ANOMALIES & PATH INTEGRAL

Consideriamo $\mathcal{L} = \bar{\Psi}(i\not{D} - m)\Psi$ $\not{D} = \gamma^\mu(\partial_\mu + iA_\mu)$

In dimensioni d ← PARI

fermioni di Dirac in rap. R

abeliano o non-ab. ($A_\mu = A_\mu^a t_R^a$)

Passiamo all'Euclideo (P.I. è meglio definito)

$$x^0 = -ix^d \qquad \partial_0 = i\partial_d \qquad \gamma_M^0 = -i\gamma_E^d \qquad \gamma_M^i = \gamma_E^i$$

$$A_0 = iA_d \qquad\qquad i = 1, \dots, d-1$$

$$\{\gamma_M^\mu, \gamma_M^\nu\} = 2\underbrace{\eta^{\mu\nu}}_{+---} \to \{\gamma_E^\mu, \gamma_E^\nu\} = -2\delta^{\mu\nu}$$

$$\not{\partial}_M = \gamma_M^\mu \partial_\mu = \gamma_M^0\partial_0 + \gamma_M^i\partial_i = \gamma_E^d\partial_d + \gamma_E^i\partial_i = \not{\partial}_E$$

$$(\gamma_M^5)^2 = \mathbb{1} \qquad \{\gamma_M^5, \gamma_M^\mu\} = 0 \quad \to \quad \gamma_M^5 = -(i)^{\frac{d}{2}+1}\gamma^0\gamma^1\cdots\gamma^{d-1}$$

Controlliamo $(\gamma_M^5)^2 = (-1)^{\frac{d}{2}+1}\gamma^0\gamma^1\cdots\gamma^{d-1}\gamma^0\gamma^1\cdots\gamma^{d-1} =$ $(\gamma^0)^2 = 1$

$$= (-1)^{\frac{d}{2}+1}(-1)^{d-1}\gamma^1\cdots\gamma^{d-1}\gamma^1\cdots\gamma^{d-1} =$$

$$= (-1)^{\frac{d}{2}+1}\cancel{(-1)^{d-1}}\,\underbrace{\cancel{(-1)^{d-1}}}_{(\gamma_E^i)^2}\,(-1)^{\sum_{k=1}^{d-2} k} \qquad \sum_{k=1}^{d-2} k = \frac{(\overbrace{d-2+1}^{d-1})(d-2)}{2}$$

$$= (-1)^{\frac{d}{2}+\cancel{1}+\frac{d^2}{2}-\frac{3d}{2}+\cancel{1}} = (-1)^{\frac{d^2}{2}-\underset{\text{pari}}{d}} = (-1)^{\frac{d^2}{2}} = 1$$

d pari $\Rightarrow d^2 \in 4\mathbb{Z}$ //

$$\gamma_E^5 = -(i)^{\frac{d}{2}+1}(-i)\gamma_E^d\gamma_E^1\cdots\gamma_E^{d-1} = -\,i^{d/2}\underset{1/i^{2d-2}}{(-1)^{d-1}}\gamma_E^1\cdots\gamma_E^d =$$

$$= (-i)^{3d/2}\gamma_E^1\cdots\gamma_E^d$$

Consistentemente con $(\gamma_E^\mu)^2 = -1$ possiamo prendere γ_E^μ ANTI-HERM.

$$\Rightarrow \not{D}^\dagger = (\gamma^\mu)^\dagger(-\partial_\mu - iA_\mu^\dagger) = \gamma^\mu(\partial_\mu + iA_\mu) = \not{D}$$

$\Rightarrow$ $\not{D}$ è HERMITIANO! ← qto vale in Euclideo (non in Mink.)

$$iS_M = i\int d^dx\, \bar\psi(i\not D_M - m)\psi = i(-i)\int d^dx_E\, \bar\psi(i\not D_E - m)\psi =$$

$$= -\int d^dx_E\, \bar\psi(-i\not D_E + m)\psi = -S_E$$

Consideriamo il funzionale generatore (def in Eucl.)

$$Z[A] = \int D\psi D\bar\psi\, e^{\int dx\, \bar\psi(i\not D - m)\psi}$$

$$\overset{!}{=} \det(i\not D - m)$$

$\mathcal{L}$ è invariante sotto trasf. assiali (globali) se $m=0$

Troviamo le WI relative alla simm. (globale) assiale:

consideriamo trasf. assiali locali

$$\psi \mapsto \psi' = e^{i\beta(x)\gamma_5}\psi \qquad \bar\psi \mapsto \bar\psi' = \bar\psi\, e^{i\beta(x)\gamma_5} \quad (\star) \qquad \beta = \beta^a t^a_R$$

$$\bar\psi(i\not D - m)\psi \mapsto \bar\psi'(i\not D - m)\psi' = \bar\psi(i\not D - m)\psi - 2im\beta\,\bar\psi\gamma^5\psi - (\partial_\mu\beta)\underbrace{\bar\psi\gamma^\mu\gamma^5\psi}_{j^\mu_A}$$

trasf. infinitesima

$$S' = S - \int dx\, \beta(x)\left[\partial_\mu j^\mu_A(x) - 2im\,P(x)\right]$$

$$Z[A] = \int D\psi D\bar\psi\, e^{-S[\psi,\bar\psi,A]}$$

cambiamo nome delle variab. integr.

$$\overset{!}{=} \int D\psi' D\bar\psi'\, e^{-S[\psi',\bar\psi',A]}$$

cambiamo coordinate usando $(\star)$

$$\overset{!}{=} \int D\psi D\bar\psi\, J[\beta,A]\, e^{-S[\psi,\bar\psi,A]}\, e^{\int dx\,\beta(x)\left(\partial_\mu j^\mu_A - 2imP\right)}$$

tipicamente lo Jacobiano è assunto essere $=1$ (misura invariante) e se però è uno otteniamo WI compatibili con ciò che uno si aspetta classicamente.

Ma in questo caso (trasf. assiale) $J \neq 1$ e questo porta a una VIOLAZIONE delle WI, cioè a un'ANOMALIA.

Riscriviamo lo Jacobiano come (vedi dopo)

$$J[\beta, A] = e^{-\int dx\, \beta(x)\, \mathcal{A}[A](x)}$$

Prendendo $\beta \ll 1$:

$$\underline{Z[A]} = \int D\psi D\bar{\psi}\, e^{-S} \left(1 + \int dx\, \beta(x) \left[\partial_\mu j^\mu_A - 2imP - \mathcal{A}[A] \right] \right)$$

$$= \underline{Z[A]} + \int dx\, \beta(x) \langle \partial_\mu j^\mu_A - 2imP - \mathcal{A}[A] \rangle_A$$

$$\Rightarrow \quad \partial_\mu \langle j^\mu_A \rangle_A = 2im \langle P \rangle + \mathcal{A}[A]$$

ANOMALOUS
A WI

Metodo di Fujikawa per calcolare lo Jacobiano:

$\not{D}$ è HERM. $\rightarrow$ autovalori REALI

$$\not{D} \varphi_n = \lambda_n \varphi_n \qquad \lambda_n \in \mathbb{R} \qquad \varphi_n(x) = \langle x | n \rangle$$

Il set di autofunz. è ortonormale e completo:

$$\int dx\, \varphi_m^\dagger(x)\, \varphi_n(x) = \int dx \langle m | x \rangle \langle x | n \rangle = \langle m | n \rangle = \delta_{mn}$$

$$\sum_n \varphi_n(x)\, \varphi_n^\dagger(y) = \delta(x-y)$$

Espandiamo ψ e $\bar{\psi}$ in autofunz. di $\not{D}$

$$\psi(x) = \sum_n a_n \varphi_n(x) = \sum_n a_n \langle x | n \rangle$$

$$\bar{\psi}(x) = \sum_m \varphi_m^\dagger(x)\, \bar{b}_m = \sum_m \langle m | x \rangle\, \bar{b}_m$$

matrici di cambio di base

$a_n, \bar{b}_m$ sono variabili di Grassmann

φ_n sono spinori di Dirac in rap. R del gruppo G

Prendiamo la misura:

$$D\psi D\bar\psi = \prod_n da_n \prod_m d\bar b_m \Big(\underbrace{\det\langle m|x\rangle \cdot \det\langle x|n\rangle}_{\det\langle m|n\rangle = 1} \Big)^{-1}$$

numeri di Grassmann

$$= \prod_n da_n \prod_m d\bar b_m$$

Azione:

$$\int dx\, \bar\psi\, (i\not{D} - m)\psi = \sum_n (i\lambda_n - m)\, \bar b_n a_n$$

$$\hookrightarrow \det(i\not{D} - m) = \int D\psi D\bar\psi\, e^{\int dx\, \bar\psi (i\not{D} - m)\psi} = \int \prod_n da_n d\bar b_n\, e^{\sum_n (i\lambda_n - m)\bar b_n a_n}$$

$$= \prod_n (i\lambda_n - m)$$

Vediamo come trasformano a_n e $\bar b_m$ sotto trasf. assial.

$$\psi'(x) = \sum_n a_n' \varphi_n(x) \qquad\qquad \bar\psi'(x) = \dots$$

$$= (1 + i\beta(x)\gamma_5) \sum_m a_m \varphi_m(x) \qquad\qquad = \dots$$

$\Downarrow$ usiamo o.n. della base $\{\varphi_n\}$

$$a_n' = \sum_n C_{nm} a_m \qquad \text{con} \qquad C_{nm} = \delta_{nm} + i\int dx\, \beta(x)\, \varphi_n^\dagger \gamma_5 \varphi_m$$

$$\bar b_m' = \sum_k C_{km} \bar b_k$$

(Variabili di Grassmann)

$$\prod_n da_n' = (\det C)^{-1} \prod_n da_n$$

$$\prod_m d\bar b_m' = (\det C)^{-1} \prod_m d\bar b_m$$

Nel caso della QED, $\bar b$ trasforma con $(C^{-1})^t$
$[\rightarrow D\psi' D\bar\psi' = D\psi D\bar\psi]$

$$\Longrightarrow \quad \mathcal{D}\psi' \mathcal{D}\bar{\psi}' = (\det C)^{-2} \mathcal{D}\psi \mathcal{D}\bar{\psi} \quad \Rightarrow \quad J[\beta, A] = (\det C)^{-2}$$

Ora $\det C = e^{\mathrm{Tr} \ln C}$ e $\ln(1+\xi) = \xi + O(\xi^2)$

$\downarrow$

$$J[\beta, A] = e^{-2 \mathrm{Tr} \ln C} = e^{-2 \mathrm{Tr} \ln \left(\delta_{nm} + i \int dx\, \beta(x) \varphi_n^\dagger(x) \gamma_5 \varphi_m(x) \right)} \simeq$$

$$\simeq e^{-2 \mathrm{Tr}\, i \int dx\, \beta(x)\, \varphi_n^\dagger(x)\, \gamma_5\, \varphi_m(x)}$$

$$= e^{-2i \int dx\, \beta(x) \underbrace{\sum_n \varphi_n^\dagger(x) \gamma_5 \varphi_n(x)}}$$

Mal definita :

$$\sum_n \varphi_n^\dagger \gamma_5 \varphi_n = \underset{\substack{\text{spin.}\\ \text{ind.}}}{\mathrm{tr}} \Big(\gamma_5 \underbrace{\sum_n \varphi_n(x) \varphi_n^\dagger(x)}_{\delta(x-x)} \Big) \sim \mathrm{tr}\, \gamma_5 \cdot \delta(0)$$
$$\text{“} \; 0 \cdot \infty \; \text{”}$$

$\longrightarrow$ necessità di REGOLARIZZARE

Idea di Fujikawa: usare un CUTOFF GAUSSIANO

$$\sum_n \varphi_n^\dagger(x) \gamma_5 \varphi_n(x) = \lim_{M \to \infty} \sum_n \varphi_n^\dagger(x) \gamma_5 e^{-\lambda_n^2 / M^2} \varphi_n(x)$$

$$= \lim_{M \to \infty} \sum_n \varphi_n^\dagger(x) \gamma_5 e^{-\not{D}^2 / M^2} \varphi_n(x)$$

per preservare GAUGE INVARIANCE
(qto farà sì che Jacob. dipende da A_μ
anche se la trasf. di variabili di integraz.
non dipende da esso)

Si può generalizzare la regolarizzazione:

$$\sum_n \varphi_n^\dagger \gamma_5 \varphi_n \equiv \lim_{M \to \infty} \sum_n \varphi_n^\dagger \gamma_5 f\left(\not{D}^2 / M^2 \right) \varphi_n$$

Se scegliamo $f(\not{\partial}^2/M^2)$, allora otteniamo
Jacob $= 1$. Però tale
regolarizzazione rompe
l' INV. DI GAUGE.

con $f(y)$ t.c. $f^{(n)}(y) \xrightarrow[y \to \infty]{} 0$ $\quad \forall n = 0, 1, 2, \ldots$

e $f(0) = 1$

Calcoliamo la somma regolarizzata nello sp. di Fourier :

$$\varphi_n(x) = \int \frac{dk}{(2\pi)^d}\, e^{ikx}\, \tilde{\varphi}_n(k)$$

$$v^T A w = \mathrm{tr}\, v^T A w = \mathrm{tr}\, A w v^T$$

$$\text{"}\sum_n \varphi_n^\dagger \gamma_5 \varphi_n\text{"} = \lim_{M\to\infty} \int \frac{dk}{(2\pi)^d} \frac{dq}{(2\pi)^d} \sum_n \tilde{\varphi}_n^\dagger(q) e^{-iqx} \gamma_5 f\left(\frac{\not{D}^2}{M^2}\right) e^{ikx} \tilde{\varphi}_n(k)$$

$$= \lim_{M\to\infty} \int \frac{dk}{(2\pi)^d} \frac{dq}{(2\pi)^d}\, e^{-iqx}\, \mathrm{tr}\, \gamma_5 f\left(\not{D}^2/M^2\right) \underbrace{\sum_n \tilde{\varphi}_n(k) \tilde{\varphi}_n^\dagger(q)}_{(2\pi)^d \delta(k-q)}\, e^{ikx}$$

traccia su indici spinoriali e di rap. R

$$= \lim_{M\to\infty} \int \frac{dk}{(2\pi)^d}\, e^{-ikx}\, \mathrm{tr}\left[\gamma_5 f\left(\not{D}^2/M^2\right)\right] e^{ikx}$$

$$\not{D}^2 = \gamma^\mu \gamma^\nu D_\mu D_\nu = \left(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu]\right) D_\mu D_\nu =$$

$$= -D^2 + \frac{i}{4}[\gamma^\mu,\gamma^\nu] F_{\mu\nu}$$

Inoltre $\quad e^{-ikx} F(\partial_\mu) e^{ikx} = F(\partial_\mu + ik_\mu)$

$$\text{"}\sum_n \varphi_n^\dagger \gamma_5 \varphi_n\text{"} = \lim_{M\to\infty} \int \frac{dk}{(2\pi)^d}\, \mathrm{tr}\left[\gamma_5 f\left(-\frac{(D_\mu+ik_\mu)(D_\mu+ik_\mu)}{M^2} + i\frac{\gamma^\mu\gamma^\nu F_{\mu\nu}}{2M^2}\right)\right]$$

$$= \lim_{M\to\infty} \int \frac{dk}{(2\pi)^d}\, \mathrm{tr}\left[\gamma_5 f\left(\frac{k^2}{M^2} - \frac{2ik_\mu D_\mu + D^2}{M^2} + i\frac{\gamma^\mu\gamma^\nu F_{\mu\nu}}{2M^2}\right)\right]$$

- $\mathrm{tr}\,\gamma_5 = 0$, $\mathrm{tr}\,\gamma_5 \gamma^{\mu_1}\cdots\gamma^{\mu_l} = 0$ se $l \neq d$,

  $\mathrm{tr}\,\gamma_5 \gamma^{\mu_1}\cdots\gamma^{\mu_d} = i^{\frac{3d}{2}}\, 2^{d/2} \epsilon^{\mu_1\cdots\mu_d}$ $\quad(\epsilon^{12\cdots d} = 1)$

- espandiamo f in Taylor attorno a k^2/M

$$= \lim_{M\to\infty} \int \frac{dk}{(2\pi)^d} \frac{i^{\frac{3d}{2}} \cancel{2}^{d/2}}{\left(\frac{d}{2}\right)! (\cancel{2}M^2)^{d/2}} f^{(d/2)}(k^2/M^2)\, \epsilon^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_{d/2}\nu_{d/2}}\, \mathrm{tr}\, iF_{\mu_1\nu_1}\cdots iF_{\mu_{\frac{d}{2}}\nu_{\frac{d}{2}}}$$

sopravvive solo un termine at leading order in $1/M^2$

$k \to Mk$

$$= \cancel{\lim_{M\to\infty}} \int \frac{dk}{(2\pi)^d} \frac{\overbrace{i^{\frac{3d}{2}}\, i^{d/2}}^{1}}{\left(\frac{d}{2}\right)!} f^{(d/2)}(k^2)\, \epsilon^{\mu_1\cdots\nu_{d/2}}\, \mathrm{tr}\, F_{\mu_1\nu_1}\cdots F_{\mu_{d/2}\nu_{d/2}} + \mathcal{O}(1/M^2)$$

$$= \epsilon^{\mu_1\cdots\nu_{d/2}}\, \mathrm{tr}\, F\cdots F\, \frac{1}{(d/2)!} \underbrace{\int \frac{d^dk}{(2\pi)^d} f^{(d/2)}(k^2)}$$

coor. polari

$$\frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty \frac{k^{d-1}\,dk}{(2\pi)^d} f^{(d/2)}(k^2)$$

$r = k^2$

$dr = 2k\,dk$

$$\frac{\pi^{d/2}}{\left(\frac{d}{2}-1\right)!} \int_0^\infty \frac{r^{d/2-1}\,dr}{(2\pi)^d} f^{(d/2)}(r) = \frac{(-1)^{d/2}}{2^d \pi^{d/2}}$$

$$\frac{1}{(n-1)!} \int_0^\infty r^{n-1} f^{(n)}(r)\,dr = \frac{1}{(n-1)!} \int_0^\infty \left[\underbrace{\cancel{\frac{d}{dr}\left(r^{n-1} f^{(n-1)}\right)}}_{=0} - (n-1)\, r^{n-2} f^{(n-1)}(r) \right] dr$$

$$= -\frac{(n-1)}{(n-1)!} \int_0^\infty r^{n-2} f^{(n-1)}\,dr = \cdots = (-1)^{n-1} \frac{(n-1)!}{(n-1)!} \int_0^\infty f'(r)\,dr =$$

$$= (-1)^{n-1} f\Big|_0^\infty = (-1)^n$$

$$\text{“}\sum_n \varphi_n^\dagger \gamma_5 \varphi_n\text{”} = \epsilon^{\mu_1\nu_1\cdots\mu_{d/2}\nu_{d/2}}\, \mathrm{tr}\, F_{\mu_1\nu_1}\cdots F_{\mu_{d/2}\nu_{d/2}}\, \frac{(-1)^{d/2}}{2^d \left(\frac{d}{2}\right)!\, \pi^{d/2}}$$

$$J[\beta, A] = e^{-2i\int dx\, \beta(x) \sum_n \varphi_n^+ \gamma_5 \varphi_n} \qquad \left[\quad J[\beta, A] = e^{-\int dx\, \beta(x)\, \mathcal{A}[A](x)} \quad\right]$$

$$\Rightarrow \mathcal{A}[A] = 2i\ \epsilon^{\mu_1\nu_1 \cdots \mu_{d/2}\nu_{d/2}}\ \mathrm{tr}\, F_{\mu_1\nu_1} \cdots F_{\mu_{d/2}\nu_{d/2}}\ \underbrace{\frac{(-1)^{d/2}}{2^d \left(\frac{d}{2}\right)!\, \pi^{d/2}}}$$

$$\begin{cases} d=4 & \dfrac{1}{16 \cdot 2 \cdot \pi^2} = \dfrac{1}{32\pi^2} \\ d=2 & \dfrac{-1}{4 \cdot 1 \cdot \pi} = -\dfrac{1}{4\pi} \end{cases}$$

Euclideo ↓ Minkowski ↓

Per $d=4$

$$\mathcal{A}[A] = \frac{i}{16\pi^2}\, \epsilon^{\mu\nu\alpha\beta}\, \mathrm{tr}\, F_{\mu\nu} F_{\alpha\beta} \quad \rightarrow \quad -\frac{1}{16\pi^2}\, \epsilon^{\mu\nu\alpha\beta}\, \mathrm{tr}\, F_{\mu\nu} F_{\alpha\beta}$$

(lezione 22; singlet anom.)

risultati trovati col calcolo dei correlatori

(lezione 20)

Per $d=2$

$$\mathcal{A}[A] = -\frac{i}{2\pi}\, \epsilon^{\mu\nu}\, F_{\mu\nu} \quad \rightarrow \quad \frac{1}{2\pi}\, \epsilon^{\mu\nu}\, F_{\mu\nu} \qquad (\epsilon^{01} = +1)$$

caso abeliano

Continuazione a Minkowski

$\partial_0 = i\partial_d$

$A_0 = i A_d$

$$d=4: \quad \epsilon^{\mu\nu\alpha\beta}_{\text{EUCL}}\, \mathrm{tr}\, F_{\mu\nu} F_{\alpha\beta} = 4!\,(-)\,\mathrm{tr}\, F_{41} F_{23} = 4!\, i\, \mathrm{tr}\, F_{01} F_{23} = i\, \epsilon^{\mu\nu\alpha\beta}_{\text{MINK.}}\, \mathrm{tr}\, F_{\mu\nu} F_{\alpha\beta}$$

$\epsilon_E^{1234} = +1$

$$d=2: \quad \epsilon^{\mu\nu}_{\text{EUCL.}}\, F_{\mu\nu} = 2F_{12} = 2(\partial_1 A_2 - \partial_2 A_1) = -2i(\partial_1 A_0 - \partial_0 A_1) = 2i(\partial_0 A_1 - \partial_1 A_0) = 2i\, F_{01} = +i\, \epsilon^{\mu\nu}_{\text{MINK.}}\, F_{\mu\nu}$$

$\epsilon_E^{12} = +1$

convenzione $\epsilon^{01} = +1$