GEOMETRIA - SCRITTO 6/2/2023

SOLUZIONE

Domande a risposta multipla.

1. $(A|b) = \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 2 & 0 & a & 1 \\ 0 & 2a & 5 & 0 \end{array}\right) \xrightarrow[II-2I]{} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -2 & a+4 & -1 \\ 0 & 2a & 5 & 0 \end{array}\right)$

$\xrightarrow[III+aII]{} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -2 & a+4 & -1 \\ 0 & 0 & 5+a^2+4a & -a \end{array}\right) \Rightarrow rg(A) = \begin{cases} 3, \text{ se } a^2+4a+5 \neq 0 \\ 2, \text{ se } a^2+4a+5 = 0 \end{cases}$

$a^2+4a+5 = 0 \iff a = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

$\Rightarrow rg(A|b) = 3 \quad \forall a \in \mathbb{C} \Rightarrow$ (b) è la risposta corretta

2. $\det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix} = 2 \neq 0 \Rightarrow$ **(a) è corretta**

3. $A^{-1} = \underbrace{\frac{1}{\det A}}_{\| \; 1} \cdot {}^t Cof(A) = {}^t\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow$ **(c) è corretta**

4. **(b) falso** per il teorema della dimensione:

$3 = \dim(\mathbb{R}^3) = \dim \ker(f) + \underbrace{rg(f)}_{\| \; 2}$

5. **(e) Falso** perché, se esistesse,

$$f\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}f\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}f\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix}0\\-1\end{pmatrix}+\begin{pmatrix}2\\2\end{pmatrix}\right]$$

$$= \frac{1}{2}\left[\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}1\\-1\end{pmatrix}\right] \neq \begin{pmatrix}0\\0\end{pmatrix}$$

6. **(c)** perché $rg(A) = rg({}^tA)$

e $rg({}^tA) = 1 \Rightarrow$ le colonne di ${}^tA$ non sono l.i.

7. **(a)** A diagonalizzabile $\Rightarrow m_a(-1)+m_a(2)=3$

e $m_a(-1) = m_g(-1)$, $m_a(2) = m_g(2) \Rightarrow m_g(2)=1$

8. $\boxed{(c)}$

$$(A|b) = \left(\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & -1 & 2 \end{array}\right)$$

$$\xrightarrow{OE\,1} \begin{array}{c} II \\ I \\ III \\ IV \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & -1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & -1 & 2 \end{array}\right) \xrightarrow[\substack{II-2I \\ III-I \\ IV-I}]{OE\,3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -3 & -1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & -2 & 3 \end{array}\right)$$

$$\xrightarrow{OE\,1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -1 & -2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & -3 & -1 & 3 \end{array}\right) \xrightarrow{OE\,3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -1 & -2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 5 & -6 \end{array}\right)$$

$\Rightarrow rg(A) = 3$, $rg(A|b) = 4 \Rightarrow$ rette sghembe

Esercizi

1. (a) $\begin{cases} x_1 + x_2 + x_4 = 0 \\ 2x_1 - x_2 - x_3 = 0 \end{cases} \xrightarrow[II - 2I]{0E3} \begin{cases} x_1 + x_2 + x_4 = 0 \\ -3x_2 - x_3 - 2x_4 = 0 \end{cases}$

$$\Rightarrow U = \left\{ \begin{pmatrix} -x_2 - x_4 \\ x_2 \\ -3x_2 - 2x_4 \\ x_4 \end{pmatrix} \mid x_2, x_4 \in \mathbb{R} \right\} = \mathrm{Span}\left(\begin{pmatrix} -1 \\ 1 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right)$$

$$\Rightarrow \left\{ \begin{pmatrix} -1 \\ 1 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\} \text{ è una base di } U.$$

(b) $\mathbb{P}: \underbrace{\begin{cases} x_1 + x_2 + x_4 = 1 \\ 2x_1 - x_2 - x_3 = 3 \end{cases}}_{\text{eq. cartesiane}}$

$$\underbrace{\begin{cases} x_1 = 1 - s - t \\ x_2 = s \\ x_3 = -1 - 3s - 2t \\ x_4 = t \end{cases}}_{\text{eq. parametriche}} \quad s, t \in \mathbb{R}$$

(c)

$$S : \lambda(x_1+x_2+x_4-1)+\mu(2x_1-x_2-x_3-3)=0$$

$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in S \iff \mu = 0$

2. (a), (b) Sia $\mathcal{C}$ la base canonica di $\mathbb{R}^3$.

$M_{\mathcal{C}}^{\mathcal{C}}(f) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ è simmetrica, $\mathcal{C}$ è ortonormale per $\langle , \rangle$

Per il teorema spettrale f è diagonalizzabile ed $\exists$ $\mathcal{B}$ base ortonormale di $(\mathbb{R}^3, \langle , \rangle)$ diagonalizzante

$$P_f(t) = \det\begin{pmatrix} 1-t & 1 & 0 \\ 1 & -t & 1 \\ 0 & 1 & 1-t \end{pmatrix} = (1-t)\left[t^2-t-1\right]+t-1$$

$$= (1-t)(t^2-t-2) = (1-t)(t+1)(t-2)$$

$$\Rightarrow Sp(f) = \{1, -1, 2\}$$

$$V_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} = Span\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$V_{-1} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} = Span\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$V_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} = Span\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

(c) $$M_{\mathcal{B}}^{\mathcal{B}}(f) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

3. (a) $M_B(g)$ è simmetrica $\Rightarrow$ g è simmetrica.

$$g\left(\begin{pmatrix}x\\y\\z\end{pmatrix},\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = \begin{pmatrix}x & y & z\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\1 & 2 & 1\\0 & 1 & 2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} =$$

$$= \begin{pmatrix}x & y & z\end{pmatrix}\begin{pmatrix}x+y\\x+2y+z\\y+2z\end{pmatrix} = x^2+xy+yx+2y^2+yz+zy+2z^2 =$$

$$= 2xy+2y^2+2yz+2z^2 =$$

$$= (x+y)^2+(y+z)^2+z^2 \geq 0 \quad \forall \begin{pmatrix}x\\y\\z\end{pmatrix} \text{ ed } \tilde{e}$$

$= 0 \iff \begin{cases} x+y=0 \\ y+z=0 \\ z=0 \end{cases} \iff x=y=z=0.$

$\Rightarrow g$ è definita positiva $\Rightarrow$ è un prodotto scalare.

(b) $\dim V = 2 \Rightarrow \dim(V^\perp) = 1 \Rightarrow$ basta trovare un vettore $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ t.c. $g\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = 0$ e $g\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right) = 0.$

$$0 = g\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} x & y & z \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = x + y$$

$$0 = g\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = x + 2y + 2z$$

$$\Rightarrow V^{\perp} = \mathrm{Span}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = W$$

(c) Una base ortonormale di W è

$$\frac{1}{\sqrt{g\left(\begin{pmatrix}2\\-2\\1\end{pmatrix},\begin{pmatrix}2\\-2\\1\end{pmatrix}\right)}}\begin{pmatrix}2\\-2\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$$\Rightarrow P_W^{\perp}\begin{pmatrix}0\\1\\0\end{pmatrix} = g\left(\begin{pmatrix}0\\1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}2\\-2\\1\end{pmatrix}\right)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}2\\-2\\1\end{pmatrix} =$$

$$= \frac{1}{2}g\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}2\\-2\\1\end{pmatrix}\right)\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0&1&0\end{pmatrix}\begin{pmatrix}1&1&0\\1&2&1\\0&1&2\end{pmatrix}\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$$\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&2&1\end{pmatrix}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}2\\-2\\1\end{pmatrix}\cdot$$