Spring, 2001 Data Processing and Analysis (GEOP 505)

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Poles and Zeros

We showed that for any linear system relating two time functions, $x(t)$ and $y(t)$, the frequency-domein response (the transfer function) can be obtained from the governing linear differential equation with constant coefficients of a single variable

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x \; . \quad (1)$$

Setting

$$x(t) = e^{\imath 2\pi f t} \; , \quad (2)$$

$$y(t) = \Phi(f) e^{\imath 2\pi f t} \; , \quad (3)$$

and solving (1) for $\Phi(f)$ gives the transfer function

$$\Phi(f) = \frac{Y(f)}{X(f)} = \frac{\sum_{j=0}^{m} b_j (2\pi \imath f)^j}{\sum_{k=0}^{n} a_k (2\pi \imath f)^k} \equiv \frac{Z(f)}{P(f)} \quad (4)$$

where Z and P are complex polynomials in f. The values of f (or equivalently, of the angular frequency $\omega = 2\pi f$) where $Z(f) = 0$ are referred to as *zeros* of (4), as the response of the system will be zero at those frequencies, no matter what the amplitude of the input. Frequencies for which $P(f) = 0$ are referred to as *poles* of (4), as the response of the system will be infinite at those frequencies.

In general, the values of f where we have poles and zeros will be complex. It is useful to express the input function (2) as

$$x(t) = e^{\imath 2\pi f t} = e^{\imath 2\pi (f_r + \imath f_i) t} = e^{\imath 2\pi f_r t} \cdot e^{-\pi f_i t} \quad (5)$$

where $f = f_r + \imath f_i$ and f_r and f_i are real numbers.

This generalized input is:

- A constant for $f = 0$

- A sinusoid for $f_r \neq 0$ and $f_i = 0$.
- A growing exponential for $f_r = 0$ and $f_i < 0$
- A shrinking exponential for $f_r = 0$ and $f_i > 0$
- A growing exponentially weighted sinusoid for $f_r \neq 0$ and $f_i < 0$
- A shrinking exponentially weighted sinusoid for $f_r \neq 0$ and $f_i > 0$

Pole positions are frequently displayed graphically in the complex plane using the convention

$$z \equiv \imath 2\pi f \equiv \sigma + \imath\omega = 2\pi(-f + i + \imath f_r) \tag{6}$$

the positions of the poles in the z plane is especially useful in characterizing the response of the system.

Systems where all pole frequencies have $\sigma < 0$ ($f_i > 0$), so that the poles lie on the left-hand side of the z plane, are stable. In this case the only way to get an infinite output is to drive the system with an exponentially increasing sinusoidal input. The impulse response of such system will always decay back to zero.

On the other hand, systems where all pole frequencies have $\sigma > 0$ ($f_i < 0$), so that the poles lie on the right-hand side of the z plane, are unstable; we obtain an infinite output even when the input is exponentially decaying. The impulse response of such systems increases in amplitude indefinately.

Systems where $\sigma = 0$ and $\omega \neq 0$ ($f_i = 0$ and $f_r \neq 0$) have pole frequencies that are sinudoidal. Such systems will oscillate once they get (even marginally) excited at their resonant frequencies.

Figure (1) shows $z-pole$ locations and cartoon impulse responses for various 2-pole system.

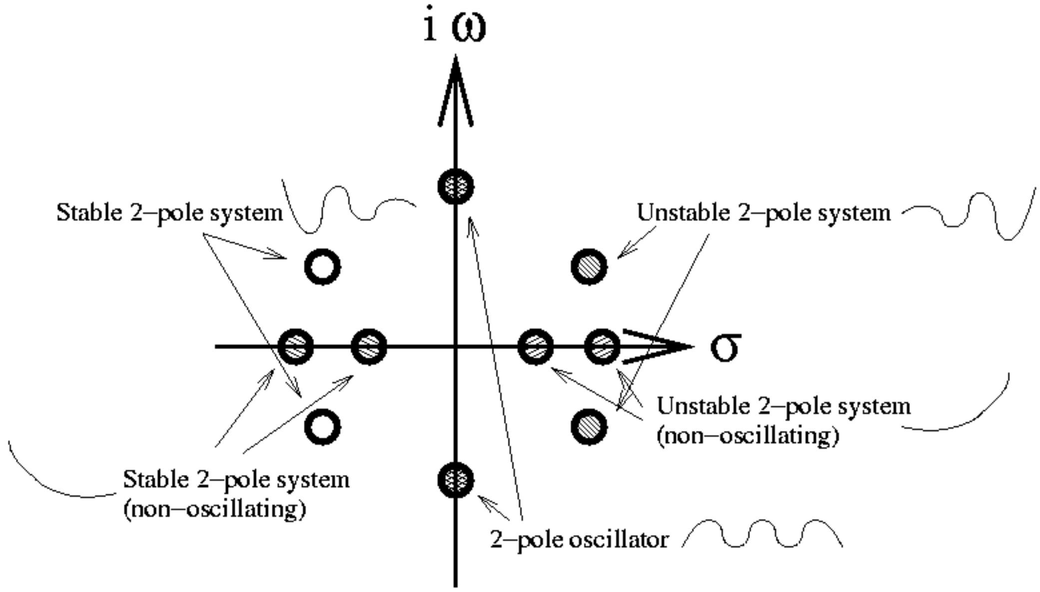

Figure 1: Pole Locations and System Stability