SEISMOLOGY

Master Degree Programme in Physics - UNITS
Physics of the Earth and of the Environment

TOWARDS WAVE EQUATION…

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What is a wave?

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

Coupling of harmonic oscillators → the disturbances can propagate, superpose, stand, and **be dispersed**

WAVE: organized **propagating imbalance**, satisfying differential equations of motion

Organization can be destroyed, when interference is destructive

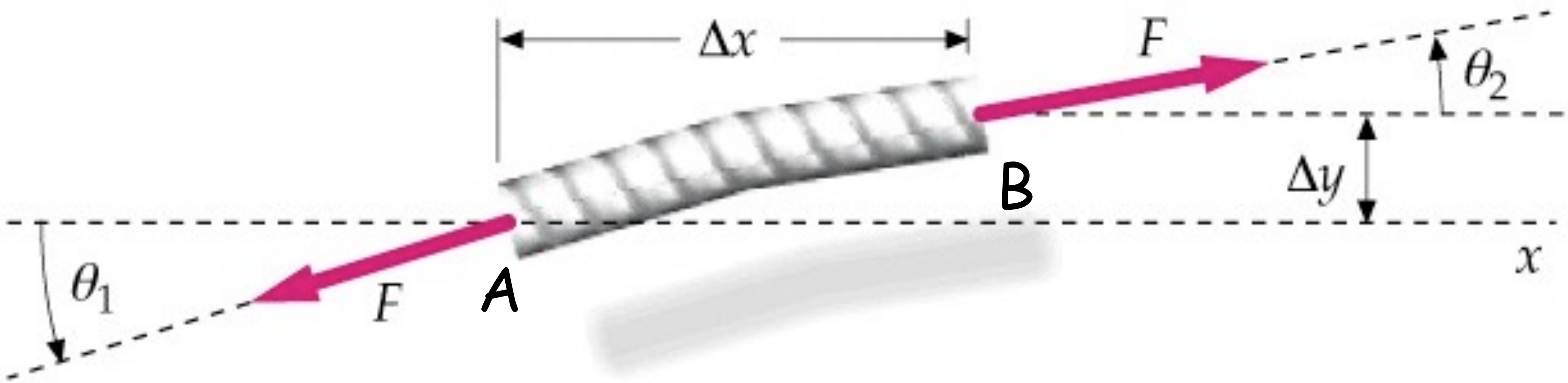
non linearity → **Turbulence**

strong scattering → **Diffusion**

Exceptions

Solitons

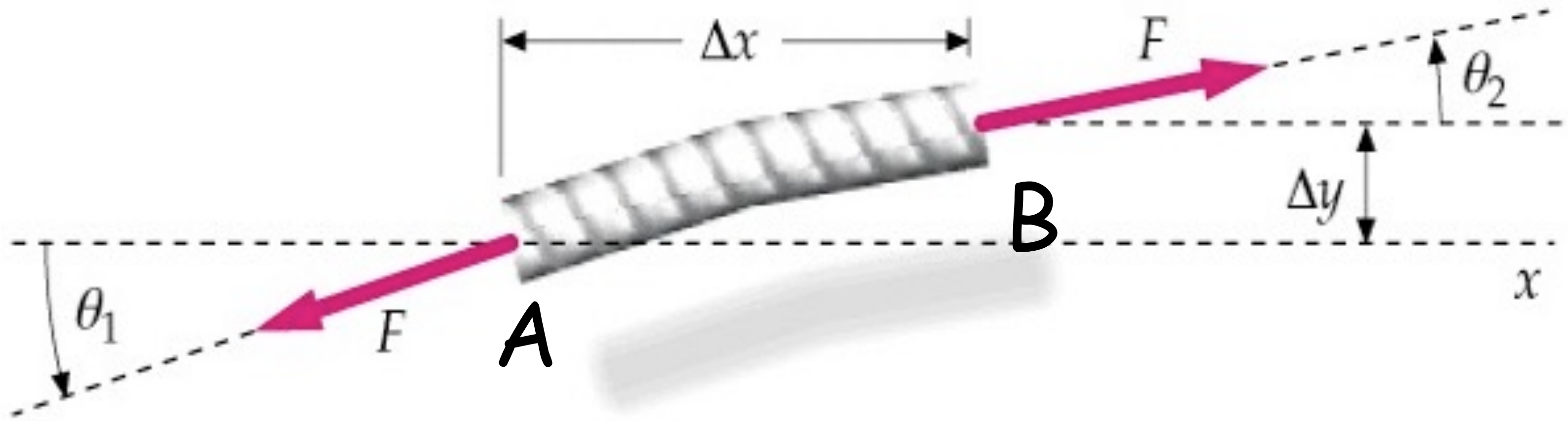
Phonons



Consider a small segment of string of length Δx and tension F

The ends of the string make **small** angles θ_1 and θ_2 with the x-axis.

The vertical displacement Δy is very **small** compared to the length of the string



Resolving forces vertically

$$\sum F_y = F\sin\theta_2 - F\sin\theta_1$$

$$= F(\sin\theta_2 - \sin\theta_1)$$

From small angle approximation

$\sin\theta \sim \tan\theta$

$$\sum F_y \approx F(\tan\theta_2 - \tan\theta_1)$$

The tangent of angle A (B) = pendence of the curve in A (B)

given by $\dfrac{\partial y}{\partial x}$

$$\therefore \quad \sum F_y \approx F\left(\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right)$$

We now apply N2 to segment

$$\sum F_y = ma = \mu\, \Delta x \left(\frac{\partial^2 y}{\partial t^2}\right)$$

$$\mu\, \Delta x \left(\frac{\partial^2 y}{\partial t^2}\right) = F\left(\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right)$$

$$\frac{\mu}{F}\left(\frac{\partial^2 y}{\partial t^2}\right) = \frac{\left[(\partial y/\partial x)_B - (\partial y/\partial x)_A\right]}{\Delta x}$$

$$\frac{\mu}{F}\left(\frac{\partial^2 y}{\partial t^2}\right) = \frac{\left[(\partial y/\partial x)_B - (\partial y/\partial x)_A\right]}{\Delta x}$$

The derivative of a function is defined as

$$\left(\frac{\partial f}{\partial x}\right) = \lim_{\Delta x \to 0} \frac{\left[f(x + \Delta x) - f(x)\right]}{\Delta x}$$

If we associate $f(x+\Delta x)$ with $(\partial y/\partial x)_B$ and $f(x)$ with $(\partial y/\partial x)_A$

as Δx -> 0

$$\frac{\mu}{F}\left(\frac{\partial^2 y}{\partial t^2}\right) = \frac{\partial^2 y}{\partial x^2}$$

This is the linear wave equation for waves on a string

Solution of the wave equation

Consider a wavefunction of the form $y(x,t) = A\sin(kx-\omega t)$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(kx-\omega t) \qquad \frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(kx-\omega t)$$

If we substitute these into the linear wave equation

$$\frac{\mu}{F}(-\omega^2 A\sin(kx-\omega t)) = -k^2 A\sin(kx-\omega t)$$

$$\frac{\mu}{F}\omega^2 = k^2$$

Using $v = \omega/k$, $v^2 = \omega^2/k^2 = F/\mu$

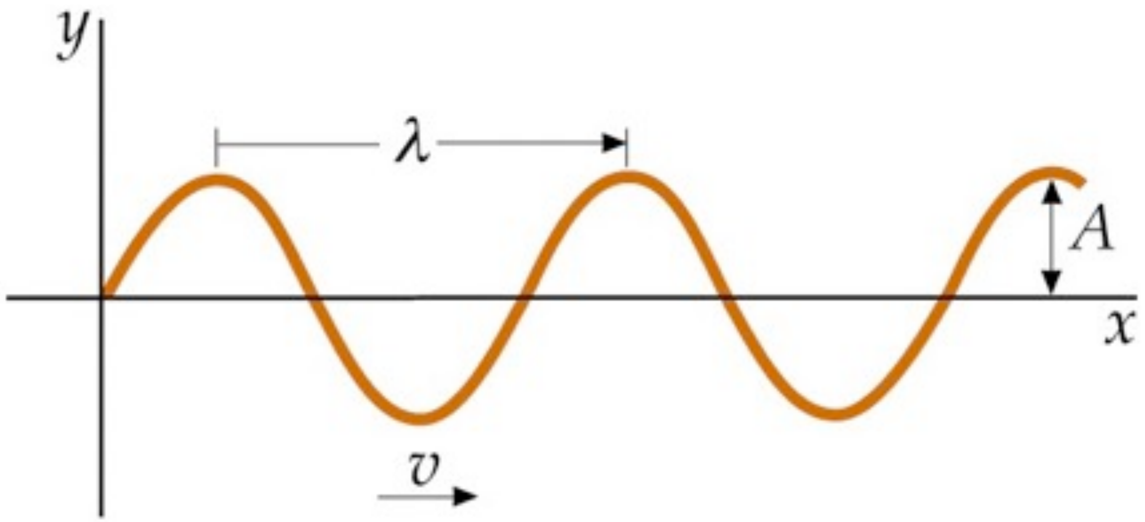
$$v = \sqrt{F/\mu}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

General form of LWE

Harmonic Waves

A harmonic wave is sinusoidal in shape, and has a displacement y at time t=0



$$y = A\sin\left(\frac{2\pi}{\lambda}x\right)$$

A is the amplitude of the wave and λ is the wavelength (the distance between two crests)

if the wave is moving to the right with speed (or phase velocity) v, the wavefunction at some t is given by

$$y = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

Time taken to travel one wavelength is the period T

Phase velocity, wavelength and period are related by

$$v = \frac{\lambda}{T} \quad \text{or} \quad \lambda = vT$$

$$\therefore \quad y = A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right]$$

The wavefunction shows the periodic nature of y:

at any time t y has the same value at x, x+λ, x+2λ.........

and at any x y has the same value at times t, t+T, t+2T......

It is convenient to express the harmonic wavefunction by defining the **wavenumber k**, and the **angular frequency** ω

$$\text{where} \quad k = \frac{2\pi}{\lambda} \quad \text{and} \quad \omega = \frac{2\pi}{T}$$

$$\therefore \quad y = A\sin(kx - \omega t)$$

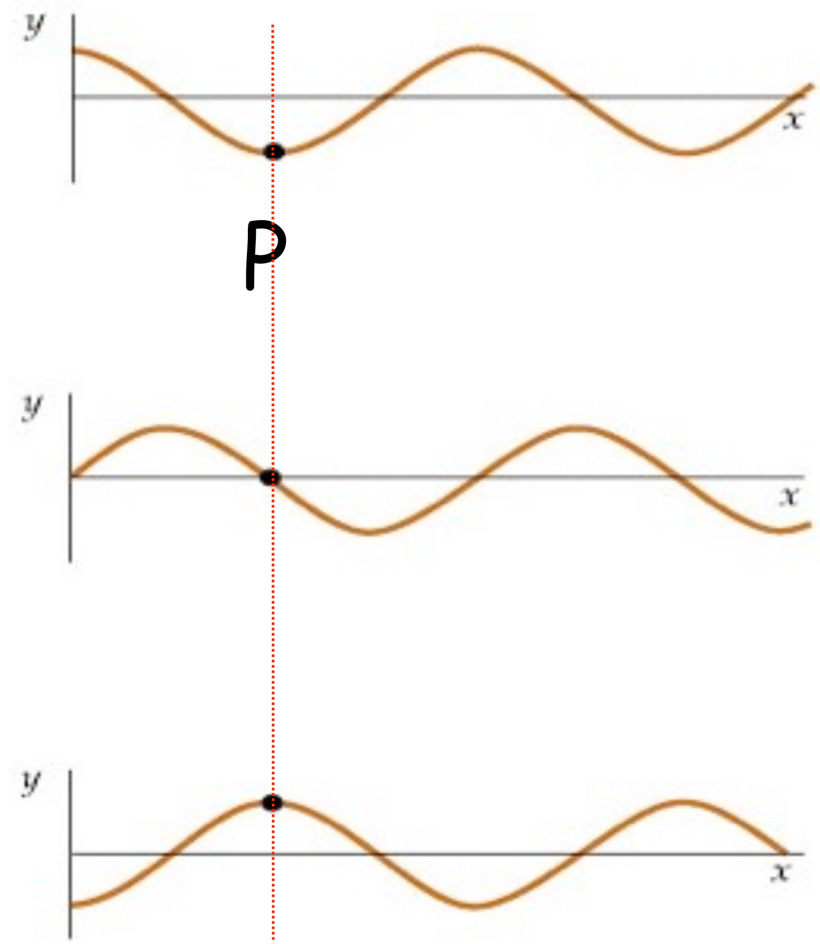
This assumes that the displacement is zero at x=0 and t=0. If this is not the case we can use a more general form

$$y = A\sin(kx - \omega t - \phi)$$

where ϕ is the **phase constant** and is determined from initial conditions

The wavefunction can be used to describe the motion of any point P.

If $y = A\sin(kx - \omega t)$



Transverse velocity v_y

$$v_y = \left.\frac{dy}{dt}\right|_{x=\text{constant}}$$

$$= \frac{\partial y}{\partial t}$$

$$= -\omega A\cos(kx - \omega t)$$

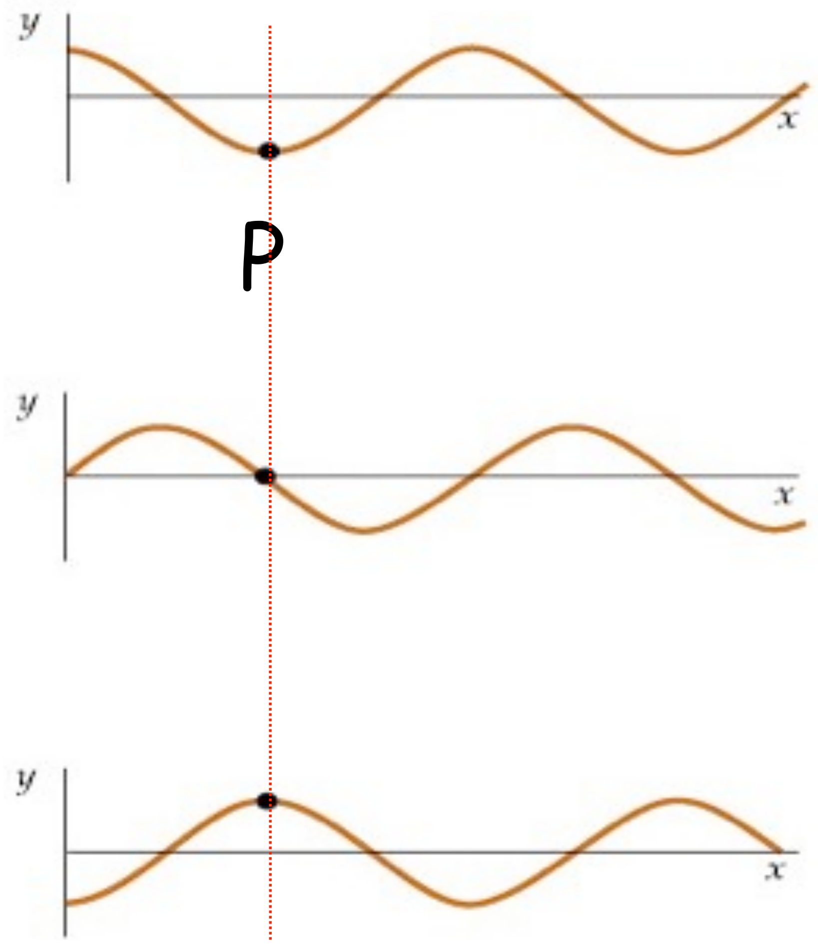
which has a maximum value $(v_y)_{max} = \omega A$ when $y = 0$

Transverse acceleration a_y

$$a_y = \left.\frac{dv_y}{dt}\right|_{x=\text{constant}}$$

$$= \frac{\partial v_y}{\partial t}$$

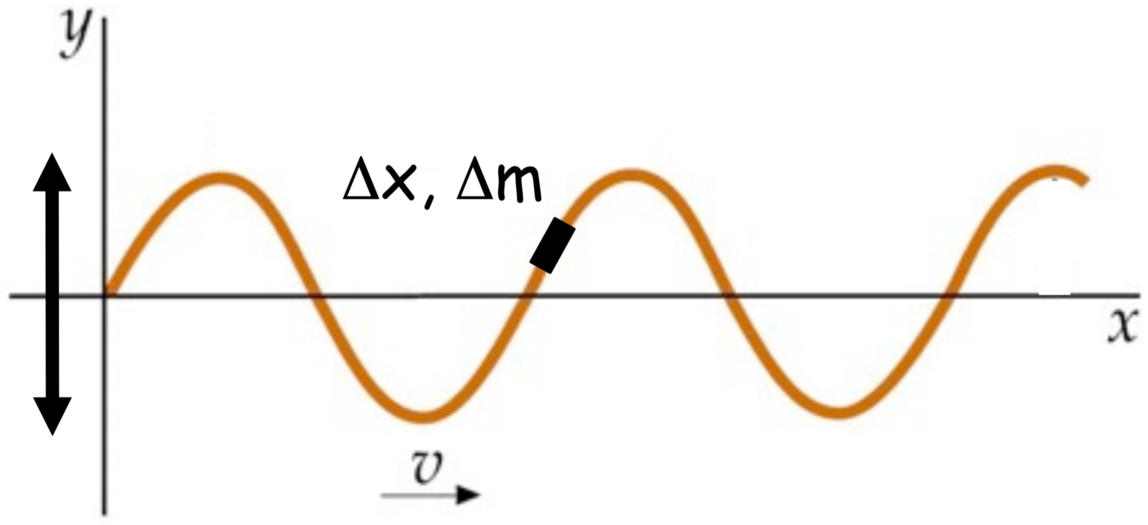
$$= -\omega^2 A \sin(kx - \omega t)$$



which has a maximum value $\left(a_y\right)_{max} = \omega^2 A$ when $y = -A$

NB: x-coordinates of P are constant

Energy of waves on a string



Consider a harmonic wave travelling on a string.

Source of energy is an external agent on the left of the wave which does work in producing oscillations.

Consider a small segment, length Δx and mass Δm.

The segment moves vertically with SHM, frequency ω and amplitude A.

Generally $E = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2A^2$ (where k is the force constant of the restoring force)

$$E = \frac{1}{2} m\omega^2 A^2$$

If we apply this to our small segment, the total energy of the element is

$$\Delta E = \frac{1}{2}(\Delta m)\,\omega^2 A^2$$

If μ is the mass per unit length, then the element Δx has mass $\Delta m = \mu\, \Delta x$

$$\Delta E = \frac{1}{2}(\mu\, \Delta x)\,\omega^2 A^2$$

If the wave is travelling from left to right, the energy ΔE arises from the work done on element Δm_i by the element Δm_{i-1} (to the left).

Similarly Δm_i does work on element Δm_{i+1} (to the right) ie. energy is transmitted to the right.

The rate at which energy is transmitted along the string is the power and is given by dE/dt.

If $\Delta x \to 0$ then

$$\text{Power} = \frac{dE}{dt} = \frac{1}{2}\left(\mu\frac{dx}{dt}\right)\omega^2 A^2$$

but dx/dt = speed

$$\therefore \quad \text{Power} = \frac{1}{2}\mu\,\omega^2 A^2 v$$

$$\text{Power} = \frac{1}{2}\mu\,\omega^2 A^2 v$$

Power transmitted on a harmonic wave is proportional to

(a) the wave speed v

(b) the square of the angular frequency ω

(c) the square of the amplitude A

All harmonic waves have the following general properties:

The power transmitted by any harmonic wave is proportional to the square of the frequency and to the square of the amplitude.

What is a **wave**? - 3

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

Coupling of harmonic oscillators → the disturbances can **propagate**, superpose and stand

WAVE: organized **propagating imbalance**, satisfying differential equations of motion

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

General form of LWE

Towards sound wave equation...

- Consider a source causing a perturbation in the gas medium rapid enough to cause a pressure variation and not a simple molecular flux.

- The regions where compression (or rarefaction), and thus the density variation of the gas, occurs are big compared to the mean free path (average distance that gas molecules travel without collisions).

- The perturbation fronts are planes and the displacement induced in the gas, X, depends only on x & t (and not on y, z).

The conventional unit for pressure is bar=10^5N/m^2 and the pressure at the equilibrium is: 1atm=1.0133bar

The pressure perturbations associated to the sound wave passage are tipically of the order of 10^{-7}bar, thus very small if compared to the value of pressure at the equlibrium.

One can thus assume that:

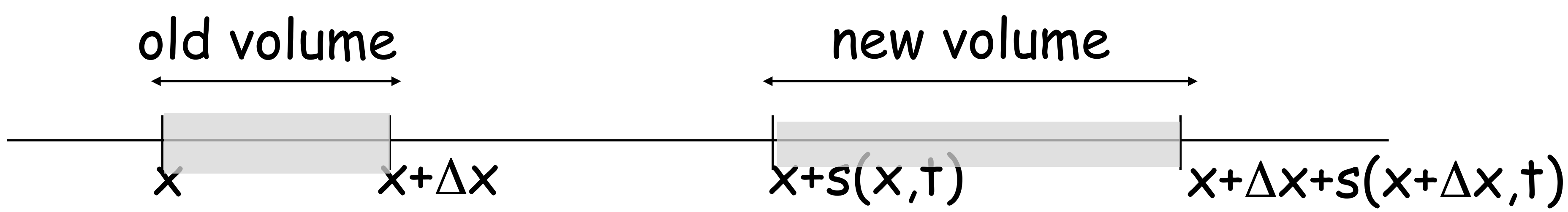
$P=P_0+\Delta P \quad \rho=\rho_0+\Delta\rho$

where ΔP and $\Delta\rho$ are the values of the (small) perturbations of the pressure and density from the equlibrium.

Sound wave equation - 1

The gas moves and causes density variations

Let us consider the displacement field, $s(x,t)$ induced by sound



and considering a unitary area perpendicular to x, direction of propagation, one has that the quantity of gas enclosed in the old and new volume is the same

$$\rho_0 \Delta x = \rho\left[x + \Delta x + s(x+\Delta x) - x - s(x)\right]$$

$$\text{where, since } \Delta x \text{ is small, } s(x+\Delta x) \approx s(x) + \frac{\partial s}{\partial x}\Delta x$$

$$\rho_0 \Delta x = (\rho_0 + \Delta\rho)\left[\Delta x + \frac{\partial s}{\partial x}\Delta x\right] = \rho_0 \Delta x + \rho_0 \frac{\partial s}{\partial x}\Delta x + \Delta\rho \Delta x + \ldots$$

thus, neglecting the second-order term, one has:

$$\Delta\rho = -\rho_0 \frac{\partial s}{\partial x}$$

relation between the variation of displacement along x with the density variation. The minus sign is due to the fact that, if the variation is positive the volume increases and the density decreases.

If the displacement field is constant the gas is simply translated without perturbation.

Sound wave equation - 2

Density variations cause pressure variations

The pressure in the medium is related to density with a relationship of the kind **$P=f(\rho)$**, that at the equilibrium is **$P_0=f(\rho_0)$**.

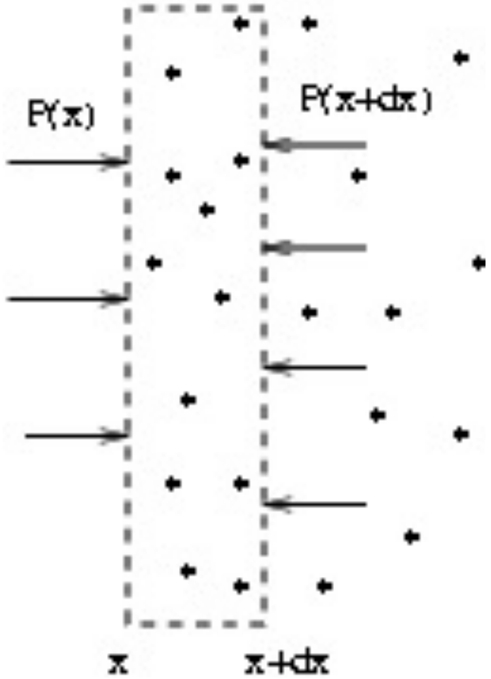
$$P = P_0 + \Delta P = f(\rho) = f(\rho_0 + \Delta\rho) \approx f(\rho_0) + \Delta\rho f'(\rho_0) = P_0 + \Delta\rho\kappa$$

and neglecting second-order terms:

$$\Delta P = \kappa \Delta\rho$$

$$\kappa = f'(\rho_0) = \left(\frac{dP}{d\rho}\right)_0$$

Pressure variations generate gas motion



The gas in the volume is accelerated by the different pressure exerted on the two sides...

$$P(x,t) - P(x+\Delta x,t) \approx -\frac{\partial P}{\partial x}\Delta x = -\frac{\partial (P_0 + \Delta P)}{\partial x}\Delta x = -\frac{\partial \Delta P}{\partial x}\Delta x$$

$$= \rho_0 \Delta x \frac{\partial^2 s}{\partial t^2}$$ for Newton's 2nd law

thus:

$$\rho_0 \frac{\partial^2 s}{\partial t^2} = -\frac{\partial \Delta P}{\partial x}$$

Sound wave equation

Using 1, 2 and 3 we have

$$\rho_0 \frac{\partial^2 s}{\partial t^2} = -\frac{\partial \Delta P}{\partial x} = -\frac{\partial\left(\kappa \Delta \rho\right)}{\partial x} = -\frac{\partial\left[\kappa\left(-\rho_0 \frac{\partial s}{\partial x}\right)\right]}{\partial x}$$

thus:

$$\frac{1}{\kappa}\frac{\partial^2 s}{\partial t^2} = \frac{\partial^2 s}{\partial x^2}$$

i.e. the typical wave equation, describing a perturbation traveling with velocity $v = \sqrt{\kappa}$

From the sound wave equation

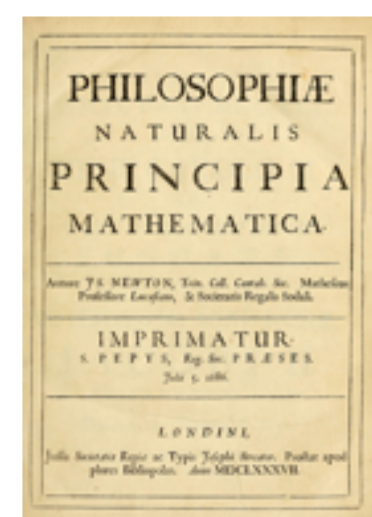
$$v = \sqrt{\kappa} = \sqrt{\left(\frac{dP}{d\rho}\right)_0}$$

Newton computed the derivative of the pressure assuming that the heat is moving from one to another region in a such rapid way that the temperature cannot vary - isotherm

PV=constant i.e. P/ρ=constant, thus

$$v = \sqrt{\left(\frac{dP}{d\rho}\right)_0} = \sqrt{(constant)_0} = \sqrt{\left(\frac{P}{\rho}\right)_0}$$

called **isothermal sound velocity**


PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA
IMPRIMATUR
LONDINI

I. Newton, "Philosophiæ Naturalis Principia Mathematica", 1687; 1713; 1728..

Laplace correctly assumed that the heat flux between a compressed gas region to a rarefied one was negligible, and, thus, that the process of the wave passage was adiabatic

PV^{γ}=constant, P/ρ^{γ}=constant, with γ ratio of the specific heats: C_p/C_v

$$v = \sqrt{\left(\frac{dP}{d\rho}\right)_0} = \sqrt{\left(\frac{\gamma}{\rho} \mathrm{constant}\, \rho^{\gamma}\right)_0} = \sqrt{\gamma\left(\frac{P}{\rho}\right)_0}$$

called adiabatic sound velocity

P. S. Laplace, "Sur la vitesse du son dans l'air et dans l'eau" Annales de chimie, 1816, 3: 238-241.

Sound velocity in the air

Using the ideal gas law

$PV=nRT=NkT$

one can write the velocity on many ways:

$$v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma PV}{\rho V}} = \sqrt{\frac{\gamma nRT}{m}} = \sqrt{\frac{\gamma NkT}{Nm_{mol}}} = \sqrt{\frac{\gamma KT}{m_{mol}}} = \sqrt{\frac{\gamma RT}{weight_{mol}}}$$

showing that it depends on **temperature only**. If the "dry" air is considered (biatomic gas $\gamma=7/5$) one has:

$v=331.4+0.6T_c$ m/s (temperature measured in Celsius)

Bulk modulus

It corresponds to the "spring constant" of a spring, and gives the magnitude of the restoring agency (pressure for a gas, force for a spring) in terms of the change in physical dimension (volume for a gas, length for a spring). Defined as an "intensive" quantity:

$$B = -\frac{\Delta P}{\Delta V / V} = -V\frac{dP}{dV}$$

and for an adiabatic process (from the 1st principle of thermodynamics applied to an ideal gas):

$$B = \gamma\ P$$

Sound speed

Sound velocity depends on the compressibility of the medium.

If the medium has a bulk modulus B and density at the equilibrium is ρ, the sound speed is: $\mathbf{v = (B/\rho)^{1/2}}$

that can be compared with the velocity of transversal waves on a string:

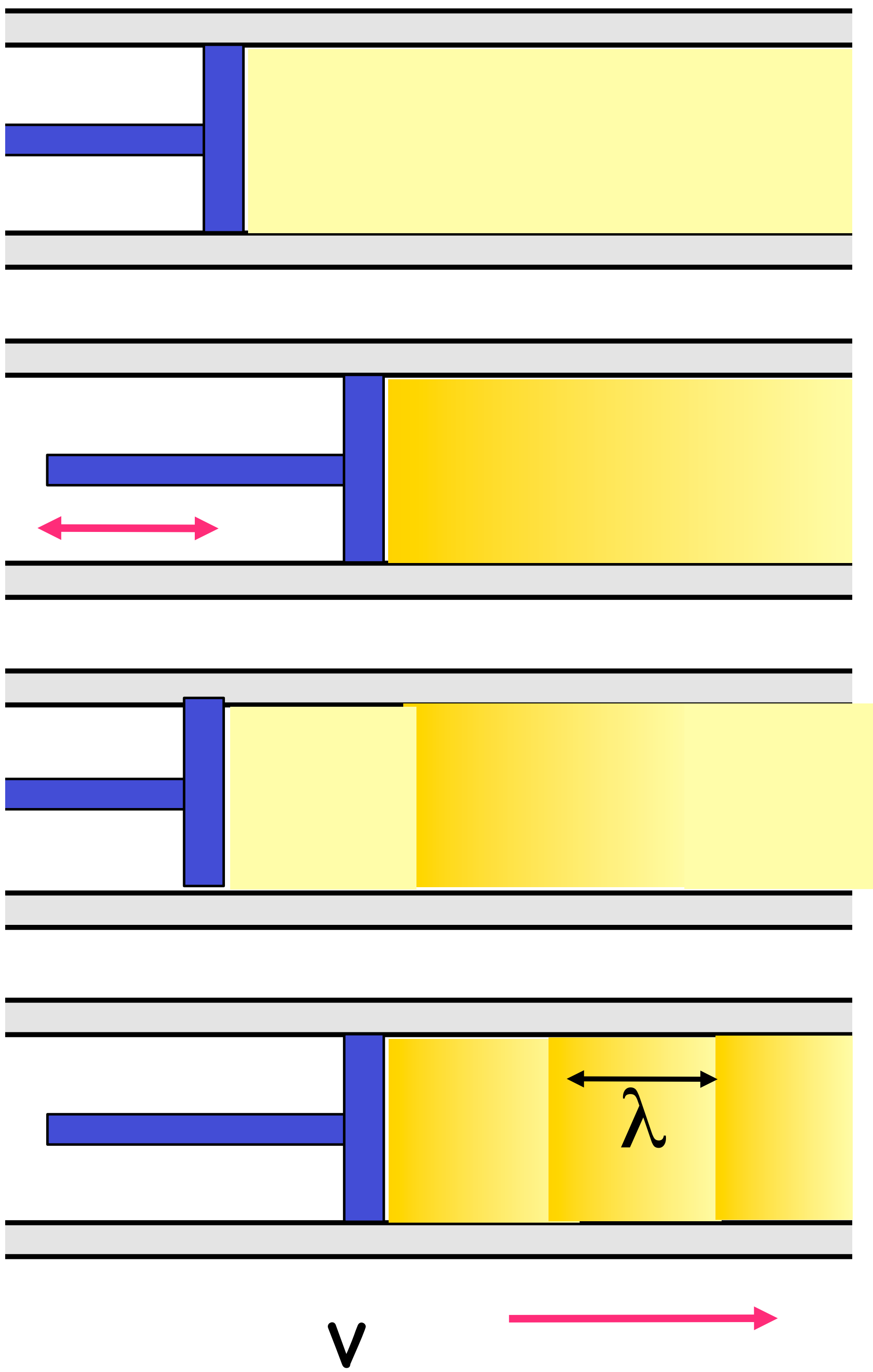
$$v = (F/\mu)^{1/2}$$

Thus, velocity depends on the elastic of the medium (B or F) and on inertial (ρ or μ) properties

Harmonic sound waves

If the source of a longitudinal wave (eg tuning fork,loudspeaker) oscillates with SHM the resulting disturbance will also be harmonic

Consider this system →



As the piston oscillates backwards and forwards regions of compression and rarefaction are set up.

The distance between successive compressions or rarefactions is λ.

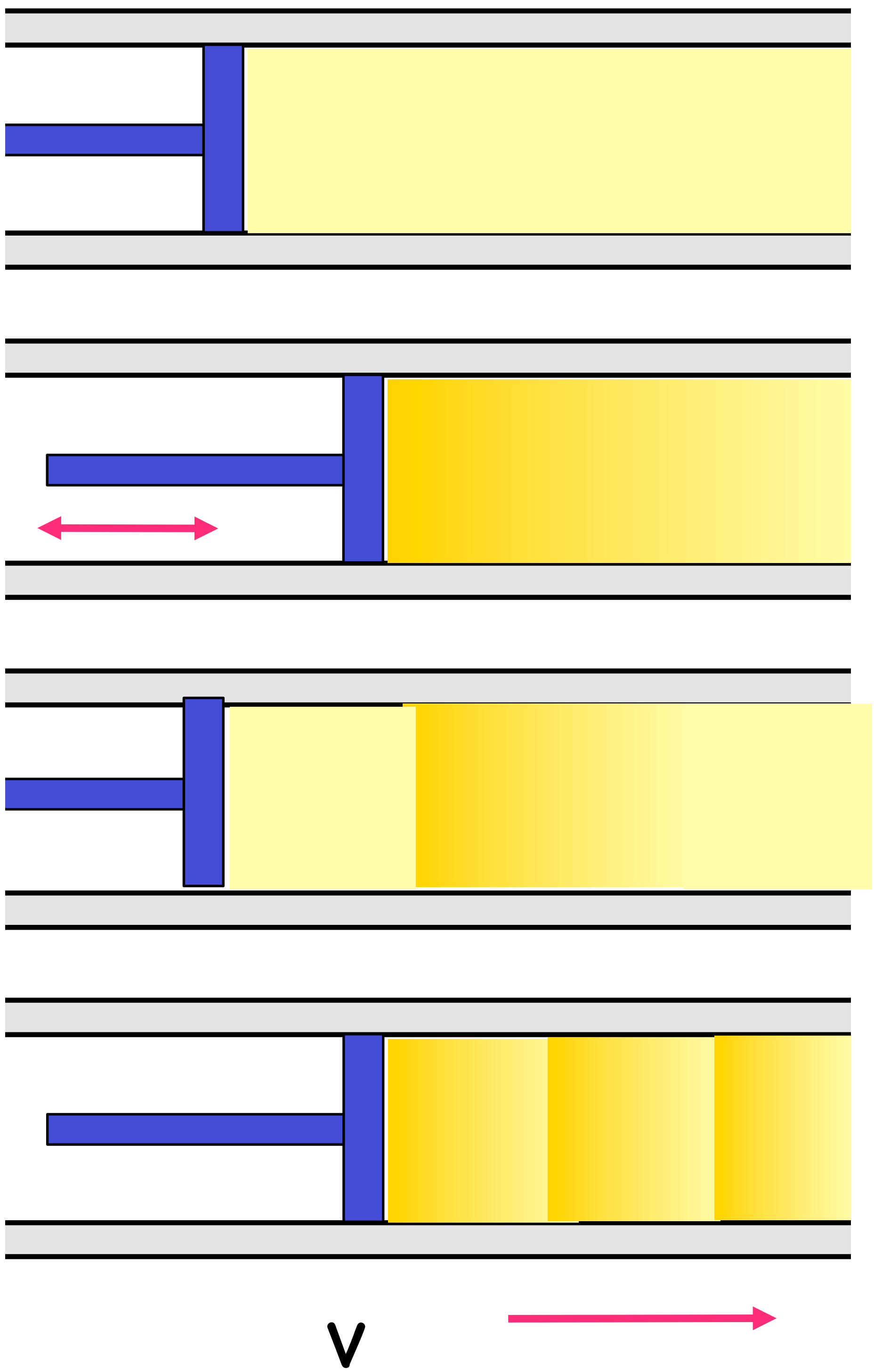
Any small region of the medium moves with SHM, given by

$$s(x,t) = s_m \cos(kx - \omega t)$$

s_m = max displacement from equilibrium

The change of the pressure in the gas, ΔP, measured relative to the equilibrium pressure

$$\Delta P = \Delta P_m \sin(kx - \omega t)$$



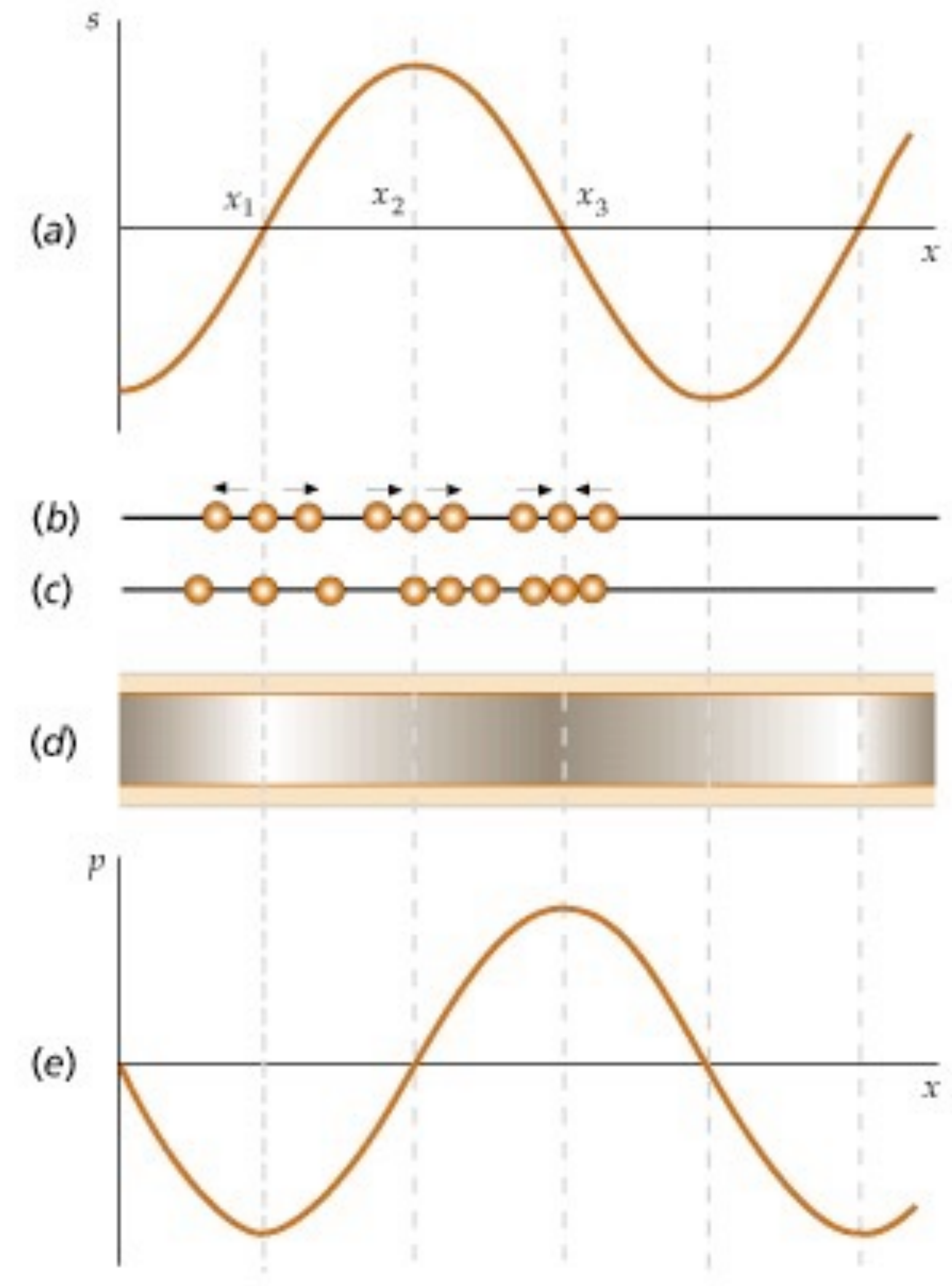
$$\Delta P = \Delta P_m \sin(kx - \omega t)$$

The pressure amplitude ΔP_m is proportional to the displacement amplitude s_m via

$$\Delta P_m = \rho v \omega s_m$$

ωs_m is the maximum longitudinal velocity of the medium in front of the piston

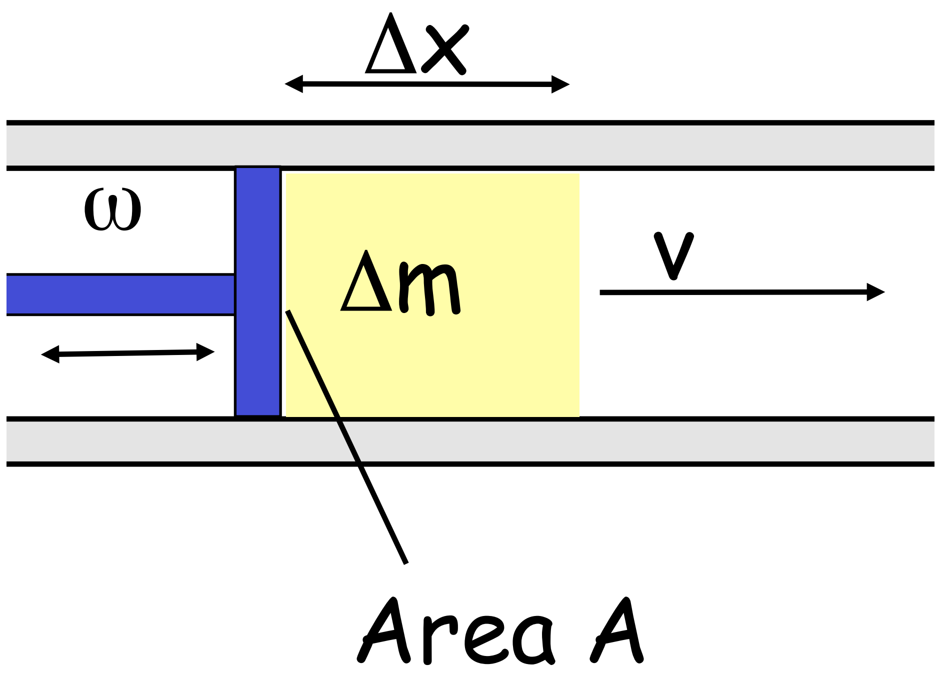
ie a sound wave may be considered as either a displacement wave or a pressure wave (90° out of phase)



Tipler fig 15-10

Consider a layer of air mass Δm and width Δx in front of a piston oscillating with frequency ω.

The piston transmits energy to the air.



In a system obeying SHM $KE_{ave} = PE_{ave}$ and $E_{ave} = KE_{max}$

$$\Delta E = \frac{1}{2}\Delta m\,(\omega s_m)^2$$

$$= \frac{1}{2}(\rho\, A\,\Delta x)\,(\omega s_m)^2$$

volume of layer

Power = rate at which energy is transferred to each layer

$$\text{Power} = \frac{\Delta E}{\Delta t}$$

$$= \frac{1}{2}\rho A\left(\frac{\Delta x}{\Delta t}\right)(\omega s_m)^2$$

velocity to right

$$= \frac{1}{2}\rho A v(\omega s_m)^2$$

$$\text{Intensity} = \frac{\text{Power}}{\text{area}} = \frac{1}{2}\rho v(\omega s_m)^2$$

$$= \frac{\Delta P_m^2}{2\rho v}$$ where $\Delta P_m = \rho v \omega s_m$

Intensity in decibels

The human ear detects sound on an approximately logarithmic scale.

We define the **intensity level** of a sound wave by

$$\beta = 10 \log\left(\frac{I}{I_o}\right)$$

where I is the intensity of the sound, I_o is the threshold of hearing (~10^{-12} W m^{-2}) and β is measured in decibels (dB).

Examples:

jet plane	150dB	conversation	50dB
rock concert	120dB	whisper	30dB
busy traffic	80dB	breathing	10dB

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