SEISMOLOGY I

Laurea Magistralis in Physics of the Earth and of the Environment

Damped & forced oscillators

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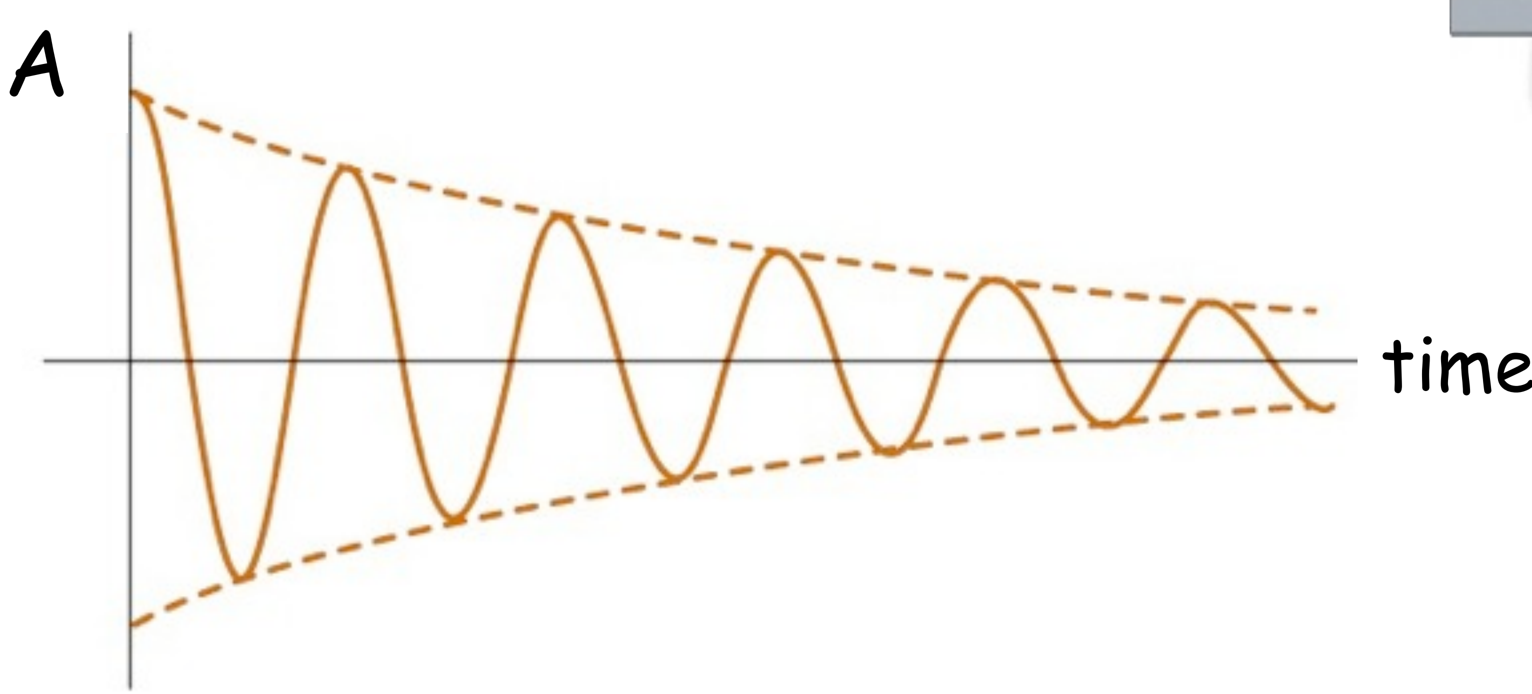
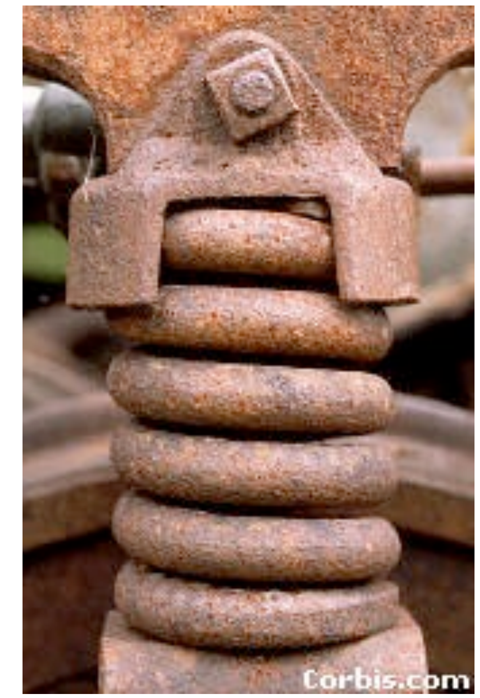
Damped Oscillations

All real oscillations are subject frictional or dissipative forces.

These forces remove energy fr the oscillating system and redu A.





Consider mass m on the end of a spring with a spring constant k

Restoring force = kx when mass is a distance x from equilibrium

drag force $\propto$ dx/dt

$$F = ma$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0$$

where $\gamma = b/m$ and $\omega^2 = k/m$

Auxiliary equation

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0$$ where $\gamma = b/m$ and $\omega_0 = (k/m)^{1/2}$

In order to find the auxiliary eq. one tries: $x(t) = e^{-\beta t}$

$$\beta^2 - \gamma\beta + \omega_o^2 = 0 \qquad \beta_{1/2} = \frac{\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}$$

1) $\gamma > 2\omega_0$ $x(t) = Ae^{-\frac{\gamma}{2}t}e^{-\frac{\sqrt{\gamma^2-4\omega_0^2}}{2}t} + Be^{-\frac{\gamma}{2}t}e^{+\frac{\sqrt{\gamma^2-4\omega_0^2}}{2}t}$

2) $\gamma = 2\omega_0$ $x(t) = Ae^{-\frac{\gamma}{2}t} + Bte^{-\frac{\gamma}{2}t}$

3) $\gamma < 2\omega_0$ $x(t) = Ae^{-\frac{\gamma}{2}t}e^{-i\frac{\sqrt{4\omega_0^2-\gamma^2}}{2}t} + Be^{-\frac{\gamma}{2}t}e^{+i\frac{\sqrt{4\omega_0^2-\gamma^2}}{2}t}$

Boundary conditions

and the constants can be determined applying the boundary conditions, e.g. $x(0)=x_0$ and $v(0)=0$.

$$1)\quad x(t) = e^{-\frac{\gamma}{2}t}\left[\left(\frac{x_0}{2} - \frac{\gamma x_0}{4\omega}\right)e^{-\omega t} + \left(\frac{x_0}{2} + \frac{\gamma x_0}{4\omega}\right)e^{+\omega t}\right]$$

overdamped

$$2)\quad x(t) = e^{-\frac{\gamma}{2}t}\left[x_0 + \frac{\gamma x_0}{2}t\right]$$

critically damped

$$3)\quad x(t) = e^{-\frac{\gamma}{2}t}\left[\left(x_0\right)\cos\omega t + \left(\frac{\gamma x_0}{2\omega}\right)\sin\omega t\right]$$

underdamped

$$\text{with}\quad \omega = \frac{\sqrt{4\omega_0^2 - \gamma^2}}{2} = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$$

Weak damping

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_o^2 x = 0 \qquad \text{where } \gamma = b/m \text{ and } \omega^2 = k/m$$

Weak damping: dissipative force is small compared to the restoring force

Oscillations continue, but gradually decrease in amplitude

Guess a solution to the differential equation above - exponential function will ensure the oscillations die at long t

first guess: $x(t) = e^{-\beta t} f(t)$

where β is a +ve constant and f(t) is to be determined

$$x = e^{-\beta t} f$$

$$\frac{dx}{dt} = -\beta e^{-\beta t} f \; + e^{-\beta t}\frac{df}{dt} \qquad = e^{-\beta t}\left(-\beta f + \frac{df}{dt}\right)$$

$$\frac{d^2x}{dt^2} = \beta^2 e^{-\beta t} f \; - \beta e^{-\beta t}\frac{df}{dt} \; - \beta e^{-\beta t}\frac{df}{dt} \; + e^{-\beta t}\frac{d^2 f}{dt^2}$$

$$= e^{-\beta t}\left(\beta^2 f - 2\beta\frac{df}{dt} + \frac{d^2 f}{dt^2}\right)$$

substitute these expressions into $\quad \dfrac{d^2x}{dt^2} + \gamma\dfrac{dx}{dt} + \omega_0^2 x = 0$

$$e^{-\beta t}\left(\beta^2 f - 2\beta\frac{df}{dt} + \frac{d^2f}{dt^2}\right) + \gamma\, e^{-\beta t}\left(-\beta f + \frac{df}{dt}\right) + \omega^2 e^{-\beta t} f = 0$$

After some tidying up we get

$$\frac{d^2f}{dt^2} + (\gamma - 2\beta)\frac{df}{dt} + (\beta^2 - \beta\gamma + \omega_o^2)f = 0$$

If $\gamma = 2\beta$ (or $\beta = \gamma/2$) we get an equation for SHM

$$\frac{d^2f}{dt^2} + (\omega_o^2 - \frac{\gamma^2}{4})\, f = 0 \qquad \frac{d^2x}{dt^2} + \omega^2 x = 0$$

ie $f = x\cos(\omega t + \delta)$ and $\omega^2 = (\omega_o^2 - \frac{\gamma^2}{4})$

when the dissipative force is small $\omega_o^2 >> \frac{\gamma^2}{4}$

and $$\omega = \left[(\omega_o^2 - \frac{\gamma^2}{4})\right]^{1/2} \approx \omega_o$$

choosing f to have its maximum value x_o at t=0

we can write $f(t) = x_o \cos \omega t$

Therefore the displacement at any time t is given by

$$x(t) = x_o\, e^{\frac{-\gamma t}{2}} \cos(\omega t)$$

Strong damping

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0$$ where $\gamma = b/m$ and $\omega^2 = k/m$

Strong damping: $\gamma > \frac{\omega_o}{20}$ oscillations rapidly cease

if $\omega_o^2 < \frac{\gamma^2}{4}$ no oscillations will occur

Our solution becomes $\frac{d^2f}{dt^2} - \alpha^2 f = 0$ with $\alpha^2 = \frac{\gamma^2}{4} - \omega_o^2$

exp(-αt) and exp(+αt) both satisfy this equation giving

$f = Ae^{-\alpha t} + Be^{+\alpha t}$ and displacement $x = e^{\frac{-\gamma t}{2}}\left(Ae^{-\alpha t} + Be^{+\alpha t}\right)$

Critical damping

$$\frac{d^2f}{dt^2} + (\omega_o^2 - \frac{\gamma^2}{4})\, f = 0$$

If $\gamma = 2\omega_o$ the mass returns to equilibrium most quickly

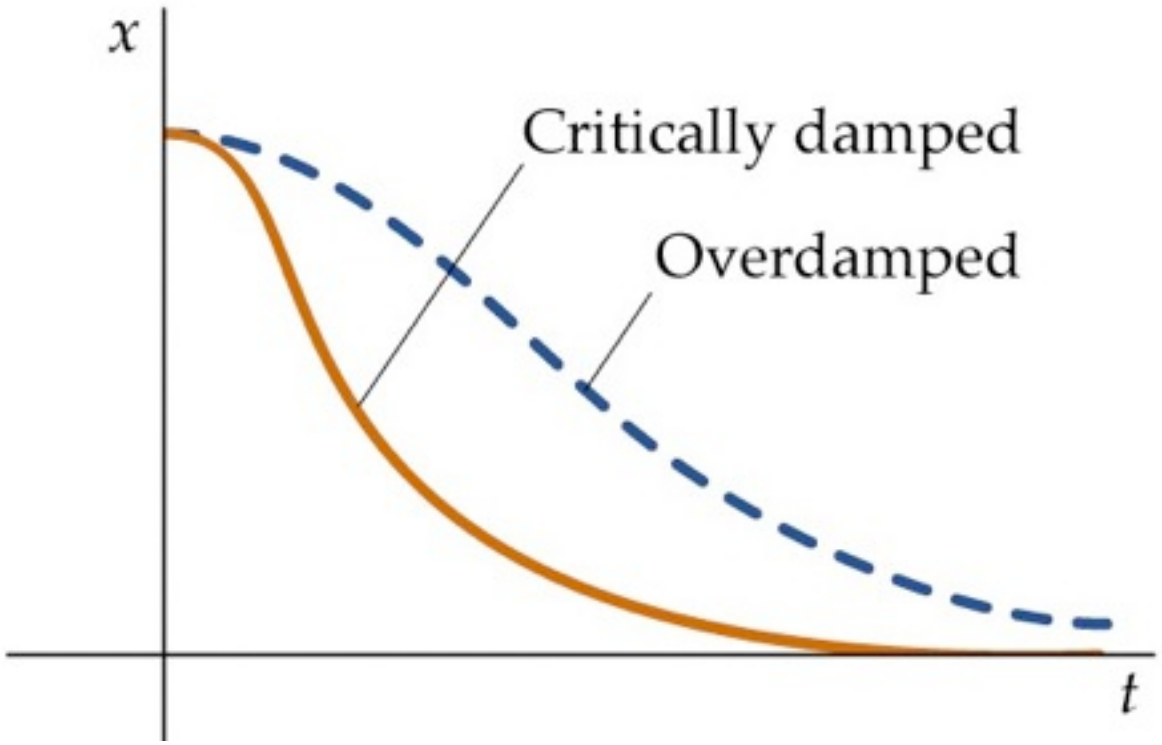
and $$\frac{d^2f}{dt^2} = 0$$

eg: shock absorbers, CD platform

$$\therefore \quad f = A + Bt$$

$df/dt=B$ $\quad d^2f/dt^2=0$

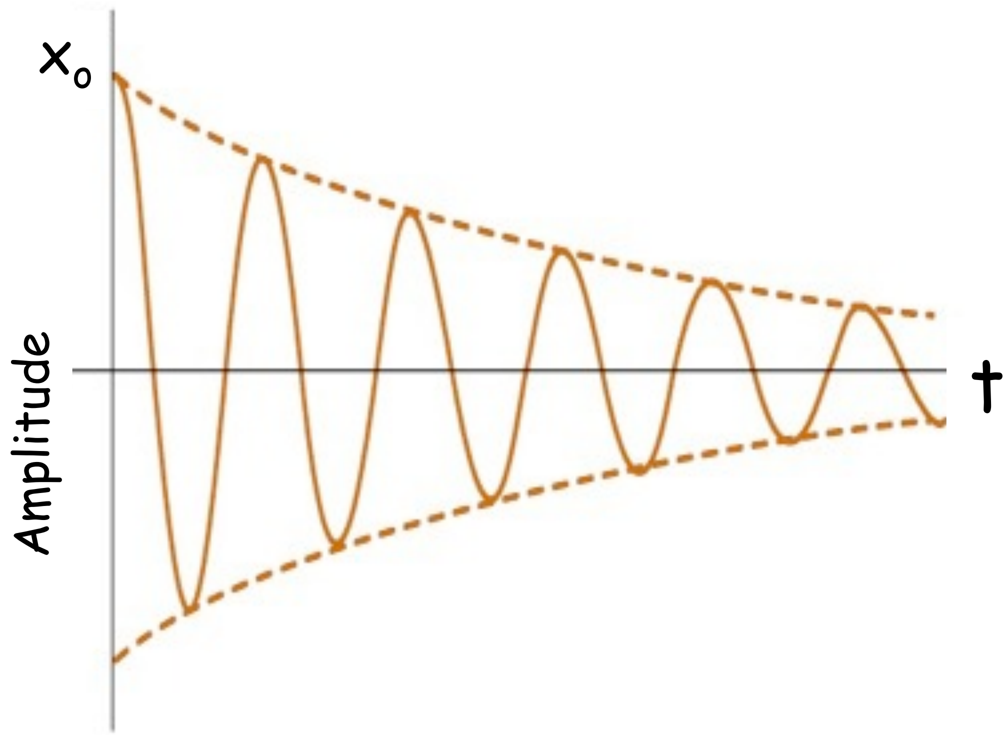
and $$x = e^{\frac{-\gamma t}{2}}(A + Bt)$$



Energy of a damped oscillator

Generally $$E = \frac{1}{2} m\omega^2 A^2$$ energy $E \propto$ amplitude A^2

if amplitude is decreasing exponentially then energy will also decrease exponentially



$$x(t) = x_o\, e^{\frac{-\gamma t}{2}} \cos(\omega t)$$

max displacement when cos=1

$$x(t) = x_o\, e^{\frac{-\gamma t}{2}}$$

$$\therefore \quad E = \frac{1}{2} m\omega^2 (x_o e^{-\frac{\gamma t}{2}})^2$$

Quality factor

A damped oscillator is often described by its quality-factor or Q-factor

$$Q = \frac{\omega_o m}{b} = \frac{\omega_o}{\gamma}$$

this can be related to the fractional energy lost per cycle

$$E = \frac{1}{2} m\omega^2 (x_o e^{-\frac{\gamma t}{2}})^2$$

$$= E_o\, e^{-\gamma t}$$

$$dE = -\gamma\, E_o\, e^{-\gamma t} dt$$

$$= -\gamma\, E\, dt$$

In a weakly damped system the energy lost / cycle is small

$$dE = \Delta E \quad \text{and} \quad dt = T$$

$$\Delta E = -\gamma E T$$

$$\frac{|\Delta E|}{E} = \gamma T$$

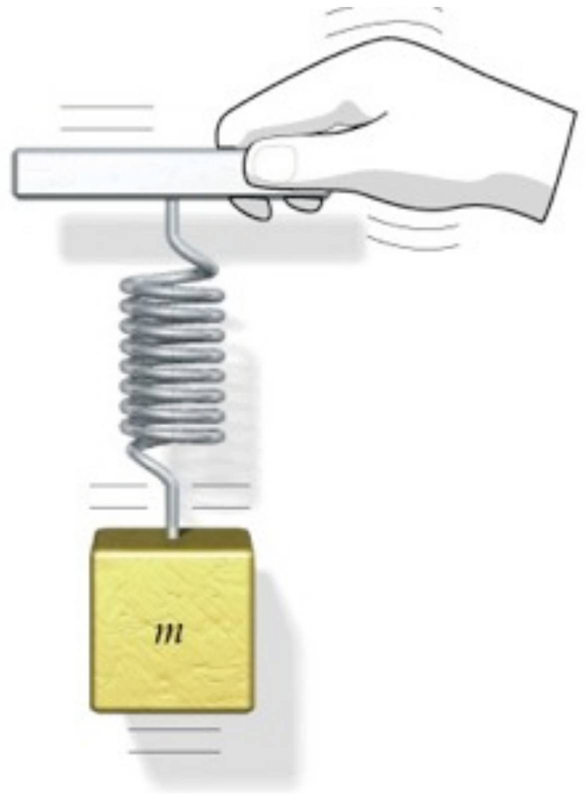
$$\frac{|\Delta E|}{E} = \frac{\gamma 2\pi}{\omega_o}$$

but $Q = \frac{\omega_o}{\gamma}$ ie $\gamma = \frac{\omega_o}{Q}$

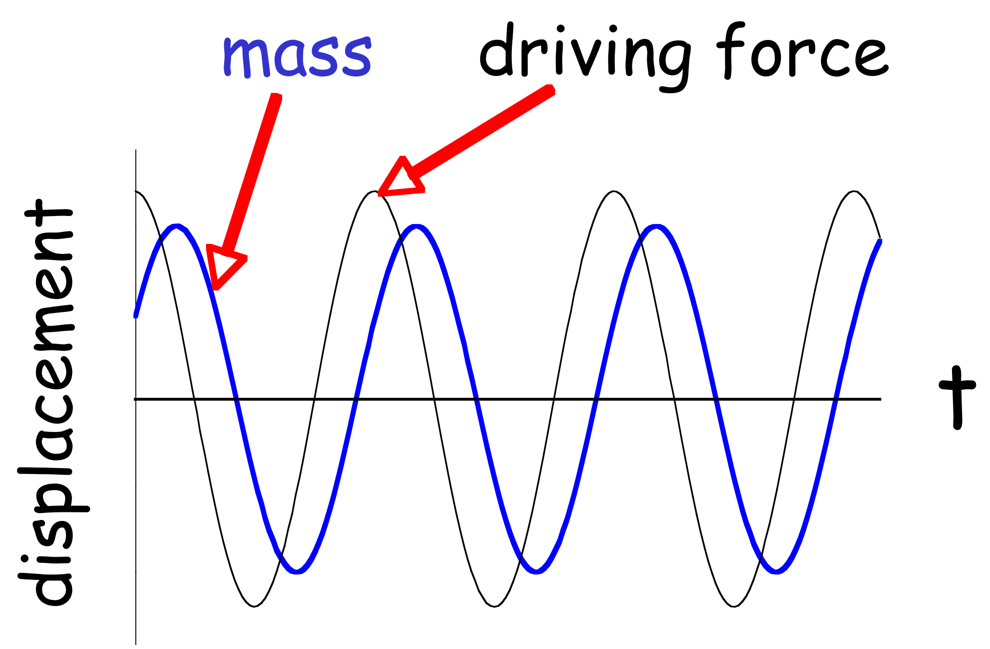
$$\boxed{\frac{|\Delta E|}{E} = \frac{2\pi}{Q}}$$

Driven oscillations

Consider the steady state behaviour of a mass oscillating on a spring under the influence of a driving force.



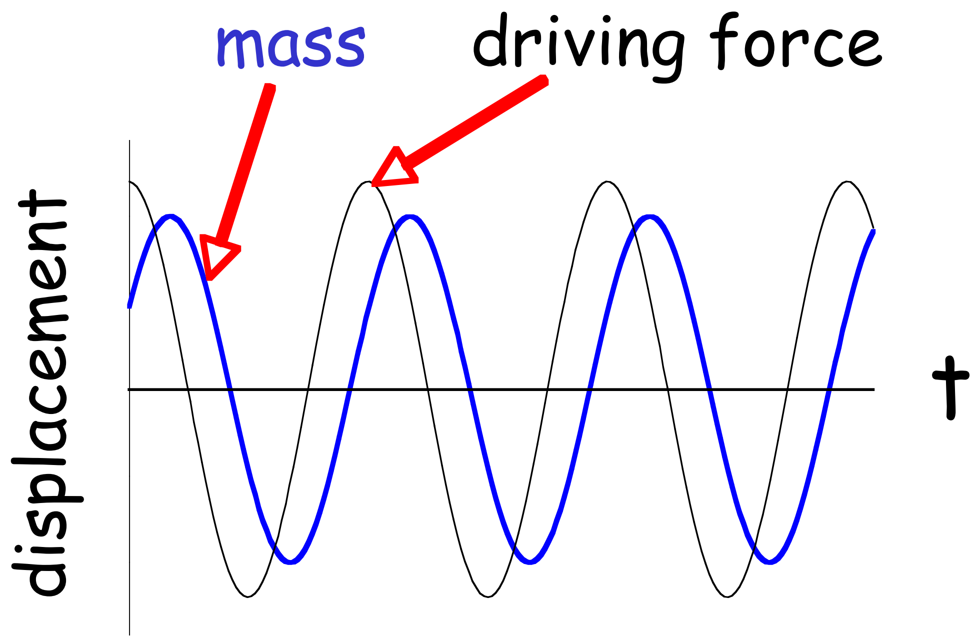
The mass oscillates at the same frequency of the driving force with a constant amplitude x_o.



The oscillations are out of phase, ie the displacement lags behind the driving force.

Force = $F_o\cos(\omega t)$ has +ve peaks at t = 0, $2\pi/\omega$, $4\pi/\omega$…………

+ve peaks of the displacement occur at t = Δt, $(2\pi/\omega)+\Delta t$, $(4\pi/\omega)+\Delta t$ …………



$$\therefore \text{the displacement } x = x_o \cos(\omega t - \phi) \quad \text{where} \quad \phi = \omega \Delta t = \frac{2\pi\Delta t}{T}$$

This describes a displacement with the same frequency as the driving force, has constant amplitude and a phase lag ϕ with respect to the driving force.

Equation of motion for a driven oscillator is

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_o^2 x = \frac{F_o}{m}\cos(\omega t) \quad \text{where } \gamma = b/m \text{ and } \omega^2 = k/m$$

Solution of this equation is $x = x_o \cos(\omega t - \phi)$

To determine the x_o and ϕ we need to substitute the solution into the equation of motion.

We need

$$\frac{dx}{dt} = -\omega x_o \sin(\omega t - \phi)$$

$$\frac{d^2x}{dt^2} = -\omega^2 x_o \cos(\omega t - \phi)$$

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_o^2 x = \frac{F_o}{m}\cos(\omega t)$$

$$-\omega^2 x_o \cos(\omega t - \phi) - \gamma\omega x_o \sin(\omega t - \phi) + \omega_o^2 x_o \cos(\omega t - \phi) = \frac{F_o}{m}\cos(\omega t)$$

$$(\omega_o^2 - \omega^2) x_o \cos(\omega t - \phi) - \gamma\omega x_o \sin(\omega t - \phi) = \frac{F_o}{m}\cos(\omega t)$$

This equation must be true at all times.

To solve for x_o and ϕ we need to consider two situations.

1. $(\omega t - \phi) = 0$ $\therefore \sin(\omega t - \phi) = 0$ and $\cos(\omega t) = \cos\phi$

2. $(\omega t - \phi) = \pi/2$ $\therefore \cos(\omega t - \phi) = 0$ and $\cos(\omega t) = \cos(\pi/2 + \phi)$

This leaves us with two simultaneous equations:

$$(\omega_o^2 - \omega^2)x_o = \frac{F_o}{m}\cos(\phi)$$

$$-\gamma\omega x_o = \frac{F_o}{m}\cos(\frac{\pi}{2} + \phi)$$

Remember $\cos(\frac{\pi}{2} + \phi) = -\sin\phi$ and $\cos^2 A + \sin^2 A = 1$

The solutions are

$$x_o = \frac{F_o / m}{\sqrt{(\omega_o^2 - \omega^2)^2 + \omega^2\gamma^2}} \qquad \tan\phi = \frac{\omega\gamma}{(\omega_o^2 - \omega^2)}$$

Resonance

The amplitude and energy of a system in the steady state depends on the amplitude and the frequency of the driver.

With no driving force the system will oscillate at its **natural frequency** ω_o

If the driving frequency $\sim \omega_o$ the energy absorbed by the oscillator is maximum and large amplitude oscillations occur

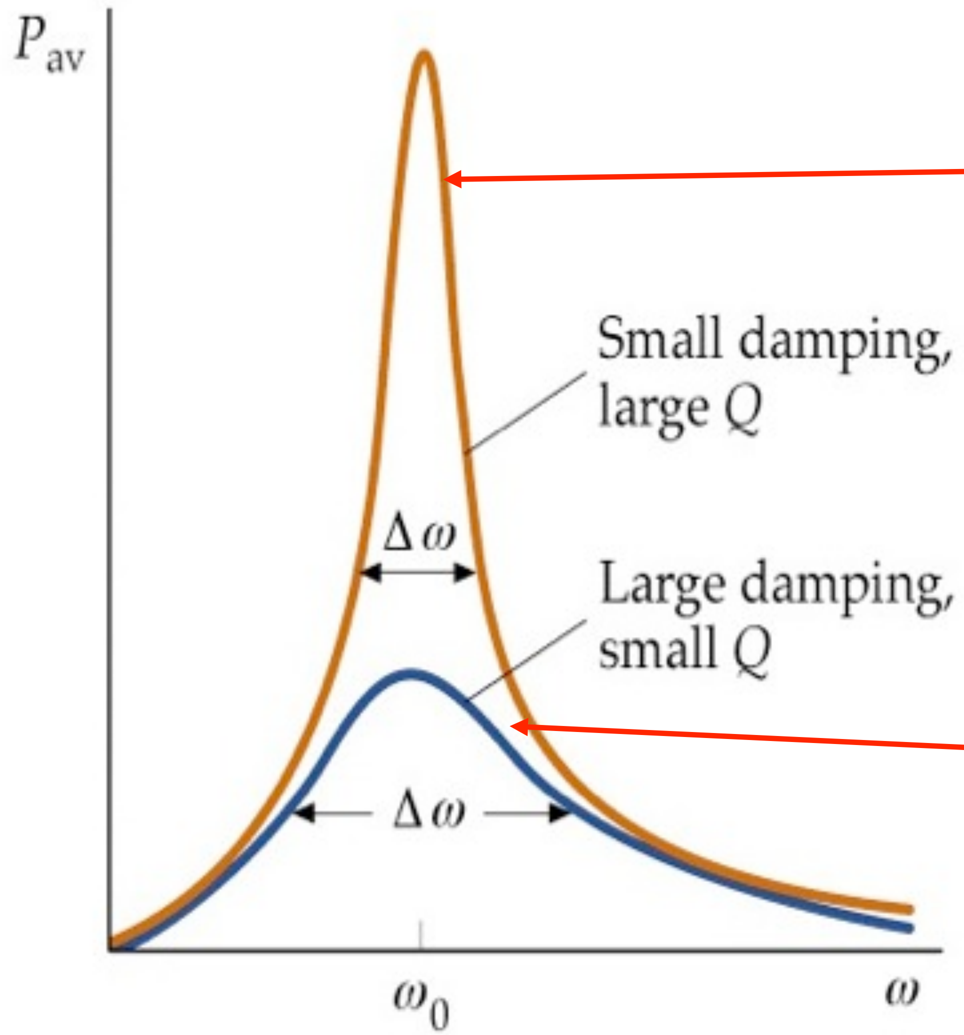
This is known as **resonance** and the natural frequency of the system is therefore called the **resonance frequency**

Resonance occurs in many systems - washing machines, breaking a glass with sound, child on a swing...........





The average rate at which power is absorbed equals the average power delivered by the driving force.



When damping is small oscillator absorbs much more energy from driving force.

Resonance peak is narrow

When damping is large oscillator resonance curve is broad

For small damping $\frac{\Delta\omega}{\omega_o} = \frac{\Delta f}{f_o} = \frac{1}{Q}$