is positively invariant and satisfies the constraints, provided that α is sufficiently small to ensure that

$$u(x) \in \mathcal{U}, \quad \forall x \in \mathcal{X}_f.$$

Notice that such a sufficiently small α does exist, because $u(0) = 0 \in \operatorname{Int}\mathcal{U}$ (the interior of $\mathcal{U}$).

It remains to show that the condition of the theorem is satisfied:

$$\begin{aligned}
&V_f(\underbrace{f(x(k), -K_{LQ}x(k))}_{(A-BK_{LQ})x(k)}) - V_f(x(k)) + \left(\|x(k)\|_Q^2 + \|K_{LQ}x(k)\|_R^2\right) = \\
&= x^\top(k)\left(A - BK_{LQ}\right)^\top P \left(A - BK_{LQ}\right) x(k) - x^\top(k) P x(k) + \\
&\quad + x^\top(k) Q x(k) + x^\top(k) K_{LQ}^\top R K_{LQ} x(k) = \\
&= x^\top(k) \underbrace{\left[(A - BK_{LQ})^\top P (A - BK_{LQ}) - P + Q + K_{LQ}^\top R K_{LQ}\right]}_{\text{zero because } P \text{ solves the Riccati equation}} x(k) = 0.
\end{aligned}$$

Quasi-infinite horizon MPC (cont.)

In the case of nonlinear systems, the auxiliary control law $K_a(x) = -K_{LQ}x$ can be obtained by solving an LQ problem for the linearized system around the origin (details in Magni and Scattolini (2014)).

References

References

Magni, L. and Scattolini, R. (2014). *Advanced and Multivariable Control.* Pitagora Bologna.

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