ESAME SCRITTO DEL 23/1/2023

SOLUZIONE

Risposte corrette alle domande a risposta multipla:

1. (c)
2. (b)
3. (c)
4. (a)
5. (c)
6. (c)
7. (b)
8. (a)

Esercizi.

1.(a) $\mathcal{E} = \{e_1, e_2, e_3\}$ base canonica.

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow f(e_1) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow f(e_2) = -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow f(e_3) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad M^{\mathcal{E}}_{\mathcal{E}}(f) = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(b) $rg(f) = \dim(im(f)) = rg(M^{\mathcal{E}}_{\mathcal{E}}(f)) = 2$

teo. della dimensione

$\Rightarrow$ $\dim(Ker(f)) = 3 - 2 = 1$

$\Rightarrow$ $Ker(f) = Span(e_1) \Rightarrow \{e_1\}$ è base di $Ker(f)$

$$im(f) = Span\left(\underbrace{\begin{pmatrix}-1\\0\\1\end{pmatrix}}_{\substack{\shortparallel\\ f(e_2)}}, \underbrace{\begin{pmatrix}1\\1\\1\end{pmatrix}}_{\substack{\shortparallel\\ f(e_3)}}\right) \Rightarrow \left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \text{ è base di } im(f)$$

(c) $Ker(f) \cap im(f) \neq \{0\} \Leftrightarrow \begin{pmatrix}1\\0\\0\end{pmatrix} \in Span\left(\begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right)$

Si verifica che questo non è vero

$\Rightarrow Ker(f) \cap im(f) = \{0\} \Rightarrow \mathbb{R}^3 = Ker(f) \oplus im(f)$

2.(a) Sia $\mathcal{C} = \{e_1, e_2, e_3\}$ la base canonica di $\mathbb{C}^3$.

$$M_{\mathcal{C}}^{\mathcal{C}}(f_c) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & c & 0 \end{pmatrix} =: A_c$$

$$P_{f_c}(t) = \det \begin{pmatrix} -t & 0 & 1 \\ 1 & -t & 0 \\ 0 & c & -t \end{pmatrix} = -t^3 + c$$

$\Rightarrow Sp(f_c) = \{ \omega^k \cdot \sqrt[3]{c} \mid k = 0, 1, 2 \}$, $\omega = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)$.

Se $c \neq 0$, f_c ha 3 autovalori distinti, quindi è diagonalizzabile.

Se $c=0$, $Sp(f_0)=\{0\}$, $m_a(0)=3$.

$$m_g(0)=\dim\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \middle| \ z=x=0\right\} = 1 \neq m_a(0)$$

Quindi f_0 non è diagonalizzabile.

(b) Siano $z_1=\omega$ e $z_2=\omega^2$

$\Rightarrow Sp(f_1)=\{1, z_1, z_2\}$

$$V_1=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \middle| \begin{pmatrix}-1 & 0 & 1\\ 1 & -1 & 0\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}=\mathrm{Span}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$V_{z_k}=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \middle| \begin{pmatrix}-z_k & 0 & 1\\ 1 & -z_k & 0\\ 0 & 1 & -z_k\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}=\mathrm{Span}\begin{pmatrix}1\\z_k^2\\z_k\end{pmatrix}$$

$$\Rightarrow \quad \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ z_1^2 \\ z_1 \end{pmatrix}, \begin{pmatrix} 1 \\ z_2^2 \\ z_2 \end{pmatrix} \right\}$$

$$M^{\mathcal{B}}_{\mathcal{B}}(f_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z_1 & 0 \\ 0 & 0 & z_2 \end{pmatrix}$$

3. (a) Risolvendo l'equazione

$$x - 3y + 2z = 0$$

si ottiene una base di V: $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right\}$.

Applichiamo Gram-Schmidt a tale base:

$$v_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad \tilde{v}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \left\langle \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \right\rangle \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \frac{6}{10} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 9/5 \\ 3/5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/5 \\ -3/5 \\ -1 \end{pmatrix}$$

$$\Rightarrow v_2 = \frac{1}{\sqrt{1/25 + 9/25 + 1}} \begin{pmatrix} 1/5 \\ -3/5 \\ -1 \end{pmatrix} = \frac{5}{\sqrt{35}} \begin{pmatrix} 1/5 \\ -3/5 \\ -1 \end{pmatrix}$$

(b) Sia s la retta cercata. Allora s ha eq. parametriche

$$s : \begin{cases} x = 1 + ta \\ y = 0 + tb \\ z = 1 + tc \end{cases}, \quad t \in \mathbb{R}$$

dove $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in V \setminus \{0\}$ ed $\underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \perp \pi}$

$$\Updownarrow$$

$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in V \setminus \{0\}$ e $a = 0$ $\quad\quad \left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{base della giacitura di } \pi} \right\rangle = 0$

$$\Rightarrow \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \quad \text{con } 3\lambda + 2\mu = 0$$

Ad esempio:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}.$$