Spring, 2001 Data Processing and Analysis (GEOP 505)

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Introduction to Linear Systems, Part 2: The Frequency Domain

Now that we have examined the processing of signals in linear systems using time as the essential variable, we come to the fundamentals of Fourier theory. The basic insight is that linear systems, being subject to superposition and scaling, can be analyzed in terms of their *frequency response*, that is, in terms of their effect on purely sinusoidal or exponential inputs, where the essential variable is frequency rather than time.

Consider the response, $g(t)$, of a linear system, $\phi(t)$, to a unit-amplitude, complex input of frequency f, $e^{\imath 2\pi f t}$. The convolution expression for the time response of any such system says that the response as a function of time will be

$$g(t) = \int_{-\infty}^{\infty} \phi(\tau) e^{\imath 2\pi f(t-\tau)}\, d\tau \ . \tag{1}$$

However, because a time shift in the argument of an exponential is equivalent to a multiplication by another exponential, we can write this as

$$g(t) = e^{\imath 2\pi f t} \int_{-\infty}^{\infty} \phi(\tau) e^{-\imath 2\pi f \tau}\, d\tau \ \equiv \ e^{\imath 2\pi f t} \cdot \Phi(f) \ . \tag{2}$$

Thus, the response of a linear system to a complex sinusoidal input is just a complex sinusoidal signal of the *same frequency*, modified in amplitude and phase by a complex factor $\Phi(f)$. As the frequency is arbitrary in (2), if an arbitrary input $\psi(t)$ can be decomposed into a sum of sinusoidal components, then, because of superposition, the relationship between $\psi(t)$ and $g(t) = \psi(t) * \phi(t)$ can be characterized by $\Phi(f)$, which is called the *transfer function* of the system, and is the *Fourier transform* or *spectrum* of the impulse response of the system, $\phi(t)$.

The Fourier transform conventions which we will use are those most commonly encountered in geophysics

$$\Phi(f) = F[\phi(t)] = \int_{-\infty}^{\infty} \phi(t) e^{-\imath 2\pi f t} dt \tag{3}$$

$$\phi(t) = F^{-1}[\Phi(f)] = \int_{-\infty}^{\infty} \Phi(f) e^{\imath 2\pi f t} df \tag{4}$$

where F denotes the Fourier transform operation, and F^{-1} denotes the *inverse Fourier transform* operation. Be aware that in some other areas of physics and in exploration geophysics the sign convention on the complex exponentials of (3) and (4) is reversed, so that the forward transform has a plus sign in the exponent and the inverse transform has a minus sign in the exponent. This will of course not affect any fundamentals of the analysis, only the phase convention.

Differential equations and Fourier theory. A particularly tractable and not uncommon situation in the physical sciences occurs when a system relating two time functions, $x(t)$ and $y(t)$, is characterizable by a linear differential equation with constant coefficients. For time functions of a single variable, the general form is

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x \; . \tag{5}$$

As none of the coefficients (the a_i and b_i) depend on t, (5) describes a time-invariant system. Because all of the terms are linear (there are no powers or other nonlinear functions of x, y, or their derivatives), it is also a linear system, which obeys superposition and scaling (because differentiation itself is a linear operation). To obtain an expression for the transfer function for (5), we substitute an exponential unit amplitude exponential of arbitrary frequency for the input, $x(t)$, and output, $y(t)$, so that

$$x(t) = e^{\imath 2\pi f t} \tag{6}$$

and, as must be the case for any linear, time-invariant system (2),

$$y(t) = \Phi(f) e^{\imath 2\pi f t} \; . \tag{7}$$

Substituting (6) and (7) into (5), dividing both sides by $e^{\imath 2\pi f t}$, and solving for $\Phi(f)$ gives the system transfer function as a ratio of two complex polynomials in f.

$$\Phi(f) = \frac{\sum_{j=0}^{m} b_j (2\pi \imath f)^j}{\sum_{k=0}^{n} a_k (2\pi \imath f)^k} \tag{8}$$

The values of f (or equivalently, of the *angular frequency*, $\omega = 2\pi f$) where the numerator is zero are referred to as *zeros* of $\Phi(f)$, as the response is zero at this frequency, regardless of the amplitude of the input signal. Frequencies for which the denominator is zero are called *poles*, as the response becomes very large at these frequencies. Note that we don't have to worry about any mysteries regarding $e^{\imath 2\pi f t}$ being a complex number, as

$$e^{\imath 2\pi f t} = \cos(\omega t) + \imath \sin(\omega t) \tag{9}$$

and we could almost have just as easily chosen to propagate the real or the imaginary part of the input signal alone through the system to reach an equivalent

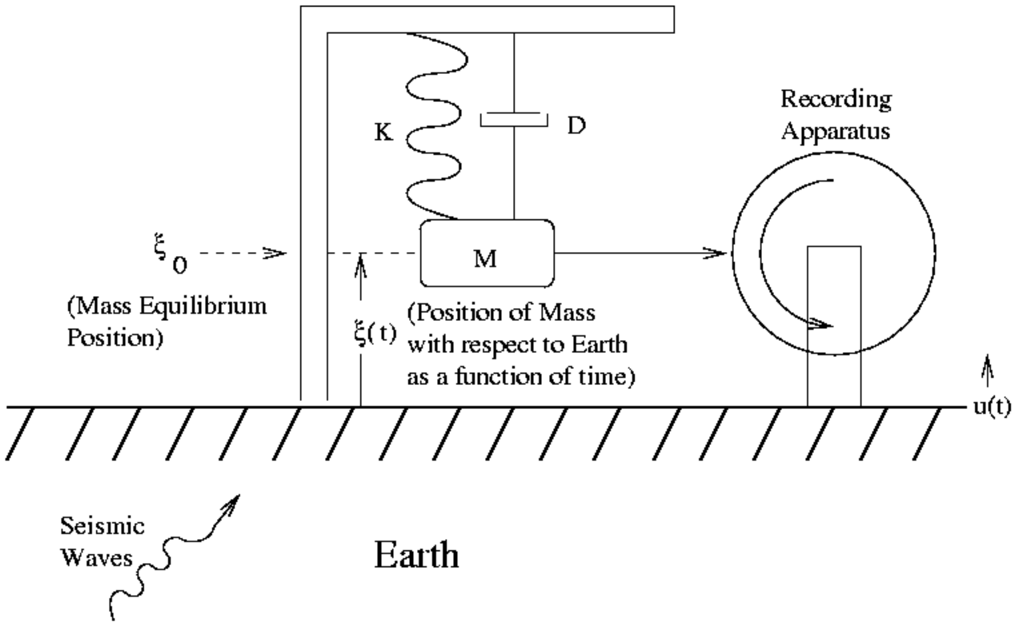

Figure 1: A Mechanical Seismometer

conclusion; in this case an input (cosine, sine) signal simply produces a scaled output (cosine, sin) with a phase shift. Note that the frequencies for which we have zero or infinite response may be imaginary or complex, in which case the corresponding input function may be an increasing or decreasing exponential, or an increasing or decreasing exponentially damped sinusoid, respectively.

Example: Response of a seismometer. As an important example of such a linear system from geophysical instrumentation, consider (Figure 1) a damped vertical harmonic oscillator with a case that is fixed to the Earth, where a mass is supported by a spring in parallel with a damping or *dashpot* component that produces Newtonian damping (i.e., a retarding force that is proportional to velocity). Intuitively, it is obvious that the motion of the mass relative to the Earth will provide some sort of representation of the true vertical ground motion. For example, if the mass somehow were completely decoupled so that it remained stationary in its inertial reference frame while the Earth moved, then the motion of the mass relative to its case (which is rigidly attached to the Earth) would be exactly the negative of the ground motion).

The differential equation of motion for such a *seismometer* can be obtained by equating the (upward) forces due to the spring and damper with the (upward) acceleration of the bob times its mass

$$F_{up} = Ma_{up} \tag{10}$$

or

$$-D\frac{d\xi(t)}{dt} + K[\xi_0 - \xi(t)] \ = M\frac{d^2}{dt^2}[\xi(t) + u(t)] \tag{11}$$

which gives rise to a homogeneous differential equation:

$$M\frac{d^2}{dt^2}[\xi(t) + u(t)] + D\frac{d\xi(t)}{dt} + K[\xi(t) - \xi_0] = 0 \ . \tag{12}$$

Here, u is the motion of the Earth (up positive), ξ is the position of the mass, which has an equilibrium position in the Earth's gravity field of ξ_0 (both measured up positive relative to the surface of the Earth), M is the mass of the inertial component, D is the dashpot constant (units of force per velocity), and K is the spring constant (units of force per distance).

We can simplify (12) somewhat by writing the equation of motion for the mass in an upward positive coordinate system (z) where $z = 0$ is the equilibrium position in the Earth's gravitational field, so that $z(t) = \xi(t) - \xi_0$. This gives

$$\ddot{z} + 2\zeta\dot{z} + \omega_s^2 z = -\ddot{u} \tag{13}$$

where the *damping coefficient* is

$$2\zeta \ \equiv \ D/M \tag{14}$$

and

$$\omega_s \ \equiv \ (K/M)^{1/2} \tag{15}$$

is the undamped or *natural* period of the system. (13) is a linear homogeneous equation describing the seismometer, where the input is the displacement of the earth, u, and the output is the deviation of the mass from its equilibrium position, z.

Using (8), we can now write the transfer function of the seismometer system (for a displacement response to a displacement input) in terms of angular frequency

$$\Phi(\omega) = \frac{z(\omega)}{u(\omega)} = \frac{-(\imath\omega)^2}{(\imath\omega)^2 + 2\zeta(\imath\omega) + \omega_s^2} = \frac{-\omega^2}{\omega^2 - 2\imath\zeta\omega - \omega_s^2} \tag{16}$$

or, in terms of the amplitude and phase

$$|\Phi(\omega)| = \frac{\omega^2}{[(\omega^2 - \omega_s^2)^2 + 4\zeta^2\omega^2]^{1/2}} \tag{17}$$

$$\theta = \arg[\Phi(\omega)] = \pi - \tan^{-1}\frac{-2\zeta\omega}{\omega^2 - \omega_s^2} \ . \tag{18}$$

At very high frequencies ($\omega \gg \omega_s$), $|\Phi(\omega)| \approx 1$, and $\theta \approx \pi$, so the seismometer displacement from equilibrium is the negative of the Earth displacement, $z \approx -u$. In this case, the Earth moves so rapidly that the mass cannot follow the motion at all, and the position of the mass relative to the frame is indeed just $-u$, as described earlier.

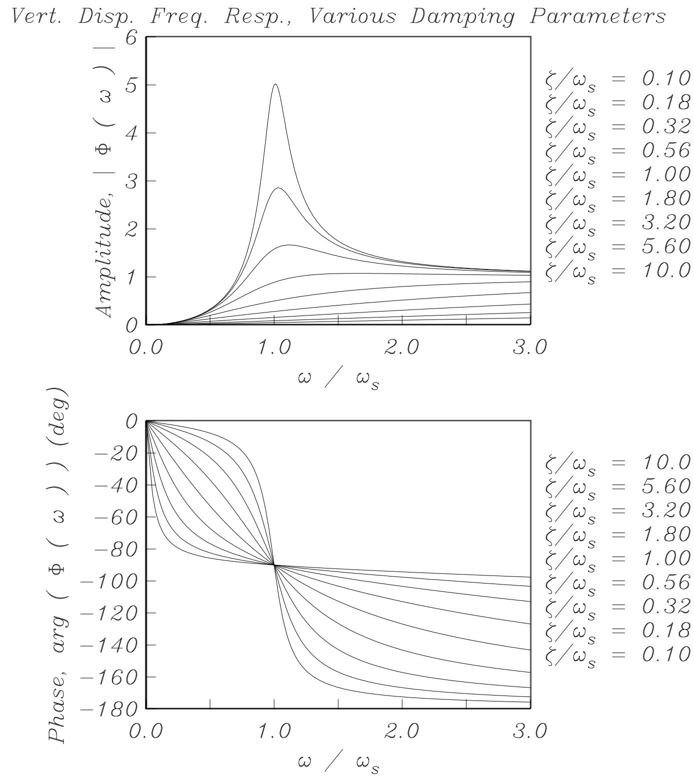

Figure 2: Frequency Response of the Mechanical Seismometer

At very low frequencies ($\omega \ll \omega_s$) we have $|\phi(\omega)| \approx \omega^2/\omega_s^2$, so that the amplitude of the response falls off quadratically with frequency. From the time domain representation (13), we see that this response is proportional to the negative of the Earth's acceleration, $z \propto -\ddot{u}$. The seismometer, if coupled to a recording displacement sensor, thus acts like a displacement sensor at short periods and as an accelerometer at long periods.

The frequency response for displacement input and displacement output [(17) and (18)] is plotted in Figure 2 for various damping factors, where the complex response is plotted in terms of its amplitude and phase.

In examining Figure 2, first consider the amplitude response. When the damping ζ is small, the system exhibits a large amplitude response for frequencies near ω_s. This occurs because the system is being excited near its natural resonant period and has little energy loss through the dashpot. When ζ becomes larger than ω_s, the resonance peak in the amplitude response disappears, and the system will no longer oscillate freely. At very long periods, the amplitude response of the system goes to zero, and at very short periods it goes to -1, as we intuited earlier.

Next consider the phase response. At the undamped resonance period, the phase is always $-90°$, implying that the output is phase-shifted by that amount (or by -$\pi/2$ radians) relative to the input. A cosine Earth motion of frequency ω_s would be phase shifted into a sine mass displacement. Regardless of damping, the phase shift approaches zero at long periods and approaches π at short periods.

Purely mechanical seismometers such as that described above were the first such instruments used to record accurate ground motion from earthquakes or other sources (they were first widely deployed starting in the 1890's). In many modern seismometers the mass motion is sensed as a voltage which is proportional to the velocity of the mass using an inductive coil and a magnetized mass. If the mass motion is small, the voltage induced in the coil is just proportional to the change in magnetic flux times the number of coils. In this situation the output of the system is a voltage that is proportional to the relative velocity, $\dot{z}$, of the mass relative to its frame or case. The voltage output is thus the time derivative of the displacement response. The system response of a differentiator, which is characterized by the differential equation

$$y(t) = \dot{x}(t) \ , \tag{19}$$

can be trivially seen by (8) to be just $i\omega$, so that the transfer function of an inductive seismometer system as voltage out versus Earth displacement is

$$\Phi_{induction}(\omega) = \frac{\dot{z}(\omega)}{u(\omega)} = \frac{-\imath\omega^3}{\omega^2 - 2\imath\zeta\omega - \omega_s^2} \tag{20}$$

Note that if we consider the Earth velocity, $\dot{u}$ instead of the Earth displacement, u as the input signal the response of the inductive seismometer is

$$\Phi_{induction}(\omega) = \frac{\dot{z}(\omega)}{\dot{u}(\omega)} = \frac{-\omega^2}{\omega^2 - 2\imath\zeta\omega - \omega_s^2} \tag{21}$$

which is identical to (16), and the same discussion of the response as above applies, except that the response is output volts relative to Earth velocity rather than output displacement relative to ground displacement. For this reason, most seismometers are referred to as *velocimiters*.

Note that the inverse Fourier transform of $\Phi(\omega)$ will give the time domain impulse response of the system (this arises because, as we shall shortly show, the Fourier transform of a delta function is 1), however we will approach the inverse Fourier transform from a different tack, as the straightforward integral for the displacement seismometer response doesn't converge (the response function of Figure 2 has infinite area under it).

In general, like the autocorrelation, the Fourier transform or its inverse will not exist if the integral definition does not converge. The following conditions are sufficient for existence:

1. $\phi(t)$ has only a finite number of maxima and minima in any finite time interval. This eliminates very wiggly functions (e.g., $sin(1/x)$).

2. $\phi(t)$ has only a finite number of finite discontinuities in any finite time interval.

3. $\phi(t)$ is has finite "energy", so that

$$\int_{-\infty}^{\infty} |\phi(t)|^2 \; dt \tag{22}$$

is bounded.

Clearly, (22) is not satisfied for the displacement transfer function in the seismometer system, and we consequently run into convergence problems if we try to directly evaluate the inverse Fourier transform of $\Phi(\omega)$ in (16). We can, however obtain the displacement response to an impulsive Earth acceleration (where $\ddot{u} = \delta(t)$) by solving

$$\ddot{a} + 2\zeta\dot{a} + \omega_s^2 a = -\delta(t) \tag{23}$$

which is shown in Figure 3 (we'll do the detailed calculation later). The inverse Fourier transform of $a(\omega)$ converges, so this will provide a tractable route to obtain an analytic expression for the displacement impulse response of the seismometer system.

Parseval's theorem. As time domain signals can be expressed as an infinite summation of complex exponentials, (this is what the inverse Fourier transform (4) says). We might therefore expect that there is a simple relationship between signal energy expressed in the time and frequency domains. Consider the total energy of a complex or real time domain signal, $\phi(t)$

$$E = \int_{-\infty}^{\infty} \phi(t)\phi^*(t)dt \tag{24}$$

where the asterisk denotes complex conjugation (which has no effect if $\phi(t)$ is real. This can be written as

$$E = \int_{-\infty}^{\infty} \phi(t) \left(\int_{-\infty}^{\infty} \Phi^*(f) e^{-\imath 2\pi f t} df \right) dt \tag{25}$$

using (4). Interchanging the order of integration, we get

$$E = \int_{-\infty}^{\infty} \Phi^*(f) \left(\int_{-\infty}^{\infty} \phi(t) e^{-\imath 2\pi f t} dt \right) df \tag{26}$$

which gives

$$E = \int_{-\infty}^{\infty} \Phi^*(f)\Phi(f) \; df = \int_{-\infty}^{\infty} \phi(t)\phi^*(t) \; dt \tag{27}$$

Equation (27) is variously referred to as *Parseval's*, *Rayleigh's* or *Plancherel's* theorem. It says that one can evaluate the total energy in a signal as either an integral of its amplitude squared time domain representation over all time, or as

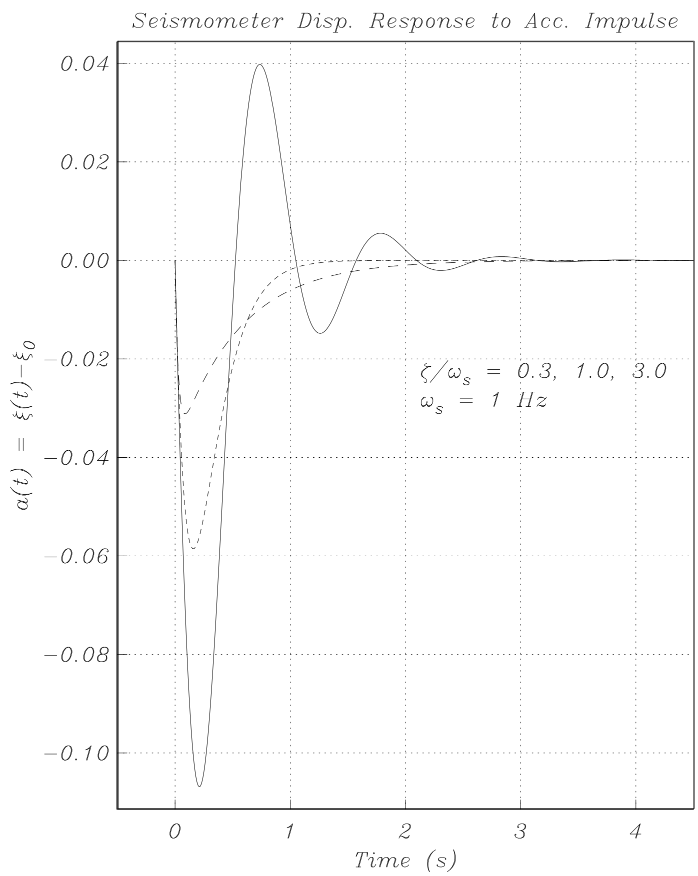

Figure 3: Response of the Mechanical Seismometer to an Acceleration Impulse

an integral across all of its amplitude squared frequency components across all frequencies. In a more general sense, Parseval's theorem says that the Fourier transform is *length preserving*, i.e., the "size" of the function (in the size-sense of the integral of the amplitude squared) is the same in the time and frequency domains.

Properties of the Fourier transform. Consider the Fourier transforms of some of our canonical functions and general symmetries and other properties. An important function in time series analysis is the boxcar function, $\Pi(t)$, which has the Fourier transform (Figure 4)

$$F[\Pi(t)] = \int_{-\infty}^{\infty} \Pi(t) e^{-\imath 2\pi f t} dt \tag{28}$$

$$= \int_{-1/2}^{1/2} e^{-\imath 2\pi f t} dt = \int_{-1/2}^{1/2} \cos(2\pi f t) dt \tag{29}$$

$$= \frac{\sin(\pi f)}{\pi f} \equiv \operatorname{sinc}(f) \ . \tag{30}$$

The corresponding inverse transform is

$$F^{-1}[\operatorname{sinc}(f)] = \int_{-\infty}^{\infty} \operatorname{sinc}(f) e^{\imath 2\pi f t} df = \Pi(t) \ . \tag{31}$$

Taking the complex conjugate and interchanging f and t, we get the Fourier transform of $\operatorname{sinc}(t)$

$$\Pi(f) = \int_{-\infty}^{\infty} \operatorname{sinc}(t) e^{-\imath 2\pi f t} dt \ . \tag{32}$$

Note that (30) and (31) show, remarkably, that we can get discontinuous functions by integrating smooth functions.

The Fourier transform of a delta function is

$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-\imath 2\pi f t} dt = 1 \ . \tag{33}$$

So a delta function contains an equal proportion of *all* frequencies with no relative phase shifts. Correspondingly

$$F^{-1}(1) = \int_{-\infty}^{\infty} e^{\imath 2\pi f t} df = \delta(t) \ . \tag{34}$$

One can get better a grasp on (34) by imagining the oscillating terms in the integral all averaging out to zero, except at $t = 0$, where they all reinforce each other, i.e.,

$$F^{-1}(1) = \lim_{\epsilon \to 0} \int_{-\infty}^{-\epsilon} e^{\imath 2\pi f t} df + \int_{\epsilon}^{\infty} e^{\imath 2\pi f t} \ df = 2 \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \cos(2\pi f t) \ df \tag{35}$$

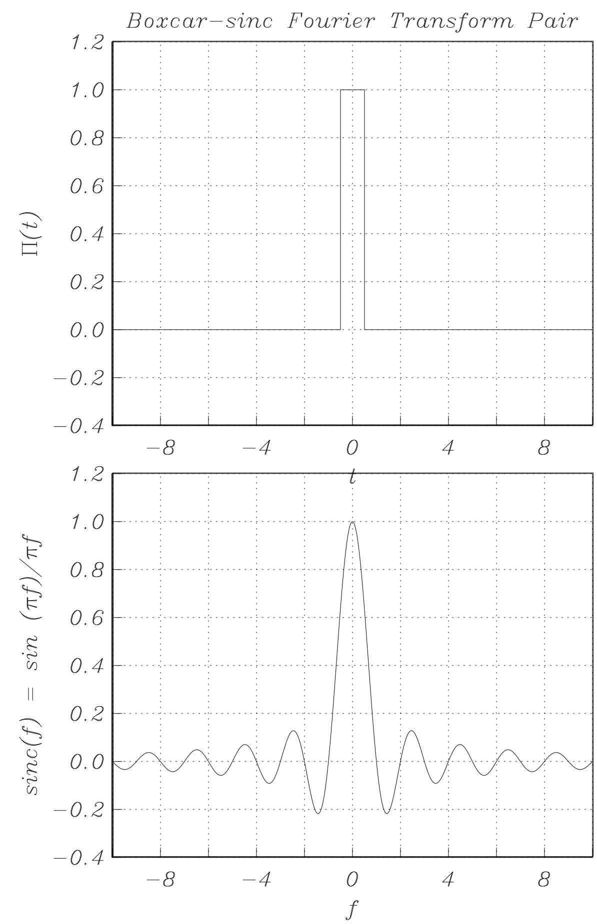

Figure 4: The Boxcar-Sinc Fourier Transform Pair

A fundamental property of the Fourier transform is its *shifting property*, which says that a simple time shift of a function only changes the phase of its Fourier transform. Consider

$$F[\phi(t-t_0)] = \int_{-\infty}^{\infty} \phi(t-t_0) e^{-\imath 2\pi f t} dt \ . \tag{36}$$

Substituting $\tau = t - t_0$, we get

$$= \int_{-\infty}^{\infty} \phi(\tau) e^{-\imath 2\pi f (\tau + t_0)} d\tau = e^{-\imath 2\pi f t_0} \int_{-\infty}^{\infty} \phi(\tau) e^{-\imath 2\pi f \tau} d\tau \tag{37}$$

$$= e^{-\imath 2\pi f t_0} \Phi(f) \tag{38}$$

so that time shifts in the time domain correspond to linear (with respect to frequency) phase shifts in the frequency domain.

Another basic relationship is *time-frequency scaling* or *similarity*, consider

$$F[\phi(\alpha t)] = \int_{-\infty}^{\infty} \phi(\alpha t) e^{-\imath 2\pi f t} dt \ . \tag{39}$$

For $\alpha > 0$, this gives

$$= \frac{1}{\alpha} \int_{-\infty}^{\infty} \phi(\tau) e^{-\imath 2\pi f \tau / \alpha} d\tau = \frac{1}{\alpha} \Phi\left(\frac{f}{\alpha}\right) \ , \tag{40}$$

using the substitution $\tau = \alpha t$. For $\alpha < 0$, the limits on the definite integral are reversed upon the change of variable, so we get

$$F[\phi(\alpha t)] = -\frac{1}{\alpha} \Phi\left(\frac{f}{\alpha}\right) \tag{41}$$

so that, in general

$$F[\phi(\alpha t)] = \frac{1}{|\alpha|} \Phi\left(\frac{f}{\alpha}\right) \ . \tag{42}$$

Thus, when we "squeeze" a function in the time domain, its Fourier transform "spreads out" in the frequency domain (and vice-versa). An extreme end member showing this behavior is the delta function, which is an infinitely squeezed function in the time domain with an infinitely spread out transform (the 1 function; (33)) in the frequency domain.

As you have probably already suspected, there is a *duality* between the two domains, the precise relationship is

$$F[\phi(t)] = \Phi(f) \tag{43}$$

$$F[\Phi(t)] = \phi(-f) \ , \tag{44}$$

where the proof is left as an exercise.

Because any signal can be decomposed into even and odd parts

$$\phi(t) = \phi_e(t) + \phi_o(t) \tag{45}$$

$$= \frac{1}{2}[\phi(t) + \phi(-t)] + \frac{1}{2}[\phi(t) - \phi(-t)] \tag{46}$$

the Fourier transform exhibits important symmetry relations. Consider the transform of a general real, even function, ϕ_e.

$$F[\phi_e(t)] = \int_{-\infty}^{\infty} \phi_e(t) e^{-\imath 2\pi f t} dt \tag{47}$$

$$= \int_{-\infty}^{\infty} \phi_e(t) \cos(2\pi f t)\ dt - \imath \int_{-\infty}^{\infty} \phi_e(t) \sin(2\pi f t)\ dt \tag{48}$$

$$= 2 \int_{0}^{\infty} \phi_e(t) \cos(2\pi f t) dt \tag{49}$$

which is even and is purely real. Similarly, for an odd, real function, ϕ_o, the Fourier transform is

$$F[\phi_o(t)] = \int_{-\infty}^{\infty} \phi_o(t) e^{-\imath 2\pi f t} dt \tag{50}$$

$$= \int_{-\infty}^{\infty} \phi_o(t) \cos(2\pi f t) dt - \imath \int_{-\infty}^{\infty} \phi_o(t) \sin(2\pi f t)\ dt \tag{51}$$

$$= -2\imath \int_{0}^{\infty} \phi_o(t) \sin(2\pi f t)\ dt \tag{52}$$

which is odd and purely imaginary. Thus, the Fourier transform of an arbitrary function may be evaluated as a superposition of (49) and (52), frequently referred to as the *cosine transform* and *sine transform*, respectively. Using superposition, one can derive a list of basic symmetry relationships between the time and frequency domains

$\phi(t)$	$\Phi(f)$
even	even
odd	odd
real, even	real, even
real, odd	imaginary, odd
imaginary, even	imaginary, even
imaginary, odd	real, odd
complex, even	complex, even
complex, odd	complex, odd
real, asymmetrical	complex, Hermitian
imaginary, asymmetrical	complex, anti-Hermitian
Hermitian	real
anti-Hermitian	imaginary
even	even

where a *Hermitian* function has an even real part and an odd imaginary part, so that $\Phi(f) = \Phi^*(-f)$. Correspondingly, an *anti-Hermitian* function has an odd real part and an even imaginary part.

We are now ready to demonstrate one of the most important relationships between the time and frequency domains, the *convolution theorem*.

Consider the Fourier transform of the convolution of two functions

$$F[\phi_1(t) * \phi_2(t)] = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \phi_1(\tau)\phi_2(t-\tau)d\tau \right) e^{-\imath 2\pi f t} dt \ . \qquad (53)$$

Reversing the order of integration gives

$$F[\phi_1(t) * \phi_2(t)] = \int_{-\infty}^{\infty} \phi_1(\tau) \left(\int_{-\infty}^{\infty} \phi_2(t-\tau) e^{-\imath 2\pi f t} dt \right) d\tau \ . \qquad (54)$$

However, by the time shift property (38), this is just

$$\int_{-\infty}^{\infty} \phi_1(\tau)\Phi_2(f) e^{-\imath 2\pi f \tau} d\tau = \Phi_1(f)\Phi_2(f) \qquad (55)$$

so that convolution in the time domain corresponds to multiplication in the frequency domain! Similarly, we can show that multiplication in the time domain corresponds to convolution in the frequency domain

$$F[\phi_1(t)\phi_2(t)] = \Phi_1(f) * \Phi_2(f) \ . \qquad (56)$$

This should make sense, as the response of a linear system at each frequency is just the complex amplitude of that frequency component in the input, times the complex value of the response function of the system at that frequency.

As already mentioned, time differentiation has a remarkably simple form in the frequency domain

$$\frac{d}{dt}\phi(t) = \frac{d}{dt}\int_{-\infty}^{\infty} \Phi(f) e^{\imath 2\pi f t} df \qquad (57)$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}[\Phi(f) e^{\imath 2\pi f t}] df = \int_{-\infty}^{\infty} 2\pi \imath f \Phi(f) e^{\imath 2\pi f t} df = F^{-1}[2\pi \imath f \Phi(f)] \qquad (58)$$

taking the Fourier transform of both sides gives:

$$= F[\frac{d}{dt}\phi(t)] = 2\pi \imath f \Phi(f) \ . \qquad (59)$$

Note that differentiation, as we'd expect, reinforces high frequency signal components relative to those at low frequency, and thus belongs to a class of operators generally referred to as *high-pass filters*.

The situation for integration is somewhat more complex

$$F\left(\int_{-\infty}^{t} \phi(\tau) d\tau \right) = \frac{\Phi(f)}{2\pi \imath f} + \frac{\delta(f)}{2} \int_{-\infty}^{\infty} \phi(t) dt \qquad (60)$$

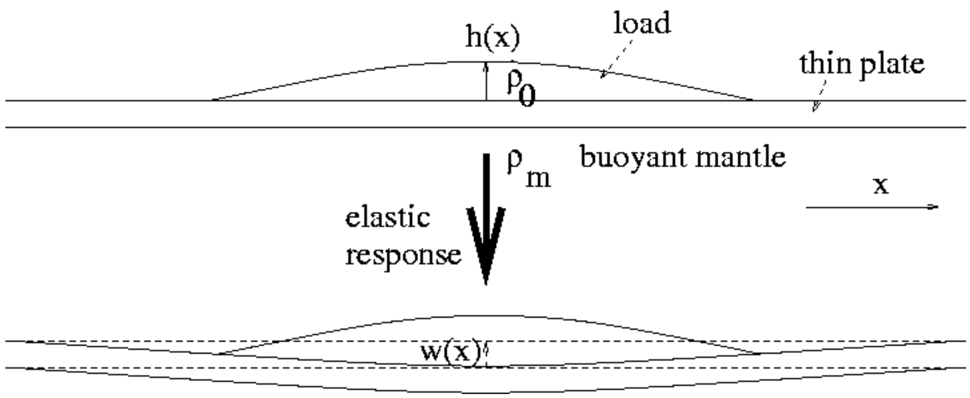

Figure 5: A Buoyant, Rigid Plate with a Spatial Load

where the delta function term accommodates the contribution of any non-zero mean value in $\phi(t)$. A definite integrator is thus a *low-pass filter*, as it reinforces low frequencies relative to high frequencies.

(59) and (60) are helpful in computing Fourier transforms, especially for discontinuous functions. Consider the step function. Using (60) gives

$$F[\mathrm{H}(t)] = F\left(\int_{-\infty}^{t} \delta(\tau)d\tau\right) \quad (61)$$

$$= \frac{1}{2\pi\imath f} + \frac{\delta(f)}{2} \ . \quad (62)$$

The Fourier transform of the sign function is thus

$$F[2\mathrm{H}(t) - 1] = \frac{1}{\pi\imath f} \ . \quad (63)$$

Example: Equilibrium elastic response of a loaded, buoyantly supported crust. The differentiation and integration properties of the Fourier transform provide a useful method for obtaining solutions to ordinary linear integrodifferential equations. An example of geophysical interest is the downward deflection of a rigid plate (such as the Earth's crust) buoyantly supported by an underlying liquid (to first order, the mantle) to a distributed load (such as an ice cap, volcano, or reservoir) (Figure 5.

The equation for the equilibrium of a deformed plate is (e.g., Banks et al., *Geophys J.*, B51, 431-452, 1977; Turcotte and Shubert, *Geodynamics*, 1982)

$$D\bigtriangledown^4 w(r) = p(r) \quad (64)$$

where $w(r)$ is the upward deflection of the plate and $p(r)$ is the upward force per unit area. The forcing term, $p(r)$, arises from a topographic load, $h_l(r)$ and from a buoyancy term due to the displaced mantle. D is the *flextural rigidity*, which depends on the thickness and elastic moduli of the plate

$$D \equiv \frac{E\tau^3}{12(1-\nu^2)} \tag{65}$$

where τ is the plate thickness, E is Young's Modulus, and ν is Poisson's Ratio.

In one spatial dimension, x, (64) becomes

$$D\frac{\partial^4 w(x)}{\partial x^4} = p(x) \ . \tag{66}$$

The total forcing function for a load of homogeneous density is

$$p(x) = -\rho_l g h_l(x) + B(x) \tag{67}$$

where ρ_l is the density of the added material, g is the acceleration of gravity, and $B(x)$ is the buoyancy term due to mantle material of density ρ_m .

$$B(x) = -w(x)\rho_m g \ . \tag{68}$$

As the net observed topography is just the topography of the load plus the topography of the deflected crust

$$h(x) = h_l(x) + w(x) \ , \tag{69}$$

we can write the forcing term in terms of $h(x)$ as

$$p(x) = -g(\rho_l h(x) + \Delta\rho w(x)) \tag{70}$$

where

$$\Delta\rho = \rho_m - \rho_c \tag{71}$$

and ρ_c is the density of the rigid crust. Now we can solve for the deformation by separating w and h and taking a spatial Fourier transform

$$[(2\pi\imath k)^4 D + g\Delta\rho]W(k) = -\rho_l g H(k) \tag{72}$$

where k is the *spatial frequency* (units of 1/length), the spatial counterpart of f. Note that $H(k)$ is the spatial Fourier transform of the input

$$H(k) = \int_{-\infty}^{\infty} h(x)e^{(-\imath 2\pi kx)}\ dx \tag{73}$$

(*not* the step function). The (spatial) frequency domain solution is thus

$$W(k) = -H(k)\frac{\left(\frac{\rho_l}{g\Delta\rho}\right)}{\left(1+\frac{16\pi^4 k^4 D}{g\Delta\rho}\right)} \ . \tag{74}$$

Note that (74) depends strongly on the wavenumber k. For k large, the response of the system becomes becomes negligible. Conversely, for k small, the response becomes increasingly significant, up to a maximum value of

$$W_{max} = -\rho_l/(g\delta\rho) \ . \tag{75}$$

Thus, for long-wavelength (small k) spatial components of the landscape, we say that we have a large degree of buoyant *compensation*, as the topographic load is primarily supported by mantle buoyancy. At short spatial wavelengths, on the other hand (large k), the landscape is almost totally supported by the flextural rigidity of the crust. The degree of compensation for a spatial component of wavelength $\lambda = 1/k$, is the deflection of the system relative to W_{max}

$$C = \frac{W(k)_{max}}{W_{max}} \ . \tag{76}$$

We can evaluate the impulse response in the x domain by taking the inverse Fourier transform of $W(k)/H(k)$ (preferably with the assistance of a table of integral transforms), to obtain

$$q(x) = F^{-1}[W(k)/H(k)] = F^{-1}\left(\frac{\rho_l}{D}\left(\frac{g\Delta\rho}{D} + (2\pi k)^4\right)^{-1}\right) = \frac{-2\rho_l}{D}\int_0^\infty \frac{\cos(2\pi kx)\ dk}{\alpha^4 + (2\pi k)^4} \tag{77}$$

where

$$\alpha = \left(\frac{g\Delta\rho}{D}\right)^{1/4} \tag{78}$$

so that (e.g., Erdelyn *et al.*, *Tables of Integral Transforms*, Volume 1, 1954):

$$q(x) = \frac{-\sqrt{2}\rho_l}{4\alpha^3 D}\ e^{\frac{-\alpha|x|}{\sqrt{2}}}\ \left(\sin\frac{\alpha|x|}{\sqrt{2}} + \cos\frac{\alpha|x|}{\sqrt{2}}\right) \ . \tag{79}$$

This function is plotted in Figure 6, which shows a central depression and an outboard peripheral bulge or upwarp. Note that (79) is the impulse response of this system, as $W(k)$ is the response for $H(k) = 1$ (74), so that *any* 1-d deformation of a rigid plate to a load (assumed to be infinitely extending in the out-of plane direction) can thus be calculated by convolving $q(x)$ and the specific linear mass distribution.

Time domain seismometer response. We can obtain a result for the displacement response of the vertical seismometer in the time domain in a similar manner by noting that the time domain response to an impulsive acceleration is characterized by $\ddot{u} = \delta(t)$, so that

$$\ddot{a} + 2\zeta\dot{a} + \omega_s^2 = -\delta(t) \ . \tag{80}$$

Taking the Fourier transform of both sides and solving for $a(\omega)$, the displacement response to an acceleration impulse input, gives the frequency domain expression

$$a(\omega) = \frac{1}{\omega^2 - 2\imath\zeta\omega - \omega_s^2} \tag{81}$$

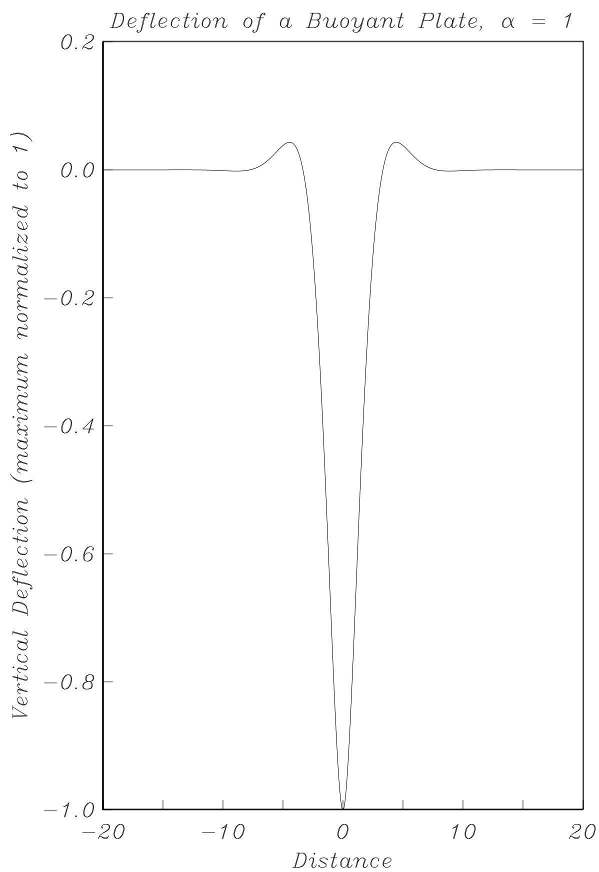

Figure 6: Response of a Buoyant, Rigid Plate to an Spatial Impulse Load

Note that this is just the response of the seismometer system to the displacement impulse (16), divided by $-\omega^2$; appropriate, as the input function has been twice differentiated in the time domain.

The time domain displacement response to an acceleration impulse input is therefore

$$\phi(t) = F^{-1}(a(\omega)) = F^{-1}\left(\frac{1}{\omega^2 - 2\imath\zeta\omega - \omega_s^2}\right) \tag{82}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\imath\omega t}\, d\omega}{\omega^2 - 2\imath\zeta\omega - \omega_s^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\imath\omega t}\, d\omega}{(\omega - \omega_1 - \imath\zeta)(\omega + \omega_1 - \imath\zeta)} \tag{83}$$

where

$$\omega_1 = \sqrt{\omega_s^2 - \zeta^2} \tag{84}$$

Solving this integral is relatively straightforward using the residue theorem, and separation into three cases. For $\omega_s > \zeta$, the system exhibits a distinct resonance near $\omega = \omega_s$ (as we have already seen from examining the frequency response) and is referred to as *underdamped*. In this case, the poles of the integrand in (83) occur at $(\omega_1, \imath\zeta)$ and $(-\omega_1, \imath\zeta)$. The time domain solution is found from the residues of the two complex poles of the integrand to be

$$a(t) = \frac{-\mathrm{H}(t)}{\omega_1} e^{-\zeta t} \sin(\omega_1 t) \ . \tag{85}$$

When $\omega_s < \zeta$, the system does not exhibit resonant behavior (the complex poles of the integrand are now on the negative real axis), and is referred to as being *overdamped*. ω_1^2 is negative in this case, and the integrand produces an impulse function that is a sum of real exponentials

$$a(t) = \frac{-\mathrm{H}(t)}{2(\zeta^2 - \omega_s^2)^{1/2}} \left(e^{-(\zeta - (\zeta^2 - \omega_s^2)^{1/2})t} - e^{-(\zeta + (\zeta^2 + \omega_s^2)^{1/2})t}\right) \tag{86}$$

The case $\omega_s = \zeta$ is referred to as *critically damped*. Because there is a double pole, a special case of the residue theorem must be applied to obtain the impulse response, which is

$$a(t) = -\mathrm{H}(t) t e^{-\zeta t} \ . \tag{87}$$

How do we evaluate the displacement impulse response of the system to Earth displacement? Remember that $a(t)$ is the time domain solution for the system response to an Earth acceleration of $a^0(t) = \delta(t)$. Because the seismometer system and differentiation are linear, we can evaluate the seismometer displacement response from a displacement impulse by twice differentiating, $a(t)$ with respect to time. For the underdamped system, for example, this gives (Figure 7)

$$d(t) = \frac{d^2 a(t)}{dt^2} = \frac{d^2}{dt^2}\left(\frac{-\mathrm{H}(t)}{\omega_1} e^{-\zeta t} \sin(\omega_1 t)\right) \tag{88}$$

$$= -2\delta(t) - \mathrm{H}(t)\frac{e^{-\zeta t}}{\omega_1}\left((\zeta^2 - \omega_1^2)\sin(\omega_1 t) \ - \ 2\zeta\omega_1 \cos(\omega_1 t)\right) \tag{89}$$

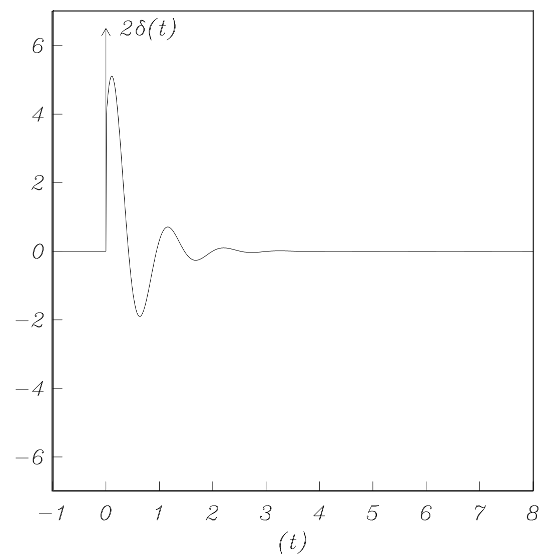

Figure 7: Displacement Response of an Underdamped Seismometer ($\zeta = 0.3\omega_s$; $\omega_0 = 2\pi$ Hz) to a Displacement Impulse

Note that as the resonant frequency, ω_1 becomes small (the resonant period becomes large), (7 approaches the perfect instrument response of a delta function. Because seismolgists frequently want to know the true ground displacement (its long-period asymptotic spectral level is proportional to the seismic moment, among other reasons), seismometers with very long natural periods are desirable and constitute the instrumental backbone of much of modern seismology.

Moment-spectral relationships. As an additional example showing the rich mathematics of Fourier theory, consider another remarkable feature of the Fourier transform is that all of the moments of the time domain function, $\phi(t)$, can be expressed in terms of the behavior of $\Phi(f)$ at the origin. Consider the n^{th} moment

$$\phi_n(t) \equiv \int_{-\infty}^{\infty} t^n \phi(t) dt \ . \tag{90}$$

The n^{th} derivative of $\Phi(f)$ with respect to f is

$$\frac{\partial^n \Phi(f)}{\partial f^n} = \int_{-\infty}^{\infty} (-2\pi\imath t)^n \phi(t) e^{-\imath 2\pi f t} dt \tag{91}$$

so that

$$\frac{1}{(-2\pi\imath)^n} \left(\frac{\partial^n \Phi(f)}{\partial f^n} \right) = \int_{-\infty}^{\infty} t^n \phi(t) e^{-\imath 2\pi f t} dt \ . \tag{92}$$

Evaluating both sides gives

$$\frac{1}{(-2\pi\imath)^n} \left(\frac{\partial^n \Phi(0)}{\partial f^n} \right) = \int_{-\infty}^{\infty} t^n \phi(t) \ dt = \phi_n(t) \ . \tag{93}$$

Thus, we can now see that the 0^{th} moment of $\phi(t)$, the total area under $\phi(t)$, is just $\Phi(0)$. Similarly, the 1^{st} moment of $\phi(t)$ is just

$$\int_{-\infty}^{\infty} t\phi(t) dt = \frac{1}{-2\pi\imath} \left(\Phi'(f) \right)_{f=0} \tag{94}$$

where

$$\Phi'(f) \equiv \frac{\partial \Phi(f)}{\partial f} \tag{95}$$

so that the slope of $\Phi(f)$ at the origin is proportional to the expectation value of t

$$< t >_{\phi(t)} = \frac{\int_{-\infty}^{\infty} t\phi(t) \ dt}{\int_{-\infty}^{\infty} \phi(t) \ dt} \ . \tag{96}$$

Time functions which which are symmetrical must therefore have Fourier transforms with zero slope at $f = 0$ (we can also see this from the aforementioned symmetry relations).

The 2^{nd} moment is

$$\int_{-\infty}^{\infty} t^2 \phi(t) dt = -\frac{1}{4\pi^2} \left(\Phi''(f) \right)_{f=0} \tag{97}$$

so that the curvature of $\Phi(f)$ at the origin is proportional to the second moment of $\phi(t)$. For functions which have an infinite second moment, the Fourier transform has a cusp at the origin, for example,

$$F\left(\frac{1}{\alpha^2+t^2}\right) = \frac{e^{-\alpha|f|}}{2\alpha} \ . \tag{98}$$

Consider the variance of $\phi(t)$

$$\sigma^2[\phi(t)] = <(t-<t>)^2>_{\phi(t)} = \frac{\int_{-\infty}^{\infty}(t^2-2t<t>+<t>^2)\phi(t)\ dt}{\int_{-\infty}^{\infty}\phi(t)\ dt} \tag{99}$$

$$= \frac{1}{\Phi(0)}\left(\frac{\Phi''(0)}{(-2\pi\imath)^2} - 2\frac{\Phi'(0)}{-2\pi\imath}\cdot\frac{\Phi'(0)}{-2\pi\imath\Phi(0)} + \frac{[\Phi'(0)]^2}{(-2\pi\imath)^2}\cdot\frac{\Phi(0)}{\Phi(0)^2}\right) \tag{100}$$

$$= \frac{1}{4\pi^2\Phi(0)}\left(-\Phi''(0) + \frac{[\Phi'(0)]^2}{\Phi(0)}\right) \ . \tag{101}$$

What is the variance, then, of $\phi_1(t)*\phi_2(t)$? Using the convolution theorem (55) makes this straightforward, as

$$\sigma^2[\phi_1(t)*\phi_2(t)] = \frac{1}{4\pi^2\Phi_1(0)\Phi_2(0)}\left(-(\Phi_1\Phi_2)''(0) + \frac{[(\Phi_1\Phi_2)'(0)]^2}{\Phi_1(0)\Phi_2(0)}\right) \tag{102}$$

$$= \frac{1}{4\pi^2}\left[-\frac{\Phi_1''(0)}{\Phi_1(0)} - \frac{\Phi_2''(0)}{\Phi_2(0)} + \left(\frac{\Phi_1'(0)}{\Phi_1(0)}\right)^2 + \left(\frac{\Phi_2'(0)}{\Phi_2(0)}\right)^2\right] = \sigma^2[\phi_1(t)] + \sigma^2[\phi_2(t)] \tag{103}$$

which gives the important result that the variance of a convolution result is just the sum of the variances of the two constituent functions.

Causal systems and the Hilbert transform. An important relationship exists between the real and imaginary parts of a real causal function, $\phi_c(t)$, that is, a real function that is zero for all $t<0$. To see this, we first decompose $\phi_c(t)$ into its even and odd parts

$$\phi_c(t) = \phi_e(t) + \phi_o(t) = 1/2(\phi_c(t)+\phi_c(-t)) + 1/2(\phi_c(t)-\phi_c(-t)). \tag{104}$$

For the causal function, we can express $\phi_o(t)$ in terms of $\phi_e(t)$, as:

$$\phi_o(t) = \phi_e(t) \quad (t>0) \tag{105}$$

and

$$\phi_o(t) = -\phi_e(t) \quad (t<0) \tag{106}$$

Thus

$$\phi_c(t) = [1+\mathrm{sgn}(t)]\phi_e(t) \ . \tag{107}$$

By superposition and the frequency domain convolution theorem (56),

$$\Phi_c(f) = \Phi_e(f) + F[\mathrm{sgn}(t)] * \Phi_e(f) \tag{108}$$

and using the Fourier transform of the sign function (63), we finally obtain the Fourier transform of $\phi_c(t)$ explicitly in terms of the Fourier transform of $\phi_e(t)$

$$\Phi_c(f) = \Phi_e(f) \cdot \frac{-\imath}{\pi f} * \Phi_e(f) \ . \tag{109}$$

Note that because $\phi_e(t)$ is real and even, so is $\Phi_e(f)$. Thus, the real and imaginary parts of $\Phi_c(f)$ are related to each other by the real convolution operator $(-1/\pi f)$. This relationship can be summarized by

$$\Im[\Phi_c(f)] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Re[\Phi_c(\xi)]}{\xi - f} \ d\xi = \Re[\Phi_c(f)] * \frac{-1}{\pi f} \ \equiv \ \mathrm{H}[\Re[\Phi_c(f)]] \ . \tag{110}$$

and conversely,

$$\Re[\Phi_c(f)] = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Im[\Phi_c(\xi)]}{\xi - f} \ d\xi = \Im[\Phi_c(f)] * \frac{1}{\pi f} \ \equiv \ \mathrm{H}^{-1}[\Im[\Phi_c(f)]] \ . \tag{111}$$

We can confirm (111) by showing that:

$$-\frac{1}{\pi f} * \frac{1}{\pi f} = \delta(f) \tag{112}$$

which is left as an exercise.

(110) is the *Hilbert transform* and (111) is the *inverse Hilbert transform* operator, acting on $\Re[\Phi_c(f)]$ and $\Im[\Phi_c(f)]$, respectively. This relationship puts constraints on the frequency response of all physical (causal) transfer functions. If we take the Hilbert transform of a *time* function, we get the associated *quadrature* function.

$$\mathrm{H}[\phi(t)] = \hat{\phi}(t) \ . \tag{113}$$

The Fourier transform of the quadrature function has the same amplitude information as the original function, but its phase is multiplied by $\imath$ sgn(f), so that it is phase shifted by $-\pi/2$ for negative frequencies and by $\pi/2$ for positive frequencies.

An *analytic signal* is one in which the real and imaginary parts are related by the Hilbert transform

$$A(t) = \phi(t) - \imath\hat{\phi}(t) \ . \tag{114}$$

The modulus of an analytic time series is useful in evaluating the instantaneous envelope of a function.

A example of a causal physical system is the attenuation which occurs when a wave propagates through a lossy medium. In seismology, such media (which of course include all real materials) are referred to as *anelastic*. The loss mechanisms need not concern us in detail here, but they include work done at grain boundaries and other irreversible changes in the material. The observational result of attenuation is that the energy arriving at the receiver is less than that which one would expect from considering the effects of geometrical spreading and other ray path effects alone.

For the idealized case of a one-dimensional plane wave propagating through a lossless medium (e.g., an electromagnetic wave propagating through a perfect vacuum, or a seismic wave propagating through a perfectly elastic medium) the signal, β, at position x and time t is simply the signal at the source delayed by the propagation time x/v

$$a(x,t) = a(t - x/v) \tag{115}$$

where v is the phase velocity. If the time function at the source is $a(t)$, then we can express the signal at an arbitrary time and place as

$$a(x,t) = a_0(t) * \delta(t - t_0) \tag{116}$$

where $t_0 = x/v$ and $a_0(t)$ is the signal at $x = 0$. We are assuming here that all frequency components propagate at a single velocity, v. Such a medium is referred to as *nondispersive*. The transfer function of a lossless, nondispersive system is therefore just that of a time delay

$$a(x,f) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-\imath 2\pi f t} \, df = e^{-\imath 2\pi f t_0} = e^{-\imath 2\pi f x/v} \; . \tag{117}$$

The quality factor, Q, of an oscillating system is given by

$$\frac{1}{Q(f)} = \frac{\delta E}{2\pi E} \tag{118}$$

where E is the peak energy of the system and δE is the energy lost in each cycle, assuming $Q \gg 1$. For a propagating sinusoidal disturbance, then, the loss relationship as a function of x is

$$\delta E = \frac{dE}{dx} \lambda \tag{119}$$

as the field goes through one oscillation in a wavelength, $\lambda = v/f$. Combining (119) and (118), we have

$$\frac{2\pi E}{Q} = \frac{dE}{dx} \lambda \tag{120}$$

which has a solution for propagating energy of

$$E(x,\ f) = E_0(t) e^{-2\pi f x/Qv} \tag{121}$$

or for propagating amplitude of

$$b(x,\ f) = b_0(t) e^{-\pi f x/Qv} \; . \tag{122}$$

The combined transfer function for the system is thus, by the convolution theorem (55)

$$c(x,\ f) = F\left(\frac{1}{a_0(t)} a(x,t) \; * \; \frac{1}{b_0(t)} b(x,t)\right) \tag{123}$$

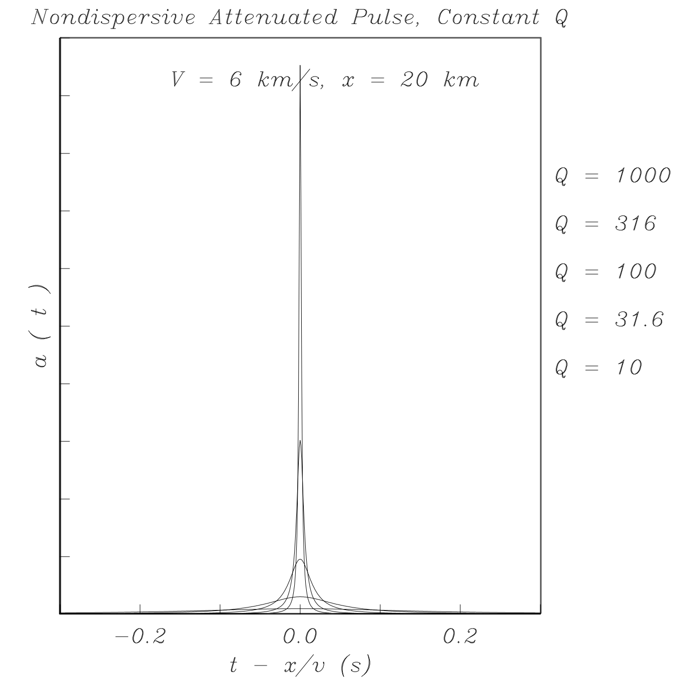

Figure 8: Attenuated Pulses, Constant Q

$$\frac{1}{a_0}a(x,\ f)\ \cdot\ \frac{1}{b_0}b(x,\ f) = e^{-\pi f x/Qv}\ \cdot\ e^{-\imath 2\pi f x/v} \tag{124}$$

Taking the inverse Fourier transform of $c(x,\ f)$ to obtain the impulse response of the system, we have (taking the absolute value of f so that negative and positive frequencies are treated equally)

$$c(x,t) = \int_{-\infty}^{\infty} e^{2\pi(-|f|t_0/2Q+\imath f(t-t_0))}\ df \tag{125}$$

$$= \int_{0}^{\infty} e^{2\pi(-ft_0/2Q+\imath f(t-t_0))}\ df + \int_{-\infty}^{0} e^{2\pi(ft_0/2Q+\imath f(t-t_0))}\ df \tag{126}$$

$$= -\frac{1}{2\pi}\left[\frac{1}{(\imath t-(\imath+1/2Q)t_0} - \frac{1}{(\imath t-(\imath-1/2Q)t_0}\right] \tag{127}$$

$$= \frac{1}{\pi}\left(\frac{(t_0/2Q)}{(t-t_0)^2+(t_0/2Q)^2}\right) \tag{128}$$

which is plotted in Figure 8

(128) is a symmetrical pulse with a maximum at $t = t_0$. Note, however, that $c(x,\ t)$ is not zero for $t < t_0$. This solution is therefore noncausal and cannot

correspond to the behavior of the real world. Reexamining our assumptions, we find that we must reassess both the nondispersiveness of the medium and the constancy of Q for all frequencies. A moment's reflection reveals that we cannot get an asymmetrical, causal pulse by simply allowing Q to vary as an even function of frequency, as the Q operator will affect positive and negative frequencies equally and hence will not alter the symmetry of the pulse. Thus, we are led to the conclusion that all real media must be dispersive!

The general transfer function for a wave propagating towards positive x is thus a generalization of (124)

$$c(x,\ f) = e^{-\pi|f|x/Q(f)v(f)} \ \cdot \ e^{-\imath 2\pi f x/v(f)} \tag{129}$$

where v and Q are now functions of f. We can write this as

$$c(x,\ f) = e^{-2\pi\imath K x} \tag{130}$$

if we define the complex wavenumber, K as

$$K = \frac{-\imath|f|}{2Q(f)v(f)} + \frac{f}{v(f)} \ \equiv \ \frac{f}{v(f)} + \imath\alpha(f) \tag{131}$$

where $\alpha(f)$ is the attenuation factor. The impulse response is thus the inverse Fourier transform of this

$$c(x,\ t) = \int_{-\infty}^{\infty} e^{\imath 2\pi(-Kx+ft)}\ df \tag{132}$$

It can be shown (e.g., Aki and Richards, v. I) that constraining $c(x,\ t)$ to be causal, i.e., equal to zero for $t < t_1 = x/v_\infty$ places the following constraint on the dispersive velocity function

$$\frac{f}{v(f)} = \frac{f}{v_\infty} + \mathrm{H}[\alpha(f)] \tag{133}$$

where v_∞ is the phase velocity at infinite frequency and H is the Hilbert transform. Finding solutions to (133) is non-trivial, and there is no solution for constant Q. If we take Q to be constant over the seismic frequency range, however, we can arrive at the useful solution proposed by Azimi *et al.* (*Izvestiya, Physics of the Solid Earth*, pp. 88-93, 1968), where the phase velocity is approximately given by

$$\frac{1}{v(f)} = \frac{1}{v_\infty} + \frac{2\alpha_0}{\pi} \ln\left(\frac{1}{2\pi f \alpha_1}\right) \tag{134}$$

where α_0 and α_1 are constants. Using

$$\alpha_0 \approx (2v_\infty Q)^{-1}. \tag{135}$$

and

$$\alpha_1 = 0.01 \text{ s} \tag{136}$$

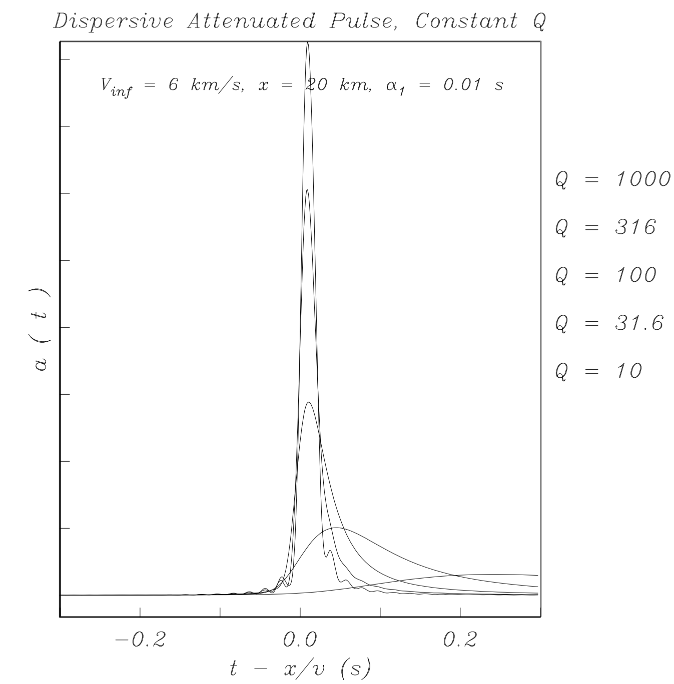

Figure 9: Attenuated Pulses, Quasi-Causal Q

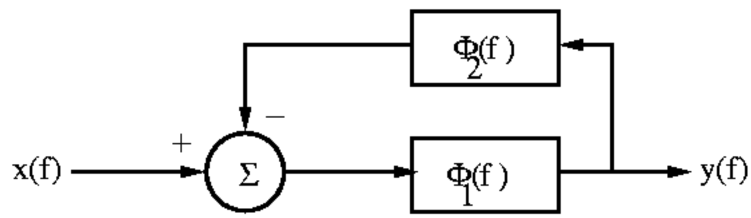

Figure 10: A linear system with feedback

Figure 9 showns the results of numerically integrating (134) for various values of Q to obtain attenuation pulses which are asymmetrical and exhibit a much better approximation to causal behavior than the nondispersive pulses of Figure 8.

The effect of feedback on the transfer function. An important phenomenon to understand is the effect of *feedback* on the transfer function of a system. Figure 10 shows the basic situation where a filtered portion of an output signal, modified by the feedback transfer function Φ_2 is subtracted from the input signal (*negative feedback*). The effect of feedback can alter the response significantly and, in the case of engineering applications, in several highly desireable ways. Consider the net transfer function for the system of Figure 10

$$y(\omega) = (x(\omega) - \Phi_2(\omega)y(\omega))\Phi_1(\omega) \tag{137}$$

which gives

$$\Phi(\omega) = \frac{y(\omega)}{x(\omega)} = \frac{\Phi_1(\omega)}{1 + \Phi_1(\omega)\Phi_2(\omega)} \ . \tag{138}$$

For example, consider Φ_1 being the displacement transfer function for a seismometer (16) with damping ζ and natural frequency ω_s, and the feed back transfer function being a constant $\Phi_2 = k$

$$\Phi_{fb}(\omega) = \frac{\frac{-\omega^2}{\omega^2 - 2\imath\zeta\omega - \omega_s^2}}{1 - \frac{k\omega^2}{\omega^2 - 2\imath\zeta\omega - \omega_s^2}} = \frac{-\omega^2}{(1-k)\omega^2 - 2\imath\zeta\omega - \omega_s^2} \tag{139}$$

which has poles at

$$p_{fb} = \imath\zeta \pm \sqrt{(1-k)\omega_s^2 - \zeta^2} \tag{140}$$

instead of the original poles given by

$$p = \imath\zeta \pm \sqrt{\omega_s^2 - \zeta^2} \equiv \imath\zeta \pm \omega_1 \ . \tag{141}$$

If the non-fedback seismometer is critically damped, for example, so that $\zeta = \omega_s/\sqrt{2}$, then the fedback poles are now at

$$p = \omega_s \left(\frac{\imath}{\sqrt{2}} \pm \sqrt{1/2 - k} \right) \tag{142}$$

and the resonant frequency of the fedback system is

$$\omega_{1_{fb}} = \omega_s \sqrt{1/2 - k}\ . \tag{143}$$

By chosing $1/2 > k > 0$, the resonant period can be substantially reduced, and hence the long-period response can be much improved. This is the essence of how modern broadband seismometers function, where feedback makes it possible to build portable stable, low noise instruments with fedback periods of as long as several hundred seconds.