SEISMOLOGY

Master Degree Programme in Physics - UNITS
Physics of the Earth and of the Environment

ELASTODYNAMIC GREEN'S FUNCTION

FABIO ROMANELLI

Department of Mathematics & Geosciences
University of Trieste
romanel@units.it

Seismic sources - 2

Elastodynamic Green function

- scalar problem
- Lamè theorem
- EGF in homogeneous media
  - near and far field
- EGF for double couple in homogeneous media
  - near, intermediate and far field
- EGF for double couple in heterogeneous media
  - surface waves in the far field

Green's function

Green's function is a basic solution to a linear differential equation, a building block that can be used to construct many useful solutions.

If one considers a linear differential equation written as:

$$L(x)u(x)=f(x)$$

where L(x) is a linear, self-adjoint differential operator, u(x) is the unknown function, and f(x) is a known non-homogeneous term, the GF is a solution of:

$$L(x)\underbrace{\boxed{u(x,s)}}_{G(x,s)}=\delta(x-s)$$

Why GF is important?

If such a function G can be found for the operator L, then if we multiply the second equation for the Green's function by f(s), and then perform an integration in the s variable, we obtain:

$$\int L(x)G(x,s)f(s)ds = \int \delta(x-s)f(s)ds = f(x) = Lu(x)$$

$$L\int G(x,s)f(s)ds = Lu(x)$$

$$u(x) = \int G(x,s)f(s)ds$$

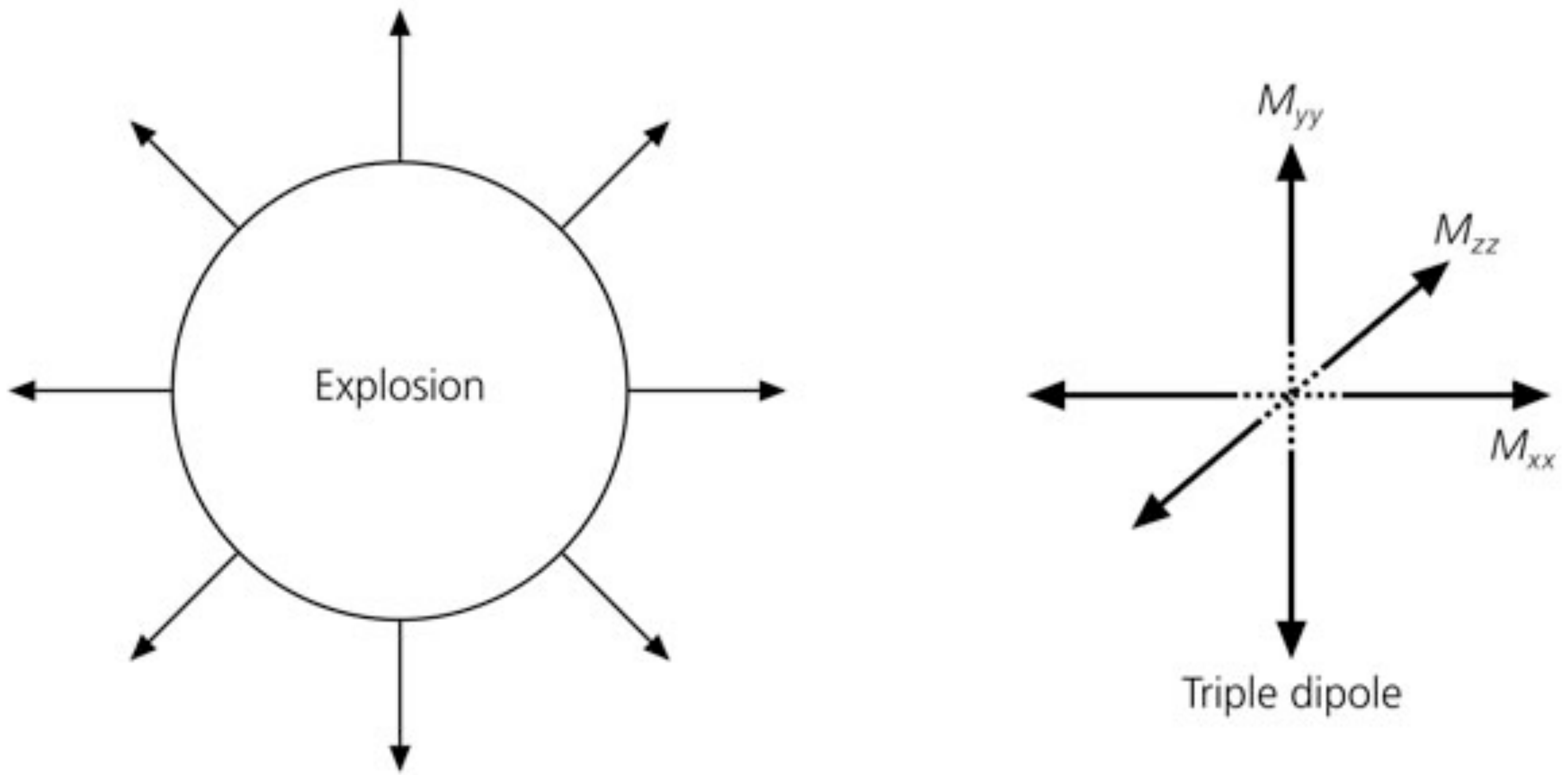
Thus, **we can obtain the function u(x) through the knowledge of the Green's function and the source term**. This process has resulted from the linearity of the operator L. See Linear System Theory (i.e. impulse response)

Inhomogeneous wave equation

Let us consider the simplest inhomogeneous scalar problem, i.e. a spherically symmetric one, to avoid the directionality of the source:

$$L(u) = \ddot{u} - c^2 \Delta u = \delta(x)\delta(t)$$

Figure 4.4-7: Modeling an explosive source as a triple force dipole.



Inhomogeneous wave equation

Let us consider the simplest inhomogeneous scalar problem, i.e. a spherically symmetric one, to avoid the directionality of the source:

$$L(u) = \ddot{u} - c^2 \Delta u = \delta(\mathbf{x})\delta(t)$$

and let us look for the solution, whose spatial dependence can be only on u=u(r,t)=u(|**x**|,t); expressing the Laplacian in spherical coordinates, one has that everywhere, except at r=0, u=f(t-r/c)/r is the general solution. At t=0, we have the Poisson equation:

$$\Delta u = \frac{\delta(\mathbf{x})}{c^2}$$ whose solution is: $$u = \frac{\delta(\mathbf{x})}{4\pi c^2}$$

Thus, the general solution is: $$u(r,t) = \frac{1}{4\pi c^2}\frac{\delta(t-r/c)}{r}$$

and the rapidly varying function depends, at any position, only on the arrival time, and its shape is the same in time as the time function at the source term.

Properties of the solution

1) If
$$L(\mathbf{u}) = \delta(\mathbf{x} - \zeta)\delta(t - \tau)$$

then
$$u(r,t) = \frac{1}{4\pi c^2}\frac{\delta(t-\tau-|\mathbf{x} - \zeta|/c)}{|\mathbf{x} - \zeta|}$$

2) If
$$L(\mathbf{u}) = \delta(\mathbf{x} - \zeta)f(t)$$

then
$$u(r,t) = \frac{1}{4\pi c^2}\frac{f(t-|\mathbf{x} - \zeta|/c)}{|\mathbf{x} - \zeta|}$$

3) If the source is extended through a volume V:
$$L(\mathbf{u}) = \frac{\Phi(\mathbf{x},t)}{\rho}$$

then
$$u(r,t) = \frac{1}{4\pi\rho c^2}\iiint_V \frac{\Phi(\zeta,t-|\mathbf{x} - \zeta|/c)}{|\mathbf{x} - \zeta|}dV$$

Helmholtz theorem

Any vector field **u=u(x)** may be separated into scalar and vector potentials

$$\boxed{\mathbf{u} = \nabla\Phi + \nabla \times \Psi}$$

Brief proof: since it is possible to solve the Poisson equation:

$$\nabla^2 \mathbf{W} = \mathbf{u}$$

$$\mathbf{W}(\mathbf{x}) = -\iiint_V \frac{\mathbf{u}(\xi)}{4\pi\left|\mathbf{x} - \xi\right|} d\xi$$

then the identity

$$\Delta = \nabla\nabla \cdot - \nabla \times \nabla \times$$

tells us that

$$\Phi = \nabla \cdot \mathbf{W} \qquad \Psi = -\nabla \times \mathbf{W}$$

Lamè theorem

The problem is to find solutions to the elastodynamic equation

$$\rho\ddot{\mathbf{u}} = \mathbf{f} + \left(\lambda + 2\mu\right)\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla\times(\nabla\times\mathbf{u})$$

for an isotropic and homogeneous elastic space,
in terms of soluble equations.

If the body terms and initial conditions can be expressed as:

$$\mathbf{f}=\nabla\boldsymbol{\Phi} + \nabla\times\boldsymbol{\Psi};\ \ \mathbf{u}(\mathbf{x},0)=\nabla\mathbf{A} + \nabla\times\mathbf{B};\ \ \dot{\mathbf{u}}(\mathbf{x},0)=\nabla\mathbf{C} + \nabla\times\mathbf{D}$$

with $$\nabla\cdot\boldsymbol{\Psi} = 0; \nabla\cdot\mathbf{B} = 0; \nabla\cdot\mathbf{D} = 0$$

then two potentials exist with the following properties:

$$\mathbf{u} = \nabla\phi + \nabla\times\psi; \nabla\cdot\psi = 0;$$

$$\ddot{\phi} = \frac{\boldsymbol{\Phi}}{\rho} + \alpha^2\Delta\phi;\ \ \ddot{\psi} = \frac{\boldsymbol{\Psi}}{\rho} + \beta^2\Delta\psi$$

Solutions for elastodynamic GF

Let us consider for example that

$$\mathbf{f} = X_0(t)\delta(\mathbf{x})\hat{\mathbf{x}}_1 = \nabla\mathbf{\Phi} + \nabla \times \mathbf{\Psi}$$

then we can build:

$$\mathbf{W} = \frac{X_0(t)}{4\pi}\iiint_V (1,0,0)\frac{\delta(\zeta)\,dV}{|\mathbf{x}-\zeta|} = -\frac{X_0(t)}{4\pi r}\hat{\mathbf{x}}_1$$

$$\mathbf{\Phi}(\mathbf{x},t) = \nabla\cdot\mathbf{W} = -\frac{X_0(t)}{4\pi}\frac{\partial}{\partial x_1}\frac{1}{r}$$

$$\mathbf{\Psi}(\mathbf{x},t) = -\nabla\times\mathbf{W} = \frac{X_0(t)}{4\pi}\left(0,\frac{\partial}{\partial x_3}\frac{1}{r},-\frac{\partial}{\partial x_2}\frac{1}{r}\right)$$

Solutions for elastodynamic GF

and we have to a) solve the wave equation for the Lamè potentials of body force and then b) to calculate the displacement.

After some heavy algebra (Stokes, 1849), generalizing from the x_j direction and using direction cosines ($\gamma_i = x_i/r = \partial r/\partial x_i$)

$$u_i = X_0(t) * G_{ij} =$$

$$= \frac{\left(3\gamma_i\gamma_j - \delta_{ij}\right)}{4\pi\rho|\mathbf{x}|^3} \int_{|\mathbf{x}|/\alpha}^{|\mathbf{x}|/\beta} \tau X_0(t-\tau)\,d\tau +$$

Near field term

$$+ \frac{\gamma_i\gamma_j}{4\pi\rho\alpha^3|\mathbf{x}|} X_0\left(t - \frac{|\mathbf{x}|}{\alpha}\right) +$$

$$+ \frac{\left(3\gamma_i\gamma_j - \delta_{ij}\right)}{4\pi\rho\beta^3|\mathbf{x}|} X_0\left(t - \frac{|\mathbf{x}|}{\beta}\right)$$

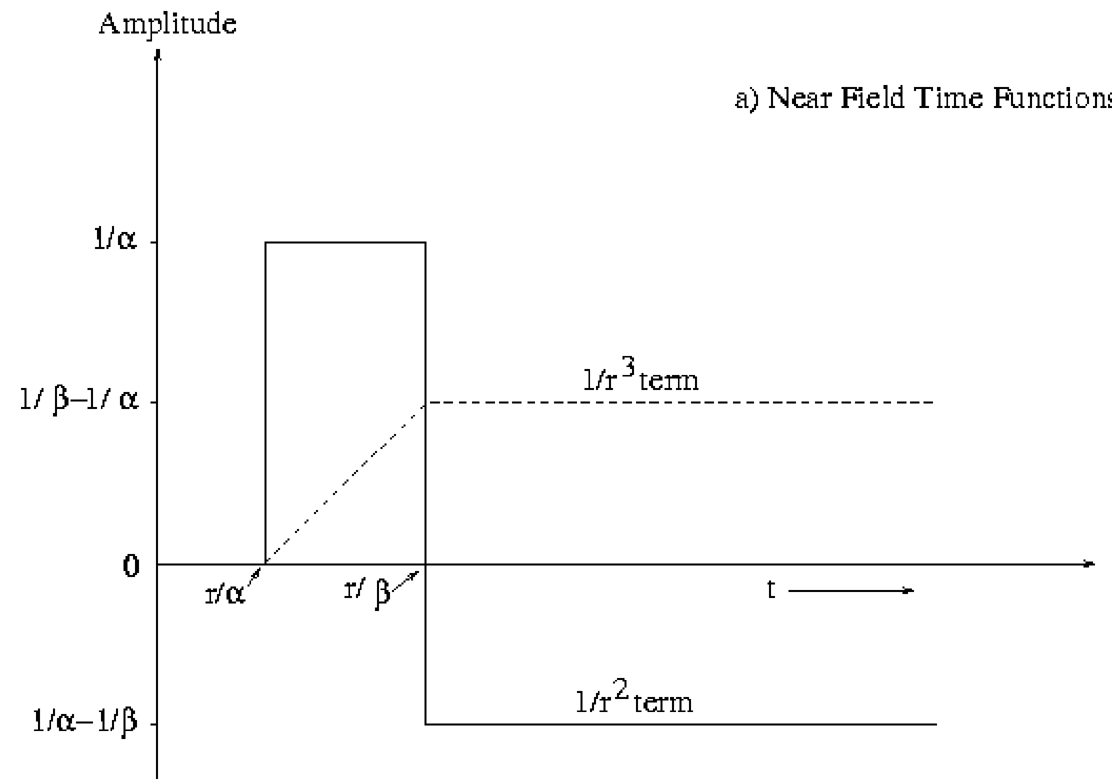
Far field term

Near field term

The **near-field expression of the point force delta function GF** is:

$$u_i^{NF} = \frac{\left(3\gamma_i\gamma_j - \delta_{ij}\right)}{4\pi\rho} \cdot$$

$$\left\{ \boxed{\frac{1}{r^3}} \left[(t-\frac{r}{\alpha})H(t-\frac{r}{\alpha}) - (t-\frac{r}{\beta})H(t-\frac{r}{\beta}) \right] + \boxed{\frac{1}{r^2}} \left[\frac{1}{\alpha} H(t-\frac{r}{\alpha}) - \frac{1}{\beta} H(t-\frac{r}{\beta}) \right] \right\}$$



and the response has a **static** (time-independent) component that corresponds to a permanent deformation of the medium, both in radial and transverse directions.

Far field term

The **far-field expressions of the point force delta function GF** are characterized by:

1) decay as 1/r;

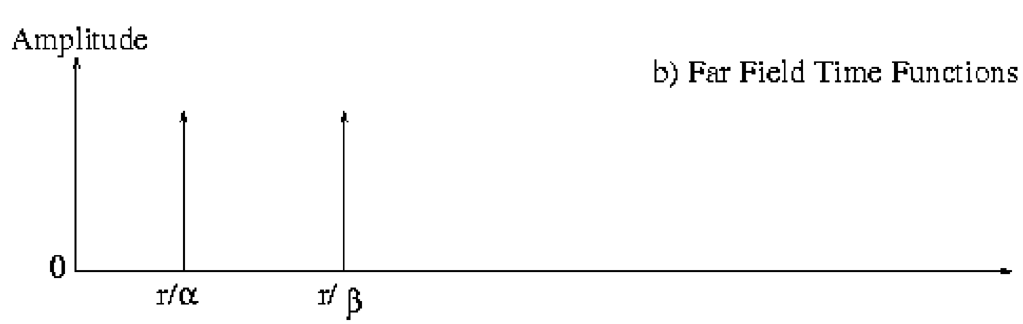
2) are made of P and S waves;

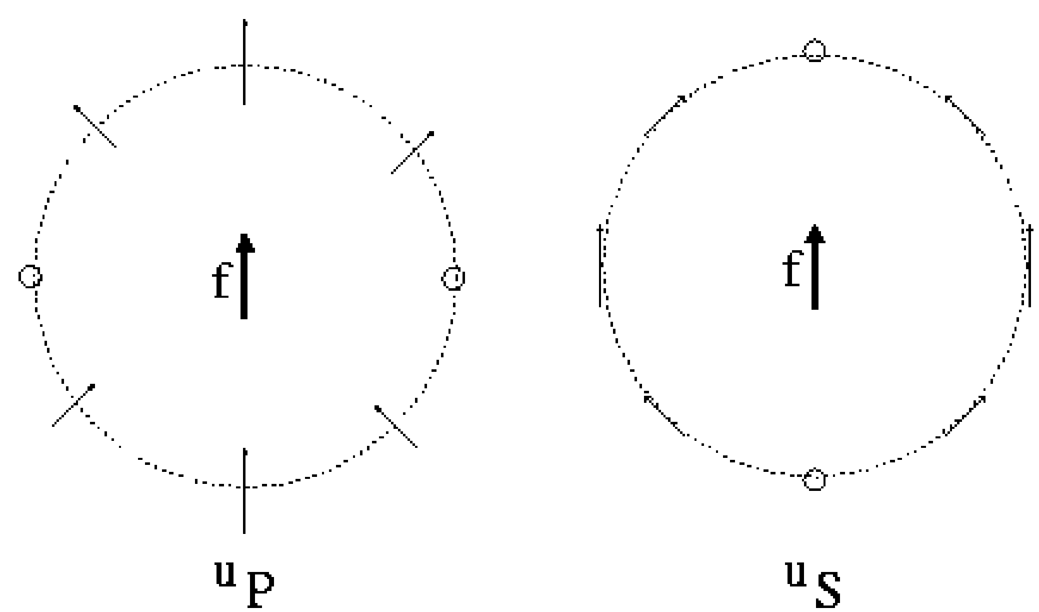
3) the displacement waveform is proportional to the applied force at the retarded time;

4) have a **radiation pattern**

$$\left|u_P^{FF}\right| \propto \gamma_1\gamma_j = \cos\theta$$

$$\left|u_S^{FF}\right| \propto -\gamma_j{}' = \sin\theta$$





GF for moment tensor

We can calculate the radiation pattern from a point source with an arbitrary moment tensor by noting that Green's function for a couple is just the spatial derivative of Green's function for a point force, so that the **displacement field from a moment tensor Mpq** is just:

$$u_n = M_{pq} * G_{np,q} = \lim_{\substack{\Delta l_q \to 0 \\ F_p \to \infty}} \Delta l_q F_p * \frac{\partial G_{np}}{\partial \zeta_q} =$$

$$= \boxed{\frac{u^{NF}}{4\pi\rho |\mathbf{x}|^4} \int_{|\mathbf{x}|/\alpha}^{|\mathbf{x}|/\beta} \tau M_{pq}(t-\tau)d\tau +}$$

Near field term

$$+ \boxed{\frac{u_P^{IF}}{4\pi\rho\alpha^2 |\mathbf{x}|^2} M_{pq}\left(t-\frac{|\mathbf{x}|}{\alpha}\right) - \frac{u_S^{IF}}{4\pi\rho\beta^2 |\mathbf{x}|^2} M_{pq}\left(t-\frac{|\mathbf{x}|}{\beta}\right)} +$$

Intermediate field term

$$+ \boxed{\frac{u_P^{FF}}{4\pi\rho\alpha^3 |\mathbf{x}|} \dot{M}_{pq}\left(t-\frac{|\mathbf{x}|}{\alpha}\right) - \frac{u_S^{FF}}{4\pi\rho\beta^3 |\mathbf{x}|} \dot{M}_{pq}\left(t-\frac{|\mathbf{x}|}{\beta}\right)}$$

Far field term

GF for double couple

An important case to consider in detail is the radiation pattern expected when the source is a double-couple. The result for a moment time function $M_0(t)$ is:

$$
\begin{aligned}
u = {} & \frac{A^{NF}}{4\pi\rho|\mathbf{x}|^4}\int_{|x|/\alpha}^{|x|/\beta}\tau M_0(t-\tau)d\tau + \\
& + \frac{A_P^{IF}}{4\pi\rho\alpha^2|\mathbf{x}|^2}M_0\left(t-\frac{|\mathbf{x}|}{\alpha}\right) - \frac{A_S^{IF}}{4\pi\rho\beta^2|\mathbf{x}|^2}M_0\left(t-\frac{|\mathbf{x}|}{\beta}\right) + \\
& + \frac{A_P^{FF}}{4\pi\rho\alpha^3|\mathbf{x}|}\dot{M}_0\left(t-\frac{|\mathbf{x}|}{\alpha}\right) - \frac{A_S^{FF}}{4\pi\rho\beta^3|\mathbf{x}|}\dot{M}_0\left(t-\frac{|\mathbf{x}|}{\beta}\right)
\end{aligned}
$$

$$A^{NF} = 9\sin 2\theta\cos\phi\hat{\mathbf{r}} - 6\left(\cos 2\theta\cos\phi\hat{\boldsymbol{\theta}} - \cos\theta\sin\phi\hat{\boldsymbol{\phi}}\right)$$

Near field term

$$A_P^{IF} = 4\sin 2\theta\cos\phi\hat{\mathbf{r}} - 2\left(\cos 2\theta\cos\phi\hat{\boldsymbol{\theta}} - \cos\theta\sin\phi\hat{\boldsymbol{\phi}}\right)$$

$$A_S^{IF} = -3\sin 2\theta\cos\phi\hat{\mathbf{r}} + 3\left(\cos 2\theta\cos\phi\hat{\boldsymbol{\theta}} - \cos\theta\sin\phi\hat{\boldsymbol{\phi}}\right)$$

Intermediate field term

$$A_P^{FF} = \sin 2\theta\cos\phi\hat{\mathbf{r}}$$

$$A_S^{FF} = \cos 2\theta\cos\phi\hat{\boldsymbol{\theta}} - \cos\theta\sin\phi\hat{\boldsymbol{\phi}}$$

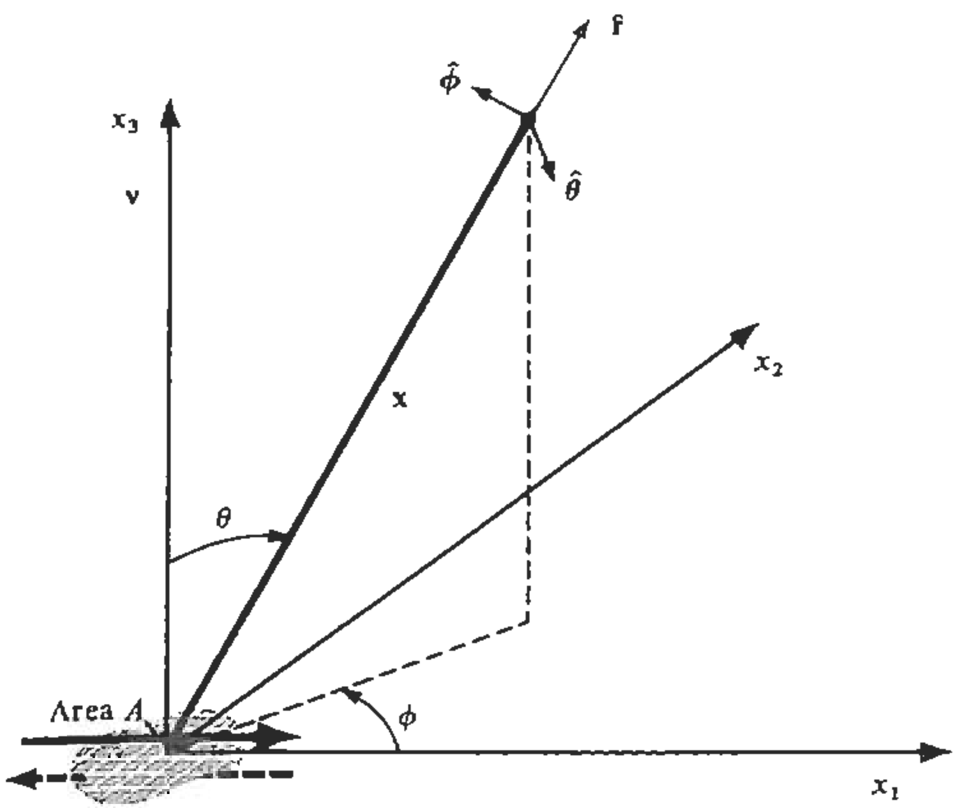
Far field term

NF DC (static) Radiation pattern

The **static** final displacement for a shear dislocation of strength M_0 is:

$$\mathbf{u} = \frac{M_0(\infty)}{4\pi\rho|\mathbf{x}|^2}\left[A^{NF}\left(\frac{1}{2\beta^2}-\frac{1}{2\alpha^2}\right)+\frac{A_P^{IF}}{\alpha^2}+\frac{A_S^{IF}}{\beta^2}\right]=$$

$$=\frac{M_0(\infty)}{4\pi\rho|\mathbf{x}|^2}\left[\left(\frac{3}{2\beta^2}-\frac{1}{2\alpha^2}\right)\sin 2\theta\cos\phi\,\hat{\mathbf{r}}+\frac{1}{\alpha^2}\left(\cos 2\theta\cos\phi\,\hat{\boldsymbol{\theta}}-\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}}\right)\right]$$



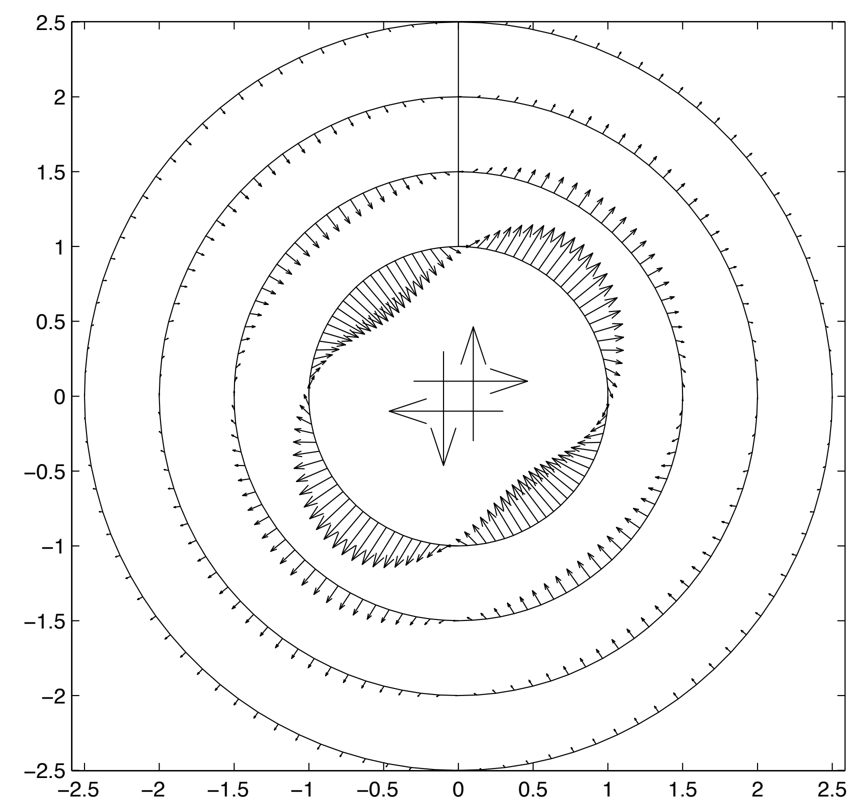
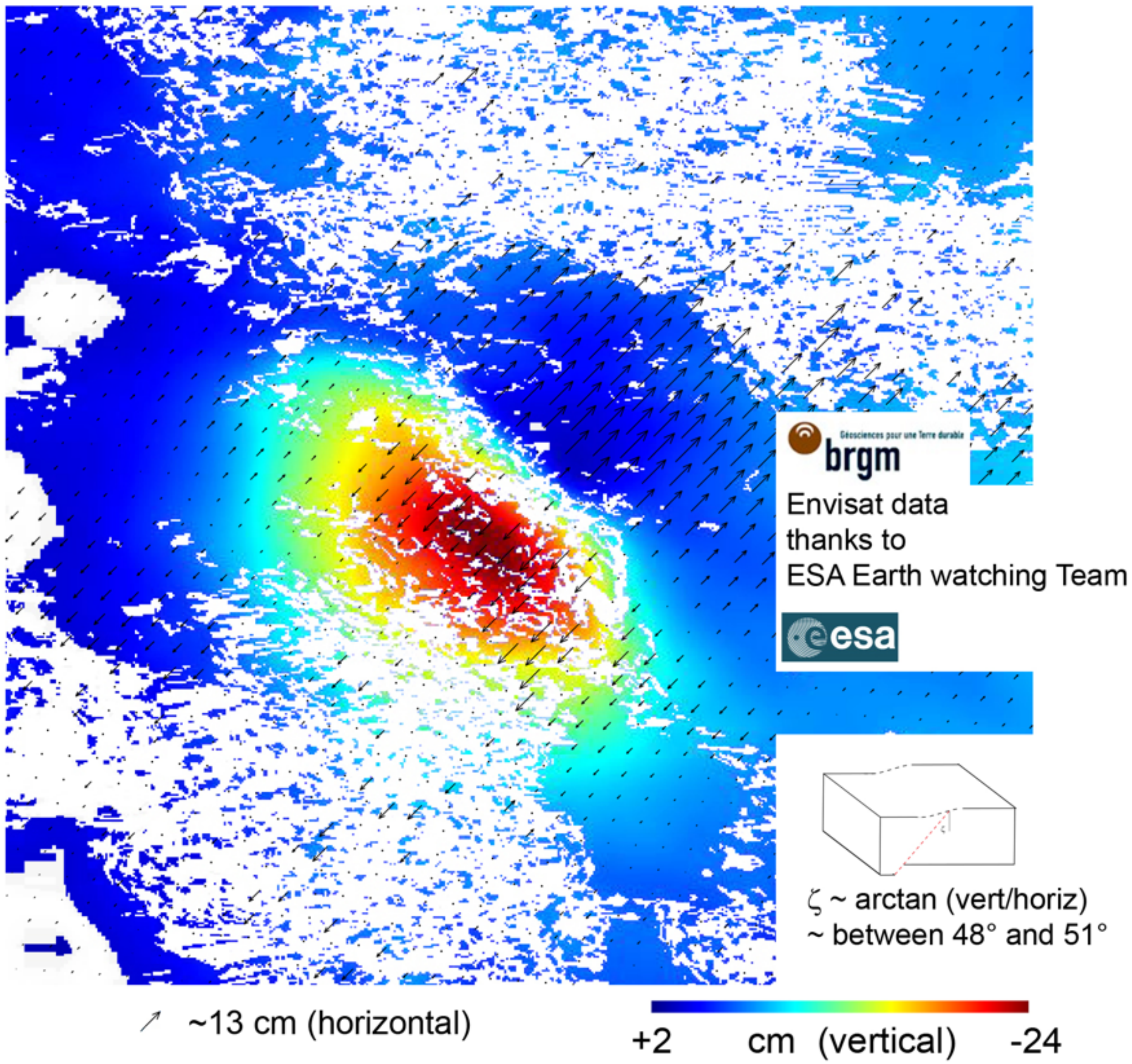
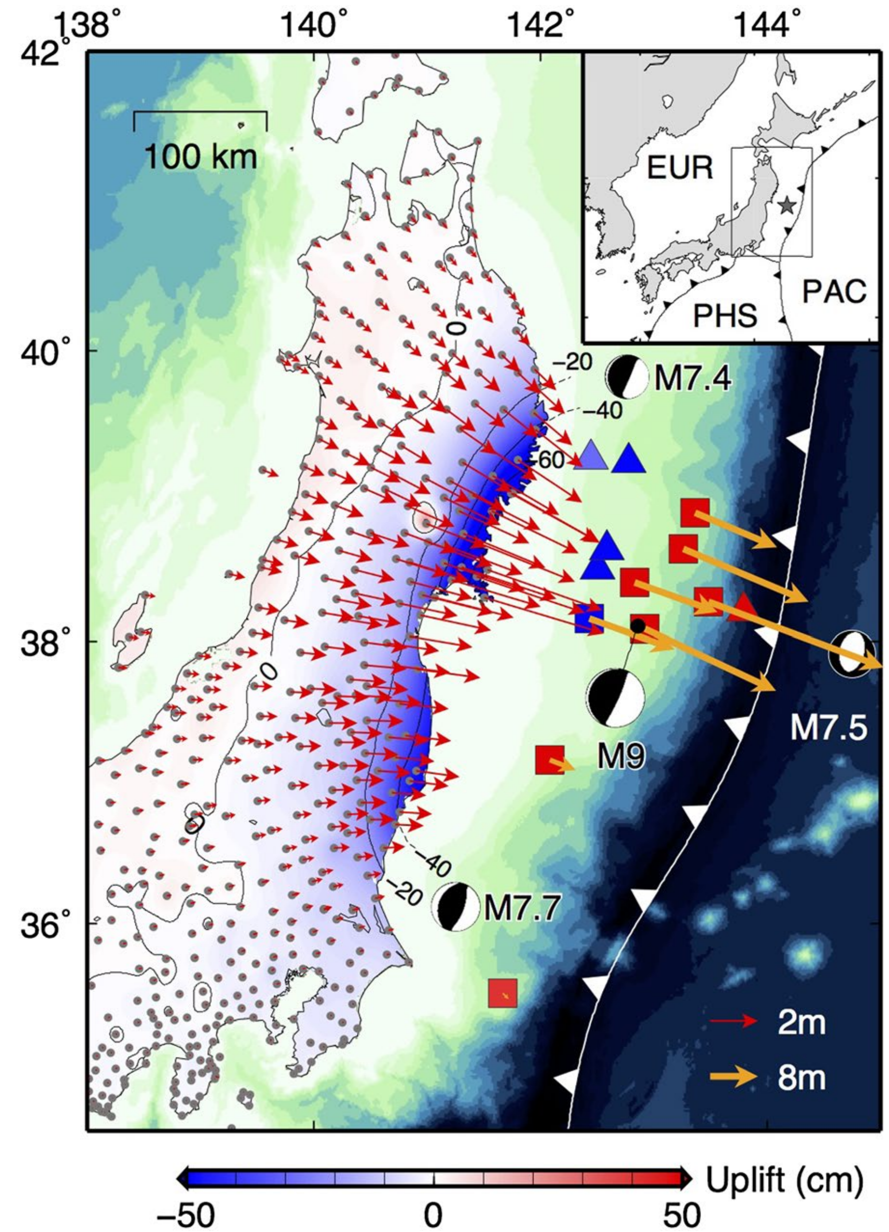


Figure 7: Near-field Static Displacement Field From a Point Double Couple Source ($\phi = 0$ plane); $\alpha = 3^{1/2}$, $\beta = 1$, $r = 0.1,\ 0.15,\ 0.20,\ 0.25$, $\rho = 1/4\pi$, $M_\infty = 1$; self-scaled displacements

Coseismic deformation

L'Aquila (Italy) earthquake, Mw 6.3.
Horizontal and Vertical surface displacement from InSAR Data
(assuming horizontal displacement is perpendicular to the fault strike ~N48W).





Coseismic displacements, seafloor topography, and plates around Japan. Red and orange arrows indicate onshore and offshore horizontal displacement vectors. Triangles and squares indicate pressure gauges and GPS/acoustic stations, respectively. Onshore uplifts are expressed by color scales and contours taken at 20-cm intervals. Offshore uplifts are indicated by color within triangles and squares. White line with triangles shows the Japan trench. Mechanisms of the M9 2011 Tohoku-oki earthquake and the three largest aftershocks are shown. The epicenter of the mainshock is represented by a black dot connected with the mechanism. Star in inset is the epicenter of the mainshock. PAC Pacific plate, EUR Eurasian plate, PHS Philippine Sea plate. Hashima et al. Earth, Planets and Space (2016) 68:159

FF DC Radiation pattern

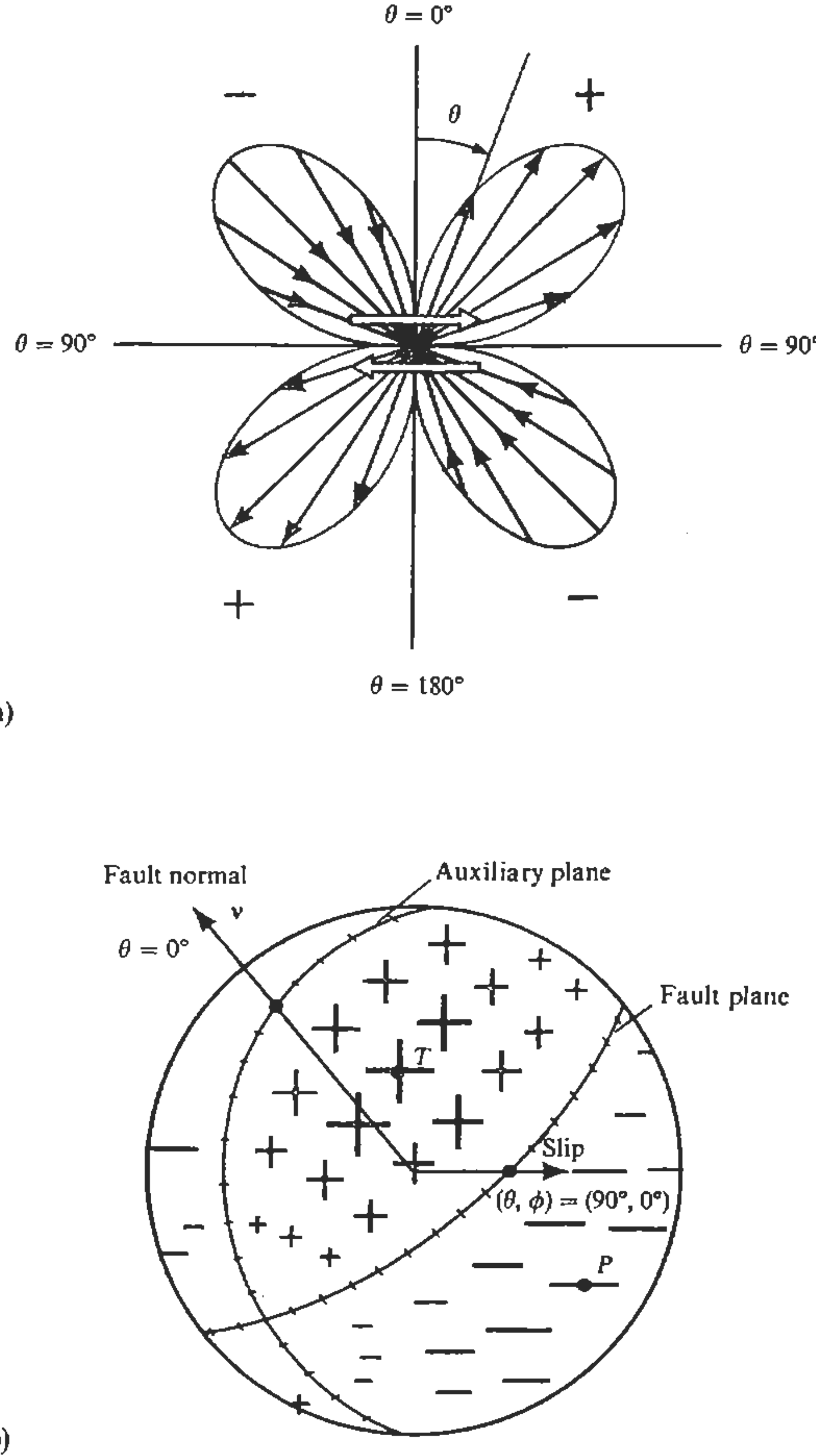


FIGURE **4.5**
Diagrams for the radiation pattern of the radial component of displacement due to a double couple, i.e., $\sin 2\theta \cos\phi\, \hat{\mathbf{r}}$. (**a**) The lobes are a locus of points having a distance from the origin that is proportional to $\sin 2\theta$. The diagram is for a plane of constant azimuth, and the pair of arrows at the center denotes the shear dislocation. Note the alternating quadrants of inward and outward directions. In terms of far-field P-wave displacement, plus signs denote outward displacement (if $\dot{M}_0(t - r/\alpha)$ is positive), and minus signs denote inward displacement. (**b**) View of the radiatiou pattern over a sphere centered on the origin. Plus and minus signs of various sizes denote variation (with θ, ϕ) of outward and inward motions. The fault plane and the auxiliary plane are nodal liues (on which $\sin 2\theta \cos\phi = 0$). An equal-area projection has been used (see Fig. 4.17). Point P marks the pressure axis, and T the tension axis.

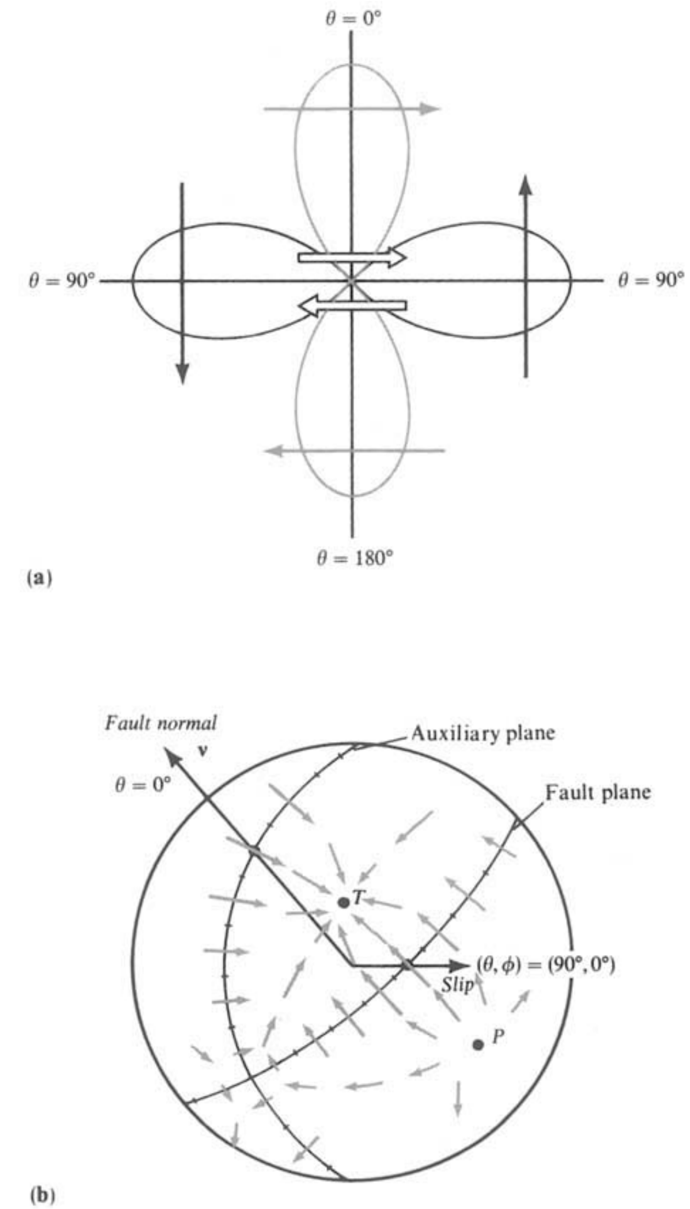


FIGURE **4.6**
Diagrams for the radiation pattern of the transverse component of displacement due to a double couple, i.e., $\cos 2\theta \cos\phi\, \hat{\boldsymbol{\theta}} - \cos\theta \sin\phi\, \hat{\boldsymbol{\phi}}$. (**a**) The four-lobed pattern in plane $\{\phi = 0, \phi = \pi\}$. The central pair of arrows shows the sense of shear dislocation, and arrows imposed on each lobe show the direction of particle displacement associated with the lobe. If applied to the far-field S-wave displacement, it is assumed that $\dot{M}_0(t - r/\beta)$ is positive. (**b**) Off the two planes $\theta = \pi/2$ and $\{\phi = 0, \phi = \pi\}$, the $\hat{\boldsymbol{\phi}}$ component is nonzero, hence (a) is of limited use. This diagram is a view of the radiation pattern over a whole sphere centered on the origin, and arrows (with varying size and direction) in the spherical surface denote the variation (with θ, ϕ) of the transverse motions. There are no nodal lines (where there is zero motion), but nodal points do occur. Note that the nodal point for transverse motion at $(\theta, \phi) = (45°, 0)$ is a maximum in the radiation pattern for longitudinal motion (Fig. 4.5b). But the maximum transverse motion (e.g., at $\theta = 0$) occurs on a nodal line for the longitudinal motion. The stereographic projection has been used (see Fig. 4.16). It is a conformal projection, meaning that it preserves the angles at which curves intersect and the shapes of small regions, but it does not preserve relative areas.