CHAPTER 5

THERMODYNAMIC DIAGRAMS

5.1 General Considerations

The primary function of a thermodynamic diagram is to provide a graphical display of the lines representing the major kinds of processes to which air may be subject, namely isobaric, isothermal, dry adiabatic, and pseudoadiabatic processes. Lines of constant value of saturation mixing ratio are also needed to permit the various kinds of graphical operations discussed in Chapter 4. We may plot on such a diagram the observed state of any set of air parcels and then be in a position to evaluate graphically the effect of any of these processes. Since energy changes are of primary importance, the first desirable characteristic of such a diagram is that the area enclosed by the lines representing any cyclic process be proportional to the change in energy or the work done during the process. This is such an important property that the designation *thermodynamic diagram* is often reserved for those in which area is proportional to work or energy.

The second desired characteristic of a diagram is that as many as possible of the fundamental lines be straight. The more a diagram satisfies this criterion the easier it will be to use.

The third desideratum is that the angle between the isotherms and the dry adiabats shall be as large as possible. As we shall see later, when soundings of the upper atmosphere are plotted on these diagrams the slope of the sounding is often compared to the slopes of the lines on the diagram. Thus the greater the difference in slope between an isotherm and an adiabat the easier it is to detect variations in slope. An isotherm-to-adiabat angle near 90° is considered very good.

We have already seen that the fundamental expression for an element of specific work, $dw = p d\alpha$, suggests that we use p and α as the coordinates in order to satisfy the first criterion. However, the angle between the isotherms and adiabats of an $\alpha, -p$ diagram is quite small so such a diagram does not satisfy the third criterion. We must seek a means of setting up other suitable diagrams in which the coordinates are two functions of thermodynamic variables, subject to the restriction that the area enclosed by any cycle in the new diagram shall be equal to the area enclosed by the same cycle on a $\alpha, -p$ diagram. Such a diagram is called an *equal-area* transformation of the $\alpha, -p$ diagram. We may then examine these new diagrams to see how well they satisfy the other two criteria.

Consider two variables A and B. Let each be a function of one or more thermodynamic variables. Since a thermodynamic variable is determined by the state of a system it suffices to know α and p for a

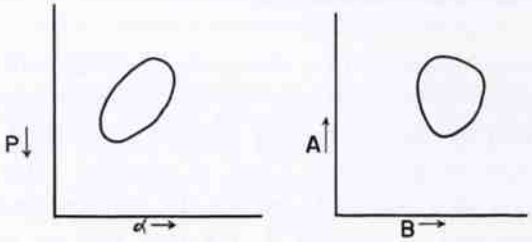


FIG. 5.1. Representation of a cycle on an $\alpha, -p$ diagram and its equal-area transformation to an A, B diagram

parcel in order to determine A and B. Thus each point on an $\alpha, -p$ diagram corresponds to a point on an A, B diagram and any closed cycle in one is a closed cycle (perhaps of different shape) on the other, as in Fig. 5.1. We shall require that the area enclosed on one diagram be equal to the area enclosed on the other. This insures that an A, B plot will be a thermodynamic diagram. Thus

$$-\oint p d\alpha = \oint A dB$$

for any given cyclic process. Thus

$$\oint (p d\alpha + A dB) = 0$$

But for this closed lines integral to be zero the integrand must be an exact differential—for example, ds:

$$p d\alpha + A dB = ds$$

where we shall look upon s as a function of α and B. But from calculus

$$ds(\alpha, B) = \left(\frac{\partial s}{\partial \alpha}\right)_B d\alpha + \left(\frac{\partial s}{\partial B}\right)_\alpha dB$$

Therefore, sufficient conditions for an equal-area transformation are

$$p = \left(\frac{\partial s}{\partial \alpha}\right)_B \quad \text{and} \quad A = \left(\frac{\partial s}{\partial B}\right)_\alpha$$

If we now differentiate the first term partially with respect to B and the second with respect to α we get

$$\left(\frac{\partial p}{\partial B}\right)_\alpha = \frac{\partial^2 s}{\partial \alpha \partial B} \quad \text{and} \quad \left(\frac{\partial A}{\partial \alpha}\right)_B = \frac{\partial^2 s}{\partial \alpha \partial B}$$

Therefore, if

$$\left(\frac{\partial A}{\partial \alpha}\right)_B = \left(\frac{\partial p}{\partial B}\right)_\alpha \tag{5.1}$$

then areas will be equal on the two diagrams. If we now specify the nature of the thermodynamic variable B, it is possible to determine what A must be in order to have an equal-area transformation from α, $-p$ to A, B.

5.2 The Emagram

Consider the case where $B = T$. This is a logical choice for one of the coordinates of a diagram, since one of the atmospheric properties we measure is the temperature. From Eq. (5.1),

$$\left(\frac{\partial A}{\partial \alpha}\right)_T = \left(\frac{\partial p}{\partial T}\right)_\alpha$$

The right side can be evaluated from the equation of state for air, $p\alpha = RT$. We find

$$\left(\frac{\partial A}{\partial \alpha}\right)_T = \frac{R}{\alpha}$$

or

$$\left(\frac{\partial A}{\partial \alpha}\right)_T d\alpha = R\frac{d\alpha}{\alpha}$$

Upon integration we obtain

$$A = R \ln \alpha + F(T) \tag{5.2}$$

where instead of a constant of integration we obtain an unspecified function of T because the partial derivative we integrated required T to be held constant. This $F(T)$ is completely at our disposal, just as a constant of integration may be specified at will. We now take the logarithm of the equation of state:

$$\ln \alpha = -\ln p + \ln R + \ln T$$

and substitute into Eq. (5.2) to get

$$A = -R \ln p + [R \ln R + R \ln T + F(T)]$$

We shall choose $F(T)$ such that the terms in brackets cancel completely. We are free to do this since the terms in brackets consist of constants and functions of temperature only. Therefore there is an $F(T)$ which will reduce the brackets to zero. Finally we obtain

$$A = -R \ln p$$
$$B = T$$

as the coordinates of a thermodynamic diagram. This plot was called the *emagram* by Refsdal as an abbreviation for "energy-per-unit-mass diagram." It consists of a linear scale of temperature along the abscissa and a logarithmic scale of pressure along the ordinate decreasing upward, as in the atmosphere. Consequently the isobars and isotherms are straight and perpendicular to each other. This is desirable since pressure and temperature are the two most commonly measured atmospheric variables.

Since pressure is on a logarithmic scale, the line $p = 0$ is at infinity and the diagram must be terminated at some conveniently low pressure such as 400 mb. This is not a great obstacle since the spacings of a logarithmic scale are repetitive. For example, the distance between the isobars 800 mb and 400 mb on this diagram is proportional to ln 800 — ln 4C0 = ln 2, while the separation between isobars 400 mb and 200 mb is also proportional to ln 2. Thus each isobar can be relabeled with any fraction of its original value. Of course the values of θ and θ_e attached to the adiabats and pseudoadiabats would have to be changed appropriately. (See Problem 1.)

The shape of the dry adiabats on an emagram can be deduced by taking the logarithm of Poisson's equation considering θ constant:

$$-\ln p = -\frac{1}{\kappa} \ln T + \text{constant}$$

Since $-\ln p$ is one of the coordinates of the diagram but $\ln T$ is not, the dry adiabats are logarithmic curves. They become steeper with decreasing temperature but do not depart markedly from straight lines in the usual meteorological range. A similar repetitive spacing exists for these logarithmic lines as for the isobars. The pseudoadiabats are markedly curved but the saturation mixing ratio lines are gently curved. Thus, considering the second criterion of a diagram, the isobars and isotherms are exactly straight, the adiabats and w_s lines are only slightly curved in the usual range of values, and the θ_e lines are more definitely curved.

The adiabat-isotherm angle can be changed by varying one or the other of the coordinate scales. However, convenience of use and economy of paper limits one's freedom here. In practice the angle between adiabats and isotherms on an emagram is near 45°. This is appreciably better than the corresponding angle on an α, $-p$ diagram.

In summary, the emagram has (1) area proportional to energy; (2) four sets of lines which are exactly or nearly straight and one set which is curved; (3) an adequately good angle between adiabats and isotherms. Therefore it is a convenient diagram which is in wide use.

5.3 The Tephigram

This diagram may be developed by letting $B = T$ as for the emagram. Thus

$$A = R \ln \alpha + F(T)$$

as before. But this time instead of substituting from the equation of state let us introduce potential temperature from Poisson's equation

$$\frac{T}{\theta} = \left(\frac{p}{1000}\right)^{\kappa} = \left(\frac{RT}{1000\alpha}\right)^{\kappa}$$

By taking logarithms and solving for $\ln \alpha$, we get

$$\ln \alpha = \frac{1}{\kappa}[\ln \theta - \ln T] + \ln T + \ln R - \ln 1000$$

or

$$R \ln \alpha = c_p \ln \theta + G(T)$$

where the function $G(T)$ includes the constants 1000 mb and R. Therefore

$$A = c_p \ln \theta + F(T) + G(T)$$

This time we shall choose the arbitrary function $F(T) = -G(T)$ so the coordinates become

$$A = c_p \ln \theta$$
$$B = T$$

Since $c_p \ln \theta$ is equal to entropy, apart from an additive constant, Sir Napier Shaw, who introduced this diagram, called it the T-ϕ diagram or *tephigram* for short.

The equation of the isobars on a tephigram may be obtained by taking the logarithm of Poisson's equation. For a constant value of p

$$\ln \theta = \ln T + \text{const}$$

Since one coordinate of this diagram is $\ln \theta$ but the other is a linear scale of T, the isobars are logarithmic curves which slope upward to the right and decrease in slope with increasing temperature. In the rather restricted range of meteorological conditions the isobars have only gentle curvature and are nearly straight. It is possible to rotate the diagram clockwise so the isobars are essentially horizontal with pressure decreasing upward as it does in the atmosphere. However, this is not absolutely necessary.

The pseudoadiabats are appreciably curved, but the saturation mixing ratio lines are nearly straight on a tephigram.

By the very nature of the diagram, the angle between isotherms and adiabats is exactly 90°. Thus this diagram is one in which changes of slope of a sounding are easily detected and comparison of slopes is readily accomplished. This large angle, which is roughly double that of the emagram, is the greatest advantage of the tephigram.

In summary, the tephigram has (1) area proportional to energy; (2) four sets of lines which are exactly or nearly straight and only one set which is quite curved; (3) an isotherm-to-adiabat angle which is large. This diagram comes very close to satisfying all three criteria perfectly and consequently it is used widely.

5.4 The Skew T-Log p Diagram

This diagram represents an attempt to modify the emagram so as to make the isotherm-adiabat angle more nearly 90°. It was first suggested by Herlofson.[1] We let $B = -R \ln p$, so this coordinate is identical with one coordinate of the emagram. Then Eq. (5.1) becomes

$$\left(\frac{\partial A}{\partial \alpha}\right)_{\ln p} = -\frac{1}{R}\left(\frac{\partial p}{\partial \ln p}\right)_{\alpha}$$

or

$$\left(\frac{\partial A}{\partial \alpha}\right)_{\ln p} = -\frac{p}{R}$$

When we multiply by $d\alpha$ and integrate, holding $\ln p$ (and therefore p) constant, we obtain

$$A = -\frac{p\alpha}{R} + F(\ln p)$$

or

$$A = -T + F(\ln p)$$

[1] N. Herlofson, The T, log p-diagram with Skew Coordinate Axes. *Meteor. Ann.*, 2, pp. 311-342, 1947.

We shall choose the arbitrary function to be

$$F(\ln p) = -K \ln p,$$

where K is a constant we may choose at will. We are not concerned with the sign of an area since this only involves the direction in which a cycle is carried out. Thus the coordinates may be written

$$A = T + K \ln p,$$
$$B = -R \ln p$$

The diagram is constructed with B as the ordinate and A as the abscissa. Consequently the ordinate is identical with that of the emagram.

On this diagram an isotherm has the equation

$$A = \text{const} + K \ln p$$

or

$$A = \text{const} - \frac{K}{R} B$$

or

$$B = -\frac{R}{K} A + \text{const}$$

That is, the isotherms are straight parallel lines whose slope depends upon the value of K selected. When K is chosen to make the isotherm-adiabat angle close to 90° then the isotherms slope upward to the right of the diagram at an angle of about 45° with the isobars.

The equation of the dry adiabats is obtained, as before, by taking the logarithm of Poisson's equation while holding θ constant:

$$\ln T = \frac{R}{c_p} \ln p + \text{const}$$

The quantity $R \ln p$ is one of the coordinates of the diagram, but $\ln T$ is not. Therefore the adiabats are not straight. In the meteorological range of conditions the adiabats are visibly but gently curved lines running from the lower right to the upper left of the diagram. They are concave upwards. As was pointed out earlier, the constant, K, is so chosen that the isotherm-adiabat angle is near 90° everywhere in the meteorological range.

As on all diagrams discussed here, the pseudoadiabats are distinctly curved. This is a characteristic which can be avoided only be sacrificing the energy-area proportionality. The saturation mixing ratio lines are essentially straight on this diagram.

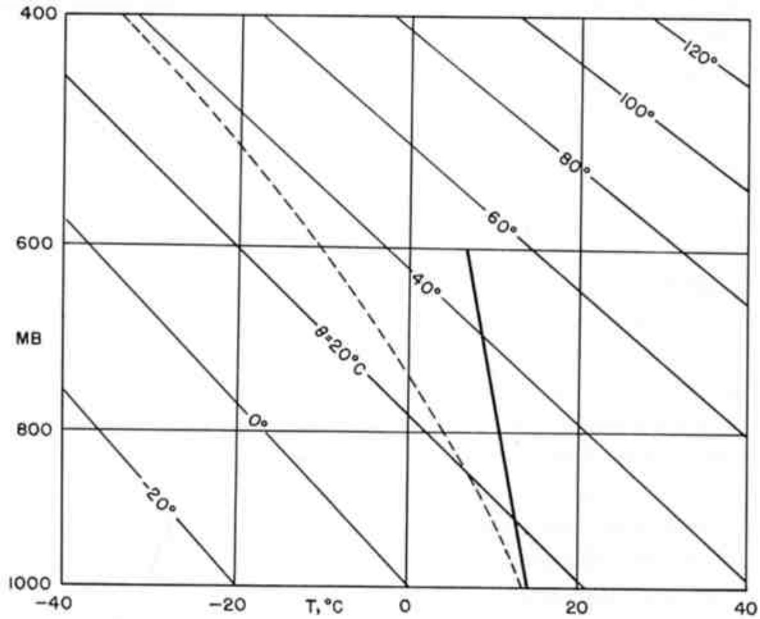
In summary, the skew T-log p diagram has (1) area proportional to energy; (2) three sets of exactly or closely straight lines, one set of gently curved lines, and one set of markedly curved lines; (3) an adiabat-isotherm angle which varies with position on the diagram but is about

90°. This diagram satisfies the three basic criteria almost as well as the tephigram and is therefore a very good thermodynamic diagram. It has been adopted for official use by the Air Weather Service of the U.S. Air Force.

Figure 5.2 gives skeleton versions of the emagram, tephigram, and skew T-log p diagram.

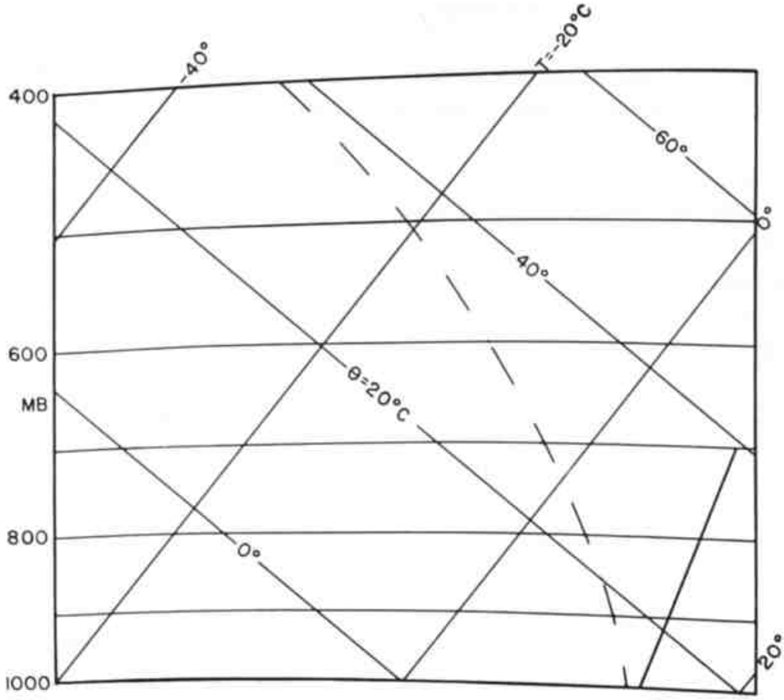
5.5 The Stüve Diagram

This diagram, which was outlined briefly in Chapter 4, consists of p^{κ} on the ordinate with p increasing downward, and T on a linear scale as the abscissa. This choice of coordinates insures that the dry adiabats will be straight lines. As usual, the pseudoadiabats are curved but the saturation mixing ratio lines are essentially straight. The adiabat-isotherm angle is usually near 45°. As may be demonstrated (see

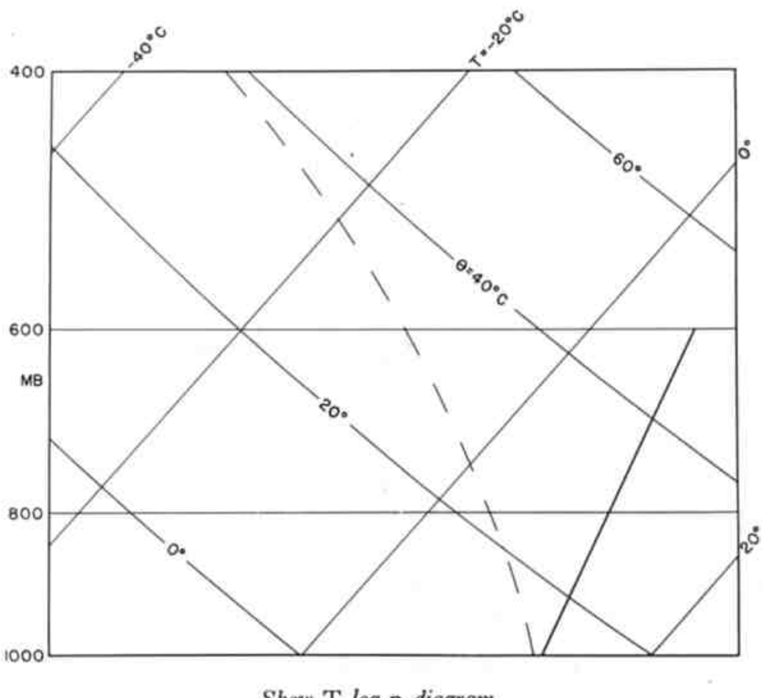


Emagram

FIG. 5.2. Isobars, isotherms, dry adiabats, saturation mixing ratio line, and pseudoadiabat on an emagram, a tephigram, a skew T-log p diagram. In all cases only the lines for $w_s = 10$ g kg^{-1} and $\theta_e = 40°$ C are shown.



Tephigram



Skew T-*log* p *diagram*

Problem 2), the Stüve diagram is *not* an equal-area transformation of the α, $-p$ diagram. That is, area is not strictly proportional to energy.

In summary, the Stüve diagram (1) does not have area proportional to energy; (2) has four sets of lines which are exactly or nearly straight and only one set which is perceptibly curved; (3) has an adiabat-to-isotherm angle of 45°. This diagram is clearly not as good as some of the others, although it was introduced very early in the history of modern meteorology. It is used enough so that one should know its properties, but it is gradually being replaced.

5.6 Choice of a Diagram

A number of diagrams other than those described here have been devised. Despite the objective criteria which have been set up to determine the desirability of a diagram, it seems true that most meteorologists have an aversion to diagrams other than the one with which they are most familiar. This is understandable and even defensible, since there really is not a great deal of difference in practice among the various diagrams. Sometimes one diagram is preferred over another because of the excellence of the printing, the skillful use of color, and minimization of eye strain. In other cases the deciding factor may be the presence or absence of some auxiliary nomogram, such as one for rapid computation of the distance between pressure levels.

From an over-all point of view, the tephigram and the skew T-log p diagram seem to be superior to all the others by a small margin. However it is likely that all the major diagrams will continue in use for many years.

PROBLEMS

1. Suppose all the isobars on an emagram were relabeled with one-half their original pressures. Show that a dry adiabat which was originally labeled with potential temperature θ_1 should be relabeled with potential temperature $\theta = 2^{\kappa}\theta_1$.

2. Show that the Stüve diagram does not satisfy the sufficient condition for a thermodynamic diagram.

3. The "Refsdal aerogram" has coordinates $-RT \ln p$ and $\ln RT$. Show that this is an equal-area transformation of the α, $-p$ diagram.