Seismic Sources

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Up to this point, we have been mostly concerned with homogeneous solutions to the wave equation, $g(\mathbf{x},\ t)$

$$\frac{\partial^2 g}{\partial t^2} - c^2 \bigtriangledown^2 g = 0 \tag{1}$$

In the study of seismic sources, however, we are interested in inhomogeneous solutions, the simplest of which is

$$\frac{\partial^2 g}{\partial t^2} - c^2 \bigtriangledown^2 g = \delta(\mathbf{x})\delta(t) \tag{2}$$

that is, where the forcing function is a delta function impulse in both space ant time.

For a homogeneous whole-space, the solution to (2) must have spherical symmetry, so spherical coordinates are the natural coordinate system for this case. In spherical coordinates, the Laplacian has the form

$$\bigtriangledown^2 g = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial g}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rg)\ . \tag{3}$$

The homogeneous wave equation for spherical symmetry (which is the solution to (2) at $\mathbf{x} = 0$ and $t = 0$), for spherical symmetry is thus

$$\frac{\partial^2}{\partial r^2}(rg) = r \bigtriangledown^2 g = \frac{1}{c^2}\frac{\partial^2 (rg)}{\partial t^2} \tag{4}$$

we thus see that rg is a simple traveling 1-dimensional wave disturbance in the $\hat{r}$ direction

$$rg = r(u_1(t - r/c) \pm u_2(t + r/c)) \tag{5}$$

where u_1 describes an outwarding expanding displacement wavefield, and u_2 describes an inward contracting wavefield. Clearly, for the case of interest here, the outward expanding wavefield is the appropriate solution.

We can see that the solution $u_1(t - r/c) = u_1(t - |x|/c)$ conserves energy, as the power in the wavefield will be proportional to the square of the displacement amplitude. We can thus see that the the spherical homogeneous solution must be of the form

$$u_1(t - r/c) = \frac{k}{|x|}\delta(t - r/c) \tag{6}$$

(6) is the isotropic homogeneous elastodynamic infinite space *impulse* response or *Green's function*; in this case it is the wavefield produced by a point impulse at the space-time origin.

To find the constant k for isotropic infinite elastic media, we consider the equation of motion for isotropic media

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} - (\lambda+\mu)\bigtriangledown(\bigtriangledown\cdot\mathbf{u}) - \mu\bigtriangledown^2\mathbf{u} = \mathbf{F} \tag{7}$$

where F is a body volume force density. We can also write the the equation of motion in terms of the body wave velocities by using the Helmholtz formulation

$$\rho\left(\frac{\partial^2 \mathbf{u}}{\partial t^2} - \alpha^2\bigtriangledown(\bigtriangledown\cdot\mathbf{u}) + \beta^2\bigtriangledown\times(\bigtriangledown\times\mathbf{u})\right) = \mathbf{F}\ . \tag{8}$$

If $\mathbf{F}$ is a purely radial force density of the form

$$\mathbf{F} = \delta(r)\delta(t) \tag{9}$$

the solution to the radial force problem is a propagating P-wave disturbance (there will be no shear waves from a pure dilatational source) of the form

$$u(r,t) = \frac{\delta(t-r/\alpha)}{4\pi\alpha^2\rho r}\ . \tag{10}$$

We can easily expand this solution somewhat by generalizing the space time coordinates, so that the time origin occurs at τ and the spatial origin occurs at ξ

$$u(\mathbf{x},t) = \frac{\delta(t-\tau-|\mathbf{x}-\xi|/\alpha)}{4\pi\alpha^2\rho|\mathbf{x}-\xi|}\ . \tag{11}$$

If the medium is linear (as it will be for small-amplitude seismic disturbances), we can further convolve the appropriate Green's function in any problem by some general time function $g(t)$ to get the solution

$$u(\mathbf{x},t) = \frac{g(t-\tau-|\mathbf{x}-\xi|/\alpha)}{4\pi\alpha^2\rho|\mathbf{x}-\xi|}\ . \tag{12}$$

Also making use of superposition, we can invoke a spatially extended source region $\Phi(\mathbf{x},t)$ over some volume V so that

$$u(\mathbf{x},t) = \iiint_V \frac{\Phi(t-\tau-|\mathbf{x}-\xi|/\alpha)dV(\xi)}{4\pi\alpha^2\rho|\mathbf{x}-\xi|}\ . \tag{13}$$

To study inhomogeneous solutions to the wave equation of a more general form, we will invoke a Helmholtz-like potential decomposition for source body forces

$$\mathbf{F} = \bigtriangledown\phi + \bigtriangledown\times\psi \tag{14}$$

in the same way that we can decompose the displacement field

$$\mathbf{u} = \bigtriangledown\Phi + \bigtriangledown\times\mathbf{\Psi}\ . \tag{15}$$

The inhomogeneous body wave equations for P and S waves in isotropic media can now be rewritten using the potential expressions for the source body force **F**, and the displacement field **u**

$$\frac{\partial^2 \Phi}{\partial^2 t} = \frac{\phi}{\rho} + \alpha^2 \bigtriangledown^2 \Phi \tag{16}$$

$$\frac{\partial^2 \mathbf{\Psi}}{\partial^2 t} = \frac{\psi}{\rho} + \beta^2 \bigtriangledown^2 \mathbf{\Psi} \tag{17}$$

Body force potentials (*Lamé potentials*), can be found using the expression (e.g., Aki and Richards, p. 69)

$$\mathbf{W}(\mathbf{x},t) = -\iiint_V \frac{\mathbf{Z}(\xi,t) dV(\xi)}{4\pi|\mathbf{x}-\xi|} \tag{18}$$

and

$$\phi(\mathbf{x},t) = \bigtriangledown \cdot \mathbf{W} \tag{19}$$

$$\psi(\mathbf{x},t) = -\bigtriangledown \times \mathbf{W} \tag{20}$$

Note that the Lamé potentials ϕ and ψ are spatially extended, even for point sources (see below).

For a body force with a delta time function $\delta(t)$ exerted in the $\hat{x}_1$ direction at $x = 0$, we thus have

$$Z(\mathbf{x},t) = \delta(t)\delta(x)\hat{x}_1 \tag{21}$$

so that

$$\mathbf{W}(x) = -\delta(t)\hat{x}_1 4\pi \iiint_V \frac{\delta(\xi) dV(\xi)}{|\mathbf{x}-\xi|} \tag{22}$$

which is just

$$\mathbf{W}(x) = -\frac{\delta(t)}{4\pi|\mathbf{x}|}\hat{x}_1 \ . \tag{23}$$

Therefore, the force potentials are

$$\phi = \bigtriangledown \cdot \mathbf{W} = -\frac{\delta(t)}{4\pi}\frac{\partial}{\partial x_1}\left(\frac{1}{|\mathbf{x}|}\right) \tag{24}$$

$$= -\frac{\delta(t)}{4\pi}\frac{\partial}{\partial x_1}(x_1^2 + x_2^2 + x_3^2)^{-1/2}$$

$$= -\frac{\delta(t)}{4\pi} \cdot \frac{-x_1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = \frac{\delta(t) x_1}{4\pi|\mathbf{x}|^3}$$

and

$$\psi = -\bigtriangledown \times \mathbf{W} = -\frac{\delta(t)}{4\pi}\left(0, \frac{\partial}{\partial x_3}\left(\frac{1}{|\mathbf{x}|}\right) \ , -\frac{\partial}{\partial x_2}\left(\frac{1}{|\mathbf{x}|}\right)\right) \tag{25}$$

$$= \frac{\delta(t)}{4\pi|\mathbf{x}|^3} \cdot (0, -x_3, x_2) \ .$$

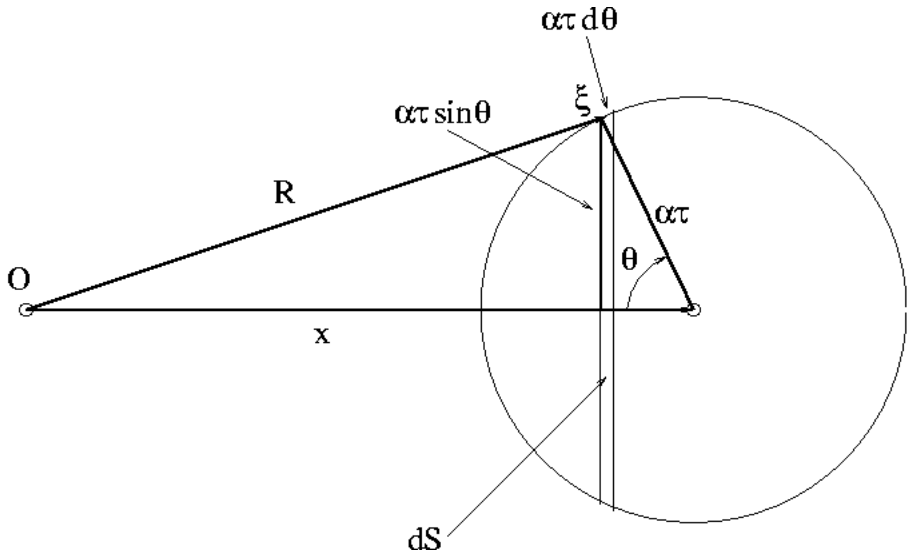

Figure 1: Integration Over a Force Potential via Spherical Shells.

We thus have the inhomogeneous P and S wave equations

$$\frac{\partial^2 \Phi}{\partial t^2} = \frac{\phi}{\rho} + \alpha^2 \bigtriangledown^2 \Phi = \frac{\delta(t) x_1}{4\pi\rho|\mathbf{x}|^3} + \alpha^2 \bigtriangledown^2 \Phi \tag{26}$$

and

$$\frac{\partial^2 \mathbf{\Psi}}{\partial t^2} = \frac{\psi}{\rho} + \beta^2 \bigtriangledown^2 \mathbf{\Psi} = \frac{\delta(t)(0, -x_3, -x_2)}{4\pi\rho|\mathbf{x}|^3} + \beta^2 \bigtriangledown^2 \mathbf{\Psi} \ . \tag{27}$$

(26) and (27) are just special cases of (13), so the solution for the P displacement potential is thus

$$\Phi(\mathbf{x}, t) = \frac{1}{(4\pi\alpha)^2\rho} \iiint_V \frac{\delta(t - |\mathbf{x} - \xi|/\alpha)}{|\mathbf{x} - \xi|} \cdot \frac{\partial}{\partial \xi_1} \left(\frac{1}{|\xi|} \right) dV(\xi) \ . \tag{28}$$

Substituting $|\mathbf{x} - \xi| = \alpha\tau$ and $dV(\xi) = \alpha d\tau dS(\xi)$, we solve the volume integral using concentric shells and a radial coordinate of $\alpha\tau$

$$\Phi(\mathbf{x}, t) = \frac{-1}{(4\pi\alpha)^2\rho} \int_0^\infty \frac{\delta(t - \tau)}{\alpha\tau} \left(\iint_S \frac{\partial}{\partial \xi_1} \left(\frac{1}{|\xi|} \right) dS(\xi) \right) \cdot \alpha d\tau \ . \tag{29}$$

The surface component of the integral is

$$h(\mathbf{x}, \tau) = \iint_{|\mathbf{x}-\xi|=\alpha\tau} \frac{\partial}{\partial \xi_1} \left(\frac{1}{\xi} \right) dS(\xi) \tag{30}$$

Using the geometry shown in Figure 1, with $\eta = -\mathbf{x}$, $R = |\xi - \eta|$, $r = |\mathbf{x}|$ and

$$dS = (2\pi\alpha\tau \sin\theta)(\alpha\tau \ d\theta) = 2\pi\alpha^2\tau^2 \sin\theta \ d\theta \tag{31}$$

we have

$$h(\mathbf{x}, \tau) = -\frac{\partial}{\partial \eta_1} \iint_S \frac{dS}{R} = -\frac{\partial}{\partial \eta_1} \int_0^\pi \frac{\sin\theta \; d\theta}{R} \cdot 2\pi\alpha^2\tau^2 \; . \tag{32}$$

Using the law of cosines, we have

$$R^2 = r^2 + \alpha^2\tau^2 - 2r\alpha\tau\cos\theta \tag{33}$$

so that

$$\frac{\sin\theta \; d\theta}{R} = \frac{dR}{r\alpha\tau} \tag{34}$$

and

$$\begin{aligned} h(\mathbf{x}, \tau) &= -\frac{\partial}{\partial \eta_1}\left(\frac{2\pi\alpha\tau}{r}\int_{R(\theta=0)}^{R(\theta=\pi)} dR\right) \\ &= -\frac{\partial}{\partial \eta_1}\left(\frac{2\pi\alpha\tau}{r}\int_{\alpha\tau-r}^{\alpha\tau+r} dR\right) \\ &= -\frac{\partial}{\partial \eta_1}\left(\frac{2\pi\alpha\tau}{r}\cdot 2r\right) = -\frac{\partial}{\partial \eta_1}(4\pi\alpha\tau) = 0 \end{aligned} \tag{35}$$

when $\alpha\tau > r$, and

$$= -\frac{\partial}{\partial \eta_1}\left(\frac{2\pi\alpha\tau}{r}\cdot 2\alpha\tau\right) = -\frac{\partial}{\partial \eta_1}\left(\frac{4\pi\alpha^2\tau^2}{r}\right) = 4\pi\alpha^2\tau^2\frac{\partial}{\partial x_1}\left(\frac{1}{|\mathbf{x}|}\right) \; .$$

when $\alpha\tau < r$. Plugging this solution for h back in, we have

$$\Phi(\mathbf{x}, t) = \frac{1}{(4\pi\alpha)^2\rho}\int_0^\infty \frac{\delta(t-\tau)}{\alpha\tau}\cdot 4\pi\alpha^2\tau^2\frac{\partial}{\partial x_1}\left(\frac{1}{|\mathbf{x}|}\right) H(|\mathbf{x}|/\alpha - \tau) \; d\tau \tag{36}$$

where $H(t)$ is the *step function* which produces zero if its argument is negative and one otherwise. The step function restricts the τ integration to values less than $|\mathbf{x}|/\alpha$, so that

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\rho}\frac{\partial}{\partial x_1}\left(\frac{1}{|\mathbf{x}|}\right)\int_0^{|x|/\alpha} \tau\delta(t-\tau) \; d\tau \; . \tag{37}$$

Following the same development for the S wave field, we have

$$\mathbf{\Psi}(\mathbf{x}, t) = \frac{1}{4\pi\rho}\left(0, \frac{\partial}{\partial x_3}\left(\frac{1}{|\mathbf{x}|}\right), -\frac{\partial}{\partial x_2}\left(\frac{1}{|\mathbf{x}|}\right)\right)\int_0^{|x|/\beta} \tau\delta(t-\tau) \; d\tau \; . \tag{38}$$

Given the above formulas for Φ and $\mathbf{\Psi}$, we can now solve for the respective displacement fields

$$\mathbf{u} = \bigtriangledown\Phi + \bigtriangledown \times \mathbf{\Psi} \tag{39}$$

or, in index notation

$$u_i = \frac{1}{4\pi\rho}\left(\frac{\partial}{\partial x_1}\bigtriangledown_i\left(\frac{1}{|\mathbf{x}|}\right)\int_0^{|x|/\alpha}\tau\delta(t-\tau)\ d\tau\right) \tag{40}$$

$$+\frac{1}{4\pi\rho}\left(\bigtriangledown\times\left(0,\frac{\partial}{\partial x_3}\left(\frac{1}{|\mathbf{x}|}\right),-\frac{\partial}{\partial x_2}\left(\frac{1}{|\mathbf{x}|}\right)\right)\int_0^{|x|/\beta}\tau\delta(t-\tau)\ d\tau\right)\ .$$

After much calculus and algebra (which we shall mercifully omit) this gives Green's function for a point force in the $\hat{x}_1$ direction in an isotropic homogeneous whole-space

$$u_i = \frac{1}{4\pi\rho}\left(-\frac{\partial^2}{\partial x_1 \partial x_i}\left(\frac{1}{|\mathbf{x}|}\right)\int_0^{|\mathbf{x}|/\alpha}\tau\delta(t-\tau)\ d\tau\right) \tag{41}$$

$$+\frac{1}{4\pi\rho}\left(\frac{\partial^2}{\partial x_1 \partial x_i}\left(\frac{1}{|\mathbf{x}|}\right)\int_0^{|\mathbf{x}|/\beta}\tau\delta(t-\tau)\ d\tau\right)$$

$$+\frac{1}{4\pi\rho}\left(\frac{1}{\alpha^2|\mathbf{x}|}\frac{\partial|\mathbf{x}|}{\partial x_1}\frac{\partial|\mathbf{x}|}{\partial x_i}\right)\delta(t-|\mathbf{x}|/\alpha)$$

$$+\frac{1}{4\pi\rho}\left(\frac{1}{\beta^2|\mathbf{x}|}\left(\delta_{i1}-\frac{\partial|\mathbf{x}|}{\partial x_1}\frac{\partial|\mathbf{x}|}{\partial x_i}\right)\right)\delta(t-|\mathbf{x}|/\beta)\ .$$

Combining the first two integral terms gives

$$u_i = \frac{1}{4\pi\rho}\left(\frac{\partial^2}{\partial x_1 \partial x_i}\left(\frac{1}{|\mathbf{x}|}\right)\int_{|\mathbf{x}|/\alpha}^{|\mathbf{x}|/\beta}\tau\delta(t-\tau)\ d\tau\right) \tag{42}$$

$$+\frac{1}{4\pi\rho}\left(\frac{1}{\alpha^2|\mathbf{x}|}\left(\frac{\partial|\mathbf{x}|}{\partial x_1}\frac{\partial|\mathbf{x}|}{\partial x_i}\right)\delta(t-|\mathbf{x}|/\alpha)\right)$$

$$+\frac{1}{4\pi\rho}\left(-\frac{1}{\beta^2|\mathbf{x}|}\left(\delta_{i1}-\frac{\partial|\mathbf{x}|}{\partial x_1}\frac{\partial|\mathbf{x}|}{\partial x_i}\right)\delta(t-|\mathbf{x}|/\beta)\right)\ .$$

Using the direction cosines

$$\gamma_i = \frac{x_i}{|\mathbf{x}|} = \frac{\partial|\mathbf{x}|}{\partial x_i} \tag{43}$$

and generalizing from the $\hat{x}_1$ direction, finally gives a compact form for Green's function for the displacement in the $\hat{x}_i$ direction for a point force in the $\hat{x}_j$ direction

$$G_{ij} = \tfrac{(3\gamma_i\gamma_j-\delta_{ij})}{4\pi\rho|\mathbf{x}|^3}\int_{|\mathbf{x}|/\alpha}^{|\mathbf{x}|/\beta}\tau\delta(t-\tau)\ d\tau \tag{44}$$

$$+\tfrac{\gamma_i\gamma_j}{4\pi\rho\alpha^2|\mathbf{x}|}\delta(t-|\mathbf{x}|/\alpha) \tag{45}$$

$$+\tfrac{(\gamma_i\gamma_j-\delta_{ij})}{4\pi\rho\beta^2|\mathbf{x}|}\delta(t-|\mathbf{x}|/\beta)\ . \tag{46}$$

The three terms (44, 45, 46), first derived by Stokes in 1848, are called the near-field, far-field P, and far-field S terms, respectively. The first term has a time dependence of

$$u^{NF} = \int_{r/\alpha}^{r/\beta} \tau\delta(t-\tau)\ d\tau = \int_{-\infty}^{\infty} \tau\delta(t-\tau)(H(\tau - r/\alpha) - H(\tau - r/\beta))\ d\tau \quad (47)$$

$$= t(H(t - r/\alpha) - tH(t - r/\beta))$$

by the sifting property of the delta function. We can write, where c is α or β,

$$tH(t - r/c) = (t - r/c)H(t - r/c) + (r/c)H(t - r/c) \quad (48)$$

for both the α and β terms to obtain a new expression for the near-field expression of the point-force, delta function Green's function (e.g., Knopoff, 1967; Udías, 1999)

$$G_{ij}^{NF} = \frac{(3\gamma_i\gamma_j - \delta_{ij})}{4\pi\rho} \cdot \quad (49)$$

$$\left(\frac{1}{r^3}\left((t - r/\alpha)H(t - r/\alpha) - (t - r/\beta)H(t - r/\beta)\right) + \frac{1}{r^2}\left(\frac{1}{\alpha}H(t - r/\alpha) - \frac{1}{\beta}H(t - r/\beta)\right)\right) .$$

The far-field expression thus depends on a combination of terms with r^{-3} and r^{-2} distance dependence. In time, the r^{-3} term depends on the difference of two ramps with unit slope. For $r/\alpha < t < r/\beta$, there is only one ramp in the expression with unit slope. For $t > r/\beta$, the difference between the two ramp functions becomes a constant with respect to time,

$$\frac{1}{\beta} - \frac{1}{\alpha} \quad (50)$$

The r^{-2} dependent terms of the near-field response have a time dependence given by a single step function (rather than a ramp) for $r/\alpha < t < r/\beta$ and the difference of two step functions for $t > r/\beta$. (Figure 2a). The spatial dependences of these two terms are such that the r^{-2} terms will eventually dominate. Note that the near-field response has a static (time-independent) component that corresponds to a permanent deformation of the medium.

The particle motions of the near field terms are both radial

$$u^{NF} \cdot \hat{\gamma} \propto (3\gamma_i\gamma_j - \delta_{ij}) \cdot \gamma_i = (3\gamma_i\gamma_j\gamma_i - \gamma_j) = 2\gamma_j \quad (51)$$

and transverse

$$u^{NF} \times \hat{\gamma} \propto (3\gamma_i\gamma_j - \delta_{ij}) \times \gamma_i \equiv -\gamma_j' \quad (52)$$

where γ_j' is the cosine of the angle between the force direction and the shear-wave particle motion (see below).

The far-field terms are the familiar traveling P and S waves. For a delta function source, the corresponding displacements are thus just delta functions arriving at the proper times r/α and r/β, respectively (Figure 2b). The polarizations of the P and S waves are parallel to and tangential to, respectively the

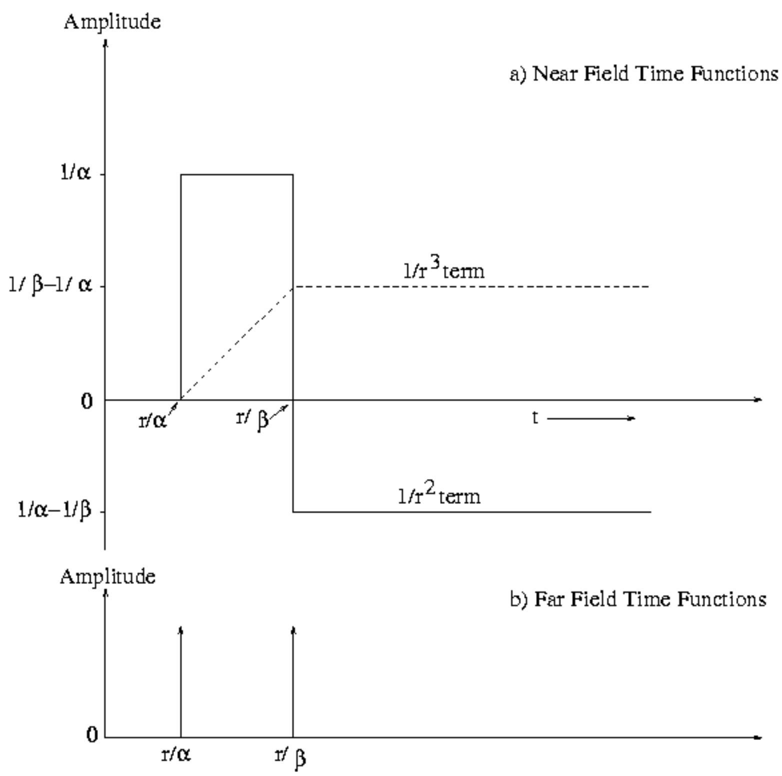

Figure 2: Time Dependence of Near Field Terms in an Isotropic Solid at a Source Distance of r. Delta Function Force in Isotropic Whole-Space.

radial direction The *radiation pattern* of the far-field radially polarized P waves for an angle θ with respect to the applied force (say, the $\hat{x}_i = \hat{x}_1$) direction is

$$|u_P| \propto \gamma_1 \gamma_j = \cos\theta \tag{53}$$

so that there is no P-wave radiation perpendicular to the direction of, and maximum radiation parallel and antiparallel to, the direction of the applied force. Similarly, the radiation pattern for the tangentially polarized far-field S-waves is

$$|u_S| \propto -\gamma_j' = \sin\theta \ . \tag{54}$$

An important source situation in seismology is the seismic radiation pattern from a slipping fault (Figure 4).

Our approach will be to first recall that we can write the general displacement response for a force in the p direction as the convolution of the forcing function with the Green function tensor (44). Second, we must recognize that faulting is an internal process, so we can never have isolated forces operating inside of a medium (Newton's third law). Thus, no net momentum or angular momentum change can take place in the Earth from a slipping fault. We can use the convenient fiction of the isolated force, however, to evaluate the Green function for *force couples*, which we shall state without proof, are the appropriate body forces to consider for the slipping fault and other situations appropriate for genuine seismic sources. The nine force couples are shown in Figure 6. Any point seismic source in a medium can be represented (and finite extent sources can be well approximated in the far field) by an appropriate superposition of these source couples. The weights for the nine elements make up the *seismic moment tensor*. Because there cannot be any net change in angular momentum for the Earth, the weighting of the moment tensor elements must be balanced, so that the moment tensor must be symmetric. For sources of finite extent, the moment concept is easily generalized using superposition into a moment density tensor.

For the case of a fault plane, Σ (Figure 4) with normal $\hat{\nu} = \hat{\xi}_3$ and slip only in the $\hat{\xi}_1$ direction, the moment tensor is simply

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & M_0 \\ 0 & 0 & 0 \\ M_0 & 0 & 0 \end{pmatrix} \tag{55}$$

where M_0 is the *seismic moment*, sometimes called the *scalar moment*, where for a planar finite fault, we have

$$M_0 = \mu A \bar{u} \tag{56}$$

where μ is the rigidity, A is the fault area, and $\bar{u}$ is the mean slip on the fault surface. The moment provides a fundamental measure of earthquake (or more generally, any seismic source) size.

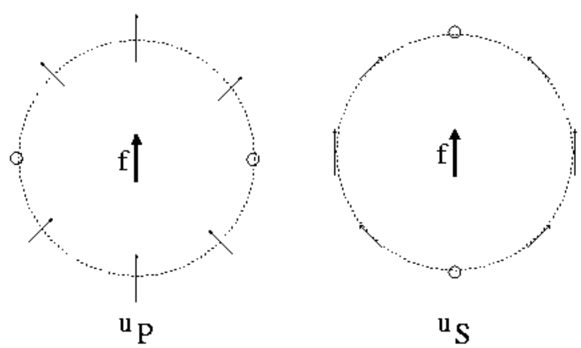

Figure 3: P and S Far-Field Displacement Radiation Patterns for a Delta Function Force in an Isotropic Whole-Space.

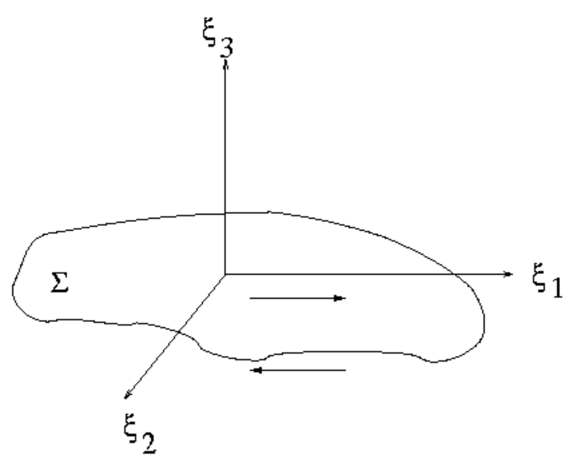

Figure 4: A Slipping Fault.

It is useful to convert the scalar moment to a *moment magnitude*, using the formula (for MKS units)

$$M_w = \frac{2}{3}\left(\log_{10} M_0 - 9.1\right) \tag{57}$$

where the constants in the formula provide reasonable consistency with the older magnitude scales based on body or surface wave amplitudes.

For a more general double-couple solution (e.g., see Stein and Wysession), if $\hat{n} = (n_x,\ n_y, n\ _z)$ is the normal to the fault plane and $\hat{d} - (d_x,\ d_y,\ d_z)$ is the unit slip vector, then the moment tensor is

$$\mathbf{M} = M_0(n_i d_j + n_j d_i) = M_0 \begin{pmatrix} 2n_x d_x & n_x d_y + n_y d_x & n_x d_z + n_z d_x \\ n_y d_x + n_x d_y & 2n_y d_y & n_y d_z + n_z d_y \\ n_z d_x + n_x d_z & n_z d_y + n_y d_z & 2n_z d_z \end{pmatrix} . \tag{58}$$

Note that the trace is just $2M_0\hat{n}\cdot\hat{d} = 0$, so that the dilational component of a double-couple moment tensor is zero regardless of fault orientation and/or coordinate system. The eigenvectors of the double couple moment tensor are called the t, p, and b axes, and they have corresponding eigenvalues of M_0, $-M_0$, and 0, respectively. One useful physical interpretation of the t and p axes is that they are that the centers of the compressional and dilational quadrants of the P-wave radiation pattern (the maximum and minimum P-wave radiation directions (see below). Conversely, the b axis is the axis of no radiated P- or S- wave energy, which occurs at the intersection of the fault plane and and its (first-motion observationally indistinguishable) counterpart, the *auxiliary plane* (Figures 5, 7). The fault and auxiliary planes of a double-couple solution lie at 45^o to the p and t axes, and the intersection of the fault and auxiliary plane denotes the b axis.

More generally, for sources of nonzero time extent, the moment tensor concept can be further generalized by letting the moment be a function of time. General components of the moment tensor per unit area for isotropic media per fault area can be calculated from

$$M_{pq} = \lambda \nu_k u_k \delta_{pq} + \mu(\nu_p u_q + \nu_q u_p) \tag{59}$$

where $\hat{\nu}$ is the normal to the fault or crack and $\mathbf{u}$ is the displacement at the source, which need not be normal to $\hat{\nu}$. Thus, a tensile crack in the plane with normal $\hat{\nu} = \hat{\xi}_3$, which has a displacement $\mathbf{u} = u_3\hat{\xi}_3$, and corresponds to only the $(3,3)$ force couple, has the moment tensor

$$\mathbf{M} = u_2 \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & (\lambda + 2\mu) \end{pmatrix} \cdot A\ , \tag{60}$$

where A is the crack area. An isotropic explosion producing a radial displacement u_r can thus be represented as the moment tensor corresponding to the sum of the (i,i) force couples

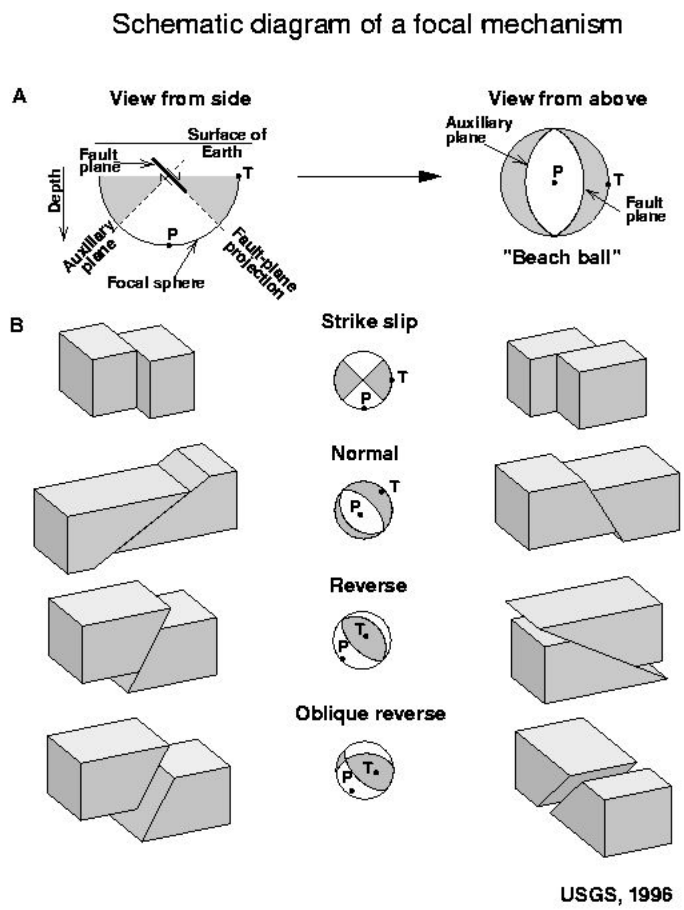

Figure 5: "Beachball" depictions of the P-wave radiation pattern (from quake.wr.usgs.gov). Note the relationship fo the p and t axes to the fault and auxiliary planes.

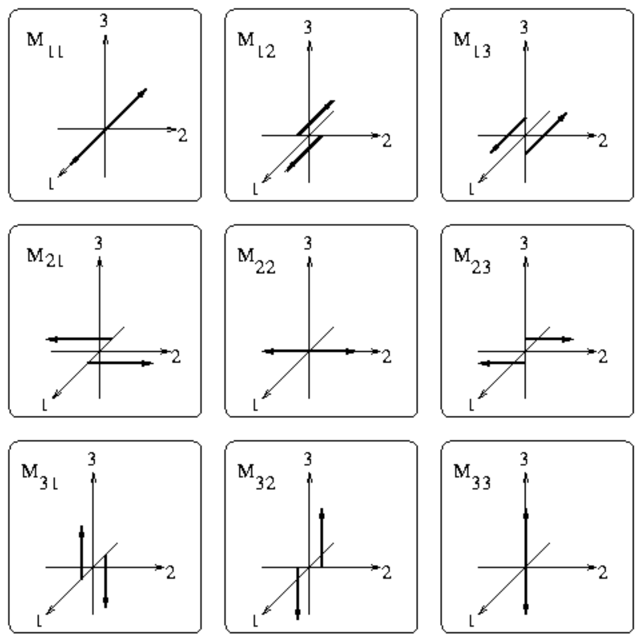

Figure 6: The nine force couples.

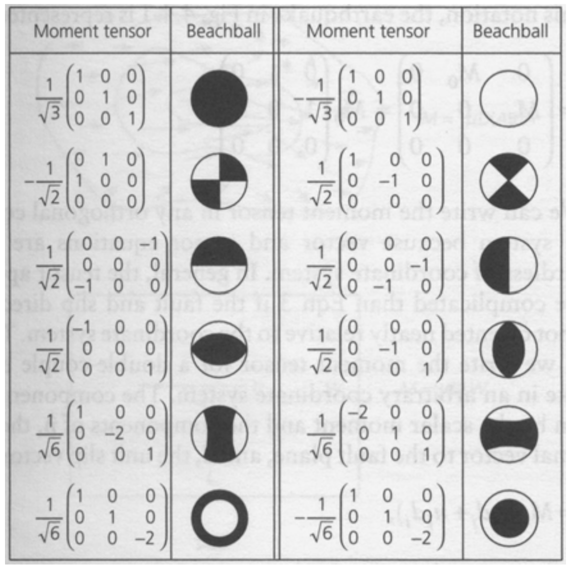

Figure 7: Depiction of "beachball" P-wave radiation patterns for various canonical moment tensors (after Dahlen and Tromp and Stein and Wysession).

$$\mathbf{M} = u_r(3\lambda + 2\mu)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot S = 3u_r\kappa \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot S \ , \qquad (61)$$

where S is the spherical surface area of the source volume.

Note that the symmetry of the moment tensor implies that there is no observational difference between a point source corresponding to slip in the $\hat{\xi}_1$ direction for a fault normal to $\hat{\xi}_3$, and a fault slipping in the $\hat{\xi}_3$ direction for a fault normal to $\hat{\xi}_1$. There reflects the inherent ambiguity in the interpretation, notably of double-couple sources, for example, a moment tensor alone cannot tell us if a strike-slip fault is right lateral on a particular vertically-dipping plane, or is actually left-lateral on an (auxiliary) vertically-dipping plane oriented 90^o away. Additional faulting information, such as pulse broadening or shortening caused by fault slip propagation or possibly direct fault geometry information is necessary to make this distinction.

Practically speaking, moment tensors must be estimated from waveform information. This can be done routinely for global earthquakes (and has been done so routinely since the late 1970's). Generally earthquake solutions derived from *moment tensor inversion* are close to, but not exact, double couples that fit the form of (58) (Stein and Wysession give a nice summary of moment tensor inversion in Chapter 4) There are several reasons why moment tensor inversions do not produce exact double couples for earthquakes. First, the forward modeling of the waveforms is not exact (e.g., unmodeled Earth heterogeneity), second, the data may be noisy, third, the earthquake itself may not be a perfect double couple. This will occur if the source region is appreciably spatially extended (as it will be for a large earthquake) and/or has a nonplanar or other complex faulting geometry, such as a curved fault plane, or nearly simultaneous rupture on adjacent faults so that the resulting signals are superimposed. A useful estimate of the scalar moment may be obtained for many earthquakes by evaluating the difference of the two largest absolute values eigenvalues of the moment tensor, divided by two (note that in the perfect double case this will give exactly M_0). Another useful method of analyzing complex moment tensors is to decompose them into a dilatational term (by extracting the trace), and the sum of a major and minor double couple to assess how earthquake point-source-like they are.

We can calculate the radiation pattern from a point source with an arbitrary moment tensor by noting that Green's function for a couple is just the spatial derivative of Green's function for a point force (44), so that the displacement field from a moment tensor M_{pq} is just (e.g., Aki and Richards, p. 78)

$$u_n = M_{pq} * G_{np,q} = \lim_{\begin{pmatrix} \Delta l_q \to 0 \\ F_p \to \infty \\ \Delta l_q F_P = M_{pq} \end{pmatrix}} \Delta l_q F_p * \frac{\partial}{\partial \xi_q} G_{np} \ , \tag{62}$$

where Δl_q is a small distance (lever arm or force separation, for off- and on-diagonal moment tensor elements, respectively) in the $\hat{\xi}_q$ direction, F_p is the force applied to each couple. Applying (62) to (44) gives the displacement field for a general moment tensor

$$\begin{aligned} u_n = M_{pq} * G_{np,q} = \left(\frac{15\gamma_n\gamma_p\gamma_q - 3\gamma_n\delta_{pq} - 3\gamma_p\delta_{nq} - 3\gamma_q\delta_{np}}{4\pi\rho r^4} \right) u^{NF} = \int_{r/\alpha}^{r/\beta} \tau M_{pq}(t-\tau)\, d\tau \\ + \left(\frac{6\gamma_n\gamma_p\gamma_q - \gamma_n\delta_{pq} - \gamma_p\delta_{nq} - \gamma_q\delta_{np}}{4\pi\rho\alpha^2 r^2} \right) M_{pq}(t - r/\alpha) \\ - \left(\frac{6\gamma_n\gamma_p\gamma_q - \gamma_n\delta_{pq} - \gamma_p\delta_{nq} - \gamma_q\delta_{np}}{4\pi\rho\beta^2 r^2} \right) M_{pq}(t - r/\beta) \\ + \frac{\gamma_n\gamma_p\gamma_q}{4\pi\rho\alpha^3 r} \dot{M}_{pq}(t - r/\alpha) \\ - \left(\frac{\gamma_n\gamma_p - \delta_{np}}{4\pi\rho\beta^3 r} \right) \gamma_q \dot{M}_{pq}(t - r/\beta) \ . \end{aligned} \tag{63}$$

The near-field has terms proportional to r^{-4} times the integral expression and to r^{-2}. The far-field terms are proportional to r^{-1} as expected and have a time history controlled by the time derivative of the moment source time function.

An important case to consider in detail is the radiation pattern expected from (63) when the source is a double-couple (55). The result for a moment time function $M_0(t)$ is

$$
\begin{aligned}
u = {} & \frac{A_N}{4\pi\rho r^4}\int_{r/\alpha}^{r/\beta}\tau M_0(t-\tau)\, d\tau \\
& + \frac{A_{IP}}{4\pi\rho\alpha^2 r^2} M_0(t - r/\alpha) \\
& + \frac{A_{IS}}{4\pi\rho\beta^2 r^2} M_{pq}(t - r/\beta) \\
& + \frac{A_{FP}}{4\pi\rho\alpha^3 r} \dot{M}_{pq}(t - r/\alpha) \\
& - \frac{A_{FS}}{4\pi\rho\beta^3 r} \dot{M}_{pq}(t - r/\beta) \ ,
\end{aligned}
\tag{64}
$$

where the polar coordinate expressions for the radiation pattern factors are

$$
\begin{aligned}
A_N &= 9\sin 2\theta\cos\phi\hat{r} - 6(\cos 2\theta\cos\phi\hat{\theta} - \cos\theta\sin\phi\hat{\phi}) \\
A_{IP} &= 4\sin 2\theta\cos\phi\hat{r} - 2(\cos 2\theta\cos\phi\hat{\theta} - \cos\theta\sin\phi\hat{\phi}) \\
A_{IS} &= -3\sin 2\theta\cos\phi\hat{r} + 3(\cos 2\theta\cos\phi\hat{\theta} - \cos\theta\sin\phi\hat{\phi}) \\
A_{FP} &= \sin 2\theta\cos\phi\hat{r} \\
A_{FS} &= \cos 2\theta\cos\phi\hat{\theta} - \cos\theta\sin\phi\hat{\phi}
\end{aligned}
$$

The far-field 2θ patterns for P and S waves are shown in Figure 8.

The near-field term gives a static displacement as $t \to \infty$

$$
\mathbf{u} = \frac{M_0(\infty)}{4\pi\rho r^2}\left(\frac{A_N}{2}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right) + \frac{A_{IP}}{\alpha^2} + \frac{A_{IS}}{\beta^2}\right) \tag{65}
$$

$$
= \frac{M_0(\infty)}{4\pi\rho r^2}\left(\frac{1}{2}\left(\frac{3}{\beta^2} - \frac{1}{\alpha^2}\right)\sin 2\theta\cos\theta\hat{r} + \frac{1}{\alpha^2}(\cos 2\theta\cos\phi\hat{\theta} - \cos\theta\sin\phi\hat{\phi})\right) \ ,
$$

where $M_0(\infty)$ is the final value of the seismic moment. Interestingly, this expression contains two terms with the same angular dependence as those for the far-field, but decays as r^{-2}. The strain field, which is the usual observable used to study such permanent near field terms, will correspondingly decay as r^{-3}.

The theory that we have developed above describes useful Green's functions for an isotropic medium. The next useful level of complexity is to consider point-source waves in a two-layer system (the top layer may be a vacuum). This famous problem in elastodynamics is commonly called *Lamb's problem* (although strictly speaking, H. Lamb only solved it for a pont impulse at a free surface). A theoretical treatment of this problem is given in chapter 6 of Aki and Richards. Practically speaking, we can usually adequately forward model or invert data involved in seismic source problems by using Green's functions that are numerically calculated, either for half space, layered or, for general structural situations, with finite element or finite difference methods.

For example, in calculating a synthetic seismogram, if one has the requisite Green's function in hand, the calculation of a synthetic seismogram in a

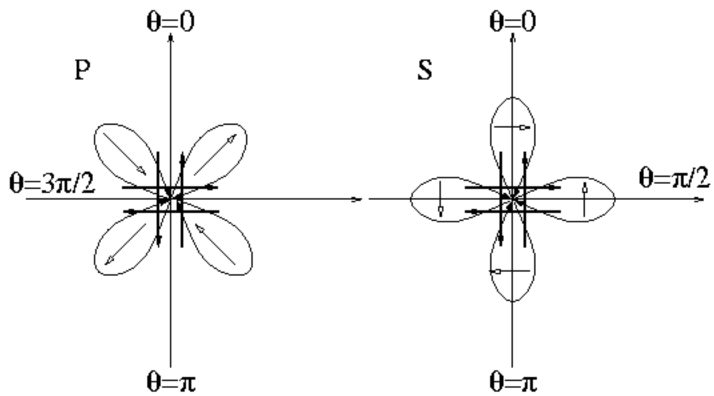

Figure 8: P and S Far Field Radiation Patterns from a Point Double Couple Source in a Poisson Solid ($\phi = 0$ plane).

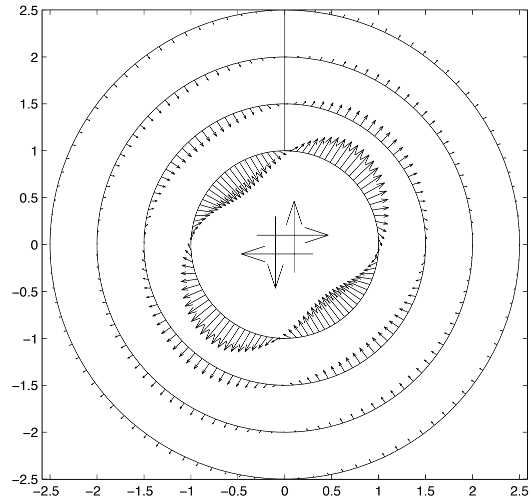

Figure 9: Near-field Static Displacement Field From a Point Double Couple Source in a Poisson Solid ($\phi = 0$ plane); $\alpha = 3^{1/2}$, $\beta = 1$, $r = 0.1,\ 0.15,\ 0.20,\ 0.25$, $\rho = 1/4\pi$, $M_\infty = 1$; Self-Scaled Displacements.

linear situation is simply found by convolving the Green's function with the appropriate moment rate function, and subsequently convolving that response with the appropriate instrument response. If the Green's function calculations produce a displacement response, then the ground velocity or acceleration responses can subsequently be calculated by successive time differentiations of the displacement result.