Studio dei sistemi dinamici tramite FdT

Risposta allo scalino

Introduzione

- Assegnato un **sistema dinamico LTI** descritto tramite una **Funzione di Trasferimento** (a tempo continuo oppure a tempo discreto), e' possibile trovare dei parametri che descrivano l'evoluzione della risposta allo scalino del sistema e che siano in qualche modo legati ai parametri tipici della FdT (zeri/poli, guadagno statico ecc.)?

Tempo continuo vs tempo discreto

- Analizziamo **separatamente** anche stavolta il caso dei **sistemi LTI a tempo continuo** ed il caso dei **sistemi LTI a tempo discreto**.

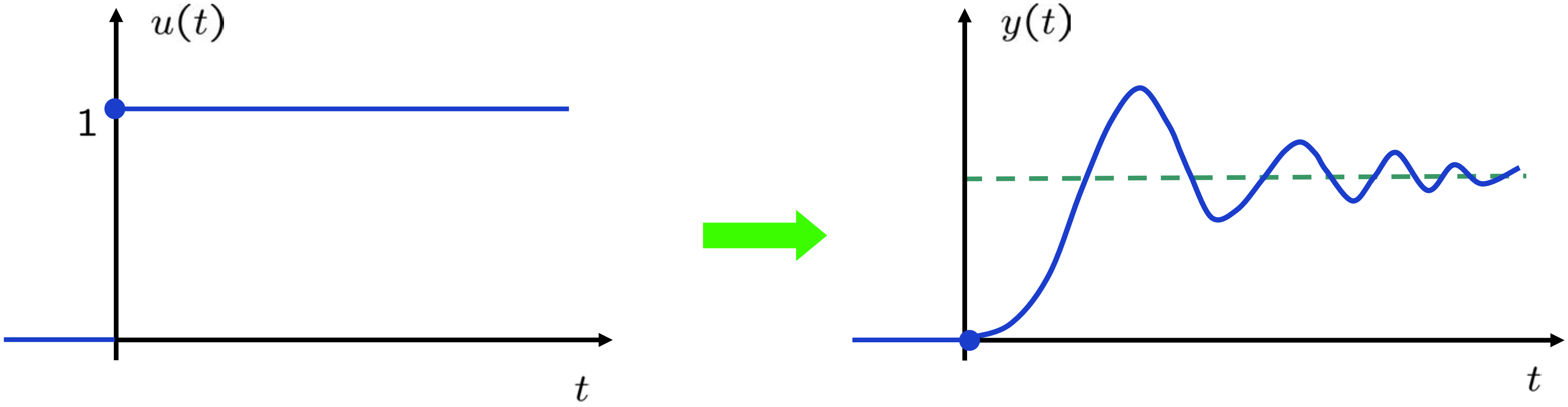
- Infatti, come vedremo, i risultati dell'analisi svolta su sistemi a tempo continuo non sono tutti direttamente portabili al caso dei sistemi a tempo discreto e viceversa esistono dei comportamenti particolari che sono peculiarità dei sistemi a tempo discreto.

Risposta allo scalino di sistemi LTI a tempo continuo

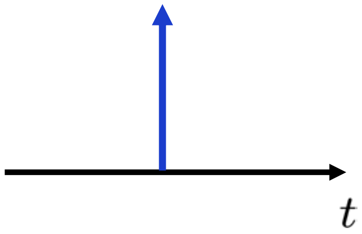
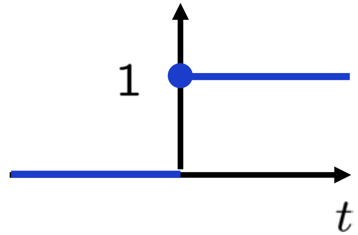
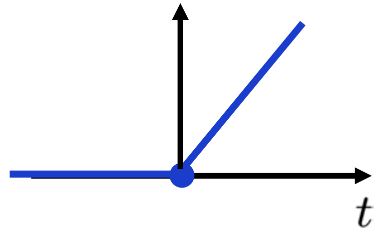
- Risposta allo scalino

$$x(0) = 0$$

$$u(t) = 1(t)$$



In sistemi asint. stabili descrive la transizione da un equilibrio ad un altro

		
$\delta(t)$	$1(t)$	$t \cdot 1(t)$
$\frac{d}{dt}1(t)$		$\int_0^t 1(\tau)d\tau$
1	$\frac{1}{s}$	$\frac{1}{s^2}$

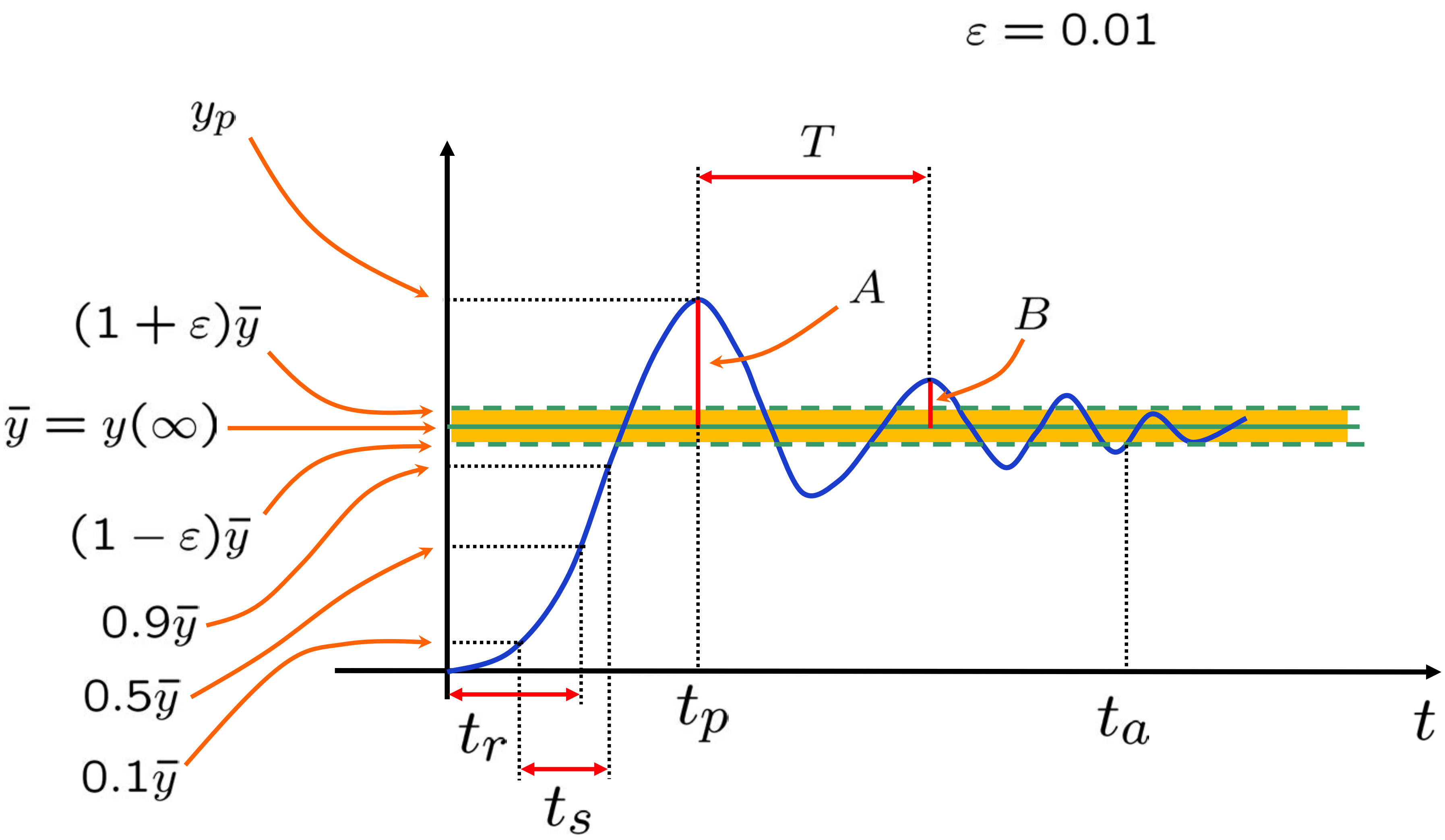
Risposta all'impulso = $\frac{d}{dt}$ (risposta allo scalino)

Risposta alla rampa = $\int_0^t$ (risposta allo scalino) $d\tau$

- Risposta allo scalino: parametri caratteristici

- $\bar{y} = y(\infty)$ — Valore di regime
- t_a — Tempo di assestamento
- t_s — Tempo di salita
- t_r — Tempo di ritardo
- t_p — Tempo di picco
- y_p — Valore di picco
- $A = y_p - y(\infty)$ — Max sovraelongazione
- $\Delta = A/y(\infty)$ — Max sovraelongazione relativa
- T — "Periodo" delle oscillazioni
- B/A — Fattore di smorzamento

- Risposta allo scalino: parametri caratteristici



- Risposta allo scalino: I ordine

A) $$G(s) = \frac{\mu}{1 + s\tau}$$ Strettamente proprio

B) $$G(s) = \frac{\mu(1 + sT)}{1 + s\tau}$$ Non strettamente proprio

$\tau > 0$ $\mu > 0$

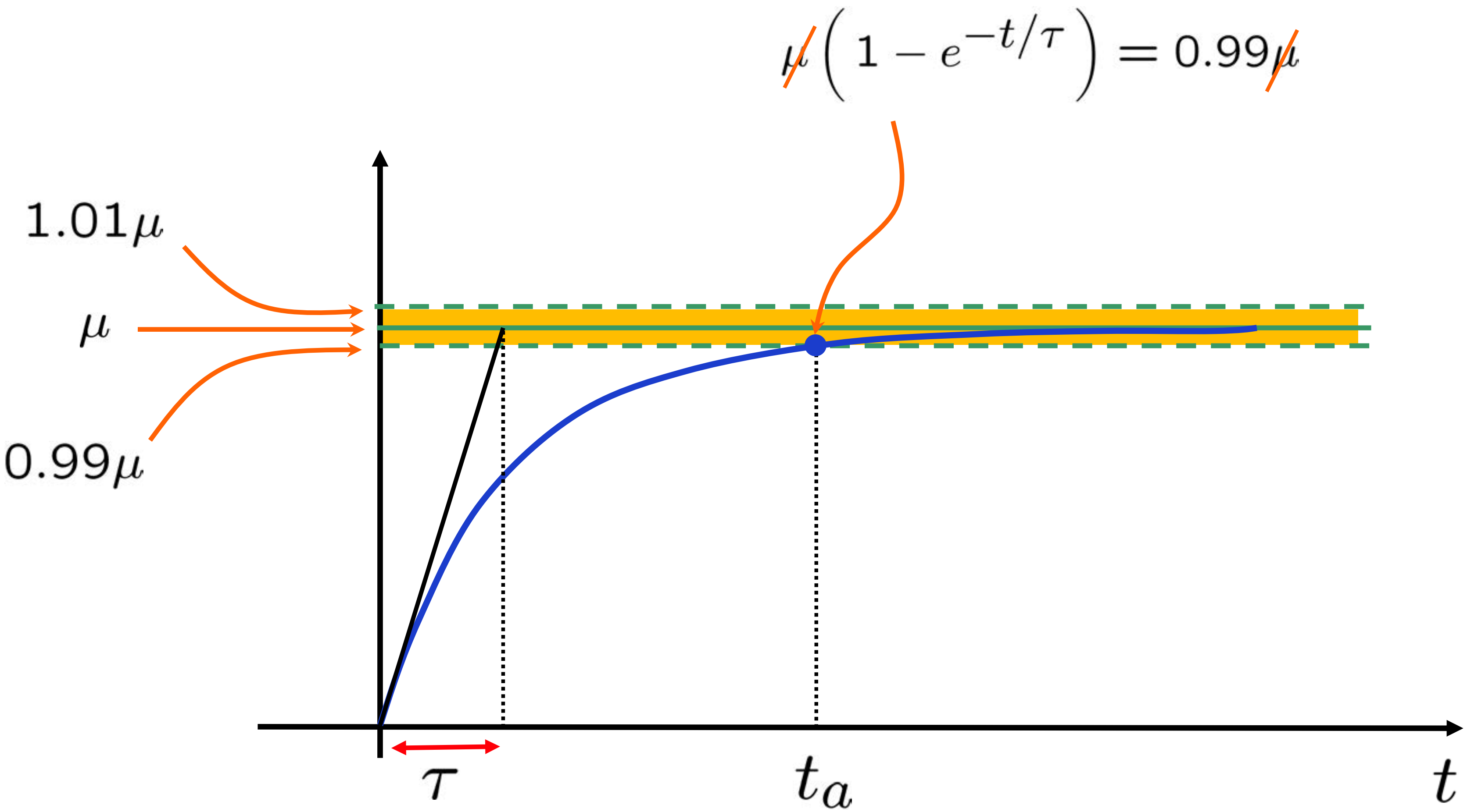
(As. Stabile)

A)

$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{\mu}{s(1+s\tau)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{\mu}{s} - \frac{\mu\tau}{1+s\tau}\right]$$

$$= \mu\left(1 - e^{-t/\tau}\right), \quad t \geq 0$$



$\mu\left(1 - e^{-t/\tau}\right) = 0.99\mu$
1.01μ
μ
0.99μ
τ
t_a
t

- Calcolo tempo di assestamento

$$1 - e^{-t/\tau} = 0.99$$

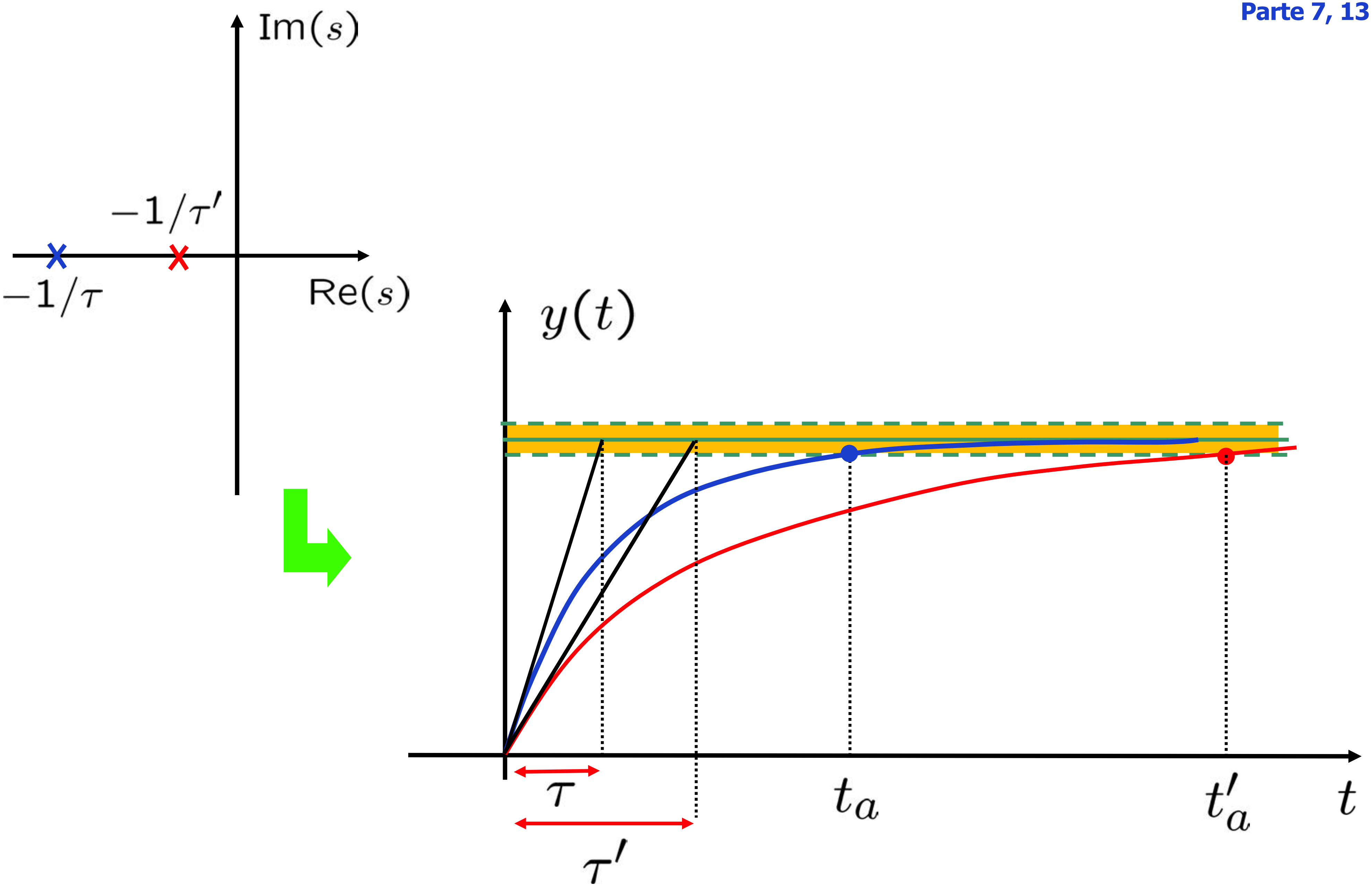
$$e^{-t/\tau} = 0.01$$

$$e^{t/\tau} = 100$$

$$t/\tau = \ln 100$$

$$t_a = \tau \ln 100 \simeq 5\tau$$

Esercizio a casa: calcolare $t_s,\ t_r$


Im(s)
−1/τ′
−1/τ
Re(s)
y(t)
τ
τ′
t_a
t′_a
t

B)

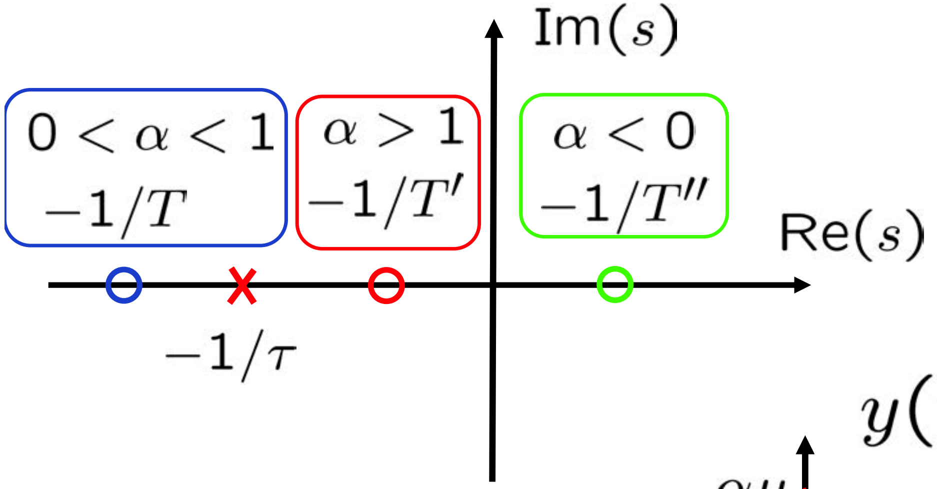
$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{\mu(1+sT)}{s(1+s\tau)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{\mu}{s} + \frac{\mu(T-\tau)}{1+s\tau}\right]$$

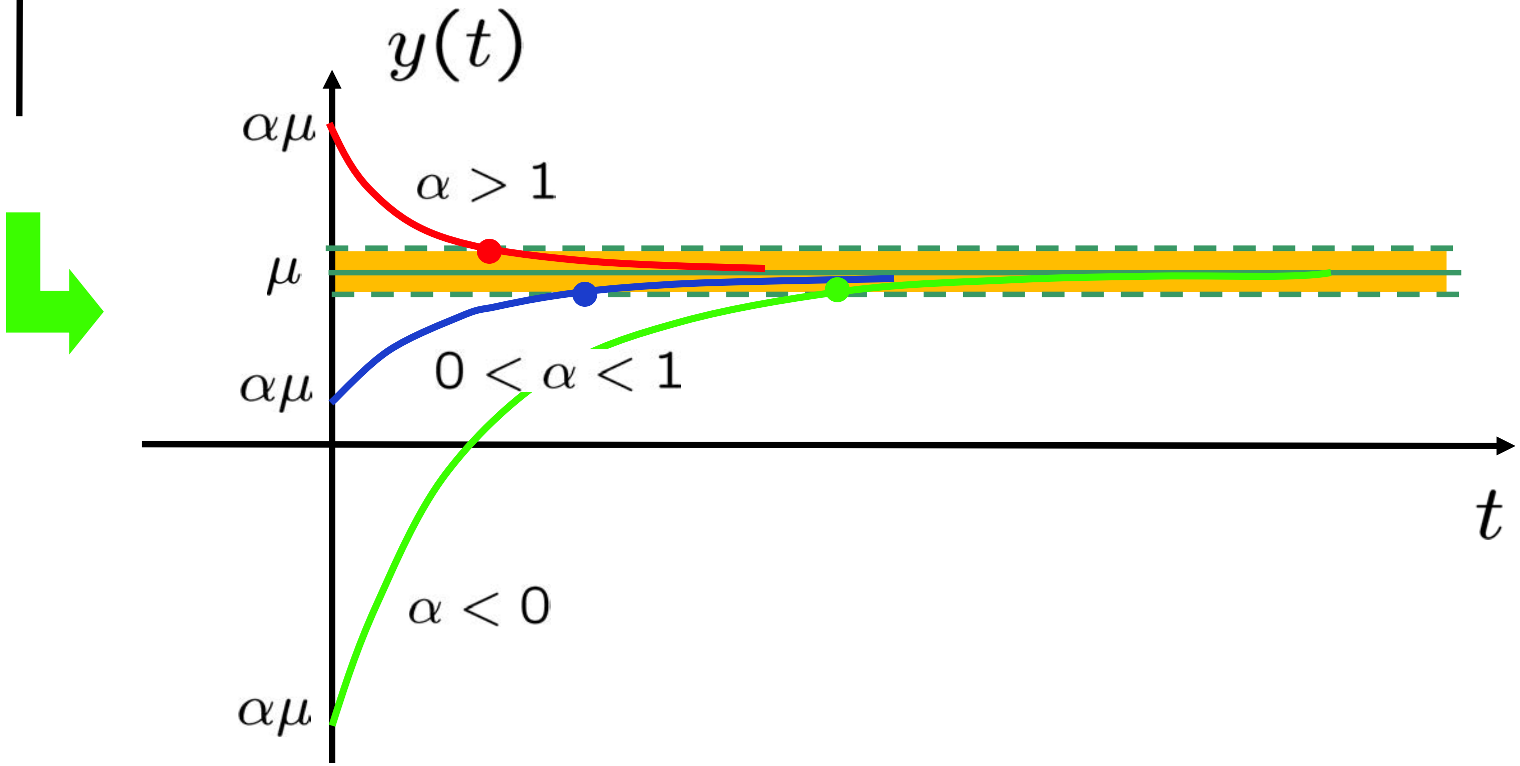
$$= \mu\left(1 + \frac{T-\tau}{\tau}e^{-t/\tau}\right), \qquad t \geq 0$$

N.B. $$\lim_{t\to 0^+} y(t) = \mu\frac{T}{\tau} \neq 0$$

Poniamo: $T = \alpha\tau$ ➡ $y(t) = \mu\left[1 + (\alpha - 1)e^{-t/\tau}\right], \quad t \geq 0$



In prima approx: $t_a \simeq 5\tau$



- Risposta allo scalino: II ordine

A) $$G(s) = \frac{\mu}{(1+s\tau_1)(1+s\tau_2)}$$ Poli reali senza zeri

B) $$G(s) = \frac{\mu(1+sT)}{(1+s\tau_1)(1+s\tau_2)}$$ Poli reali con zero

C) $$G(s) = \frac{\varrho}{(s+\sigma+j\omega)(s+\sigma-j\omega)}$$ Poli complessi senza zeri

D) $$G(s) = \frac{\varrho(1+sT)}{(s+\sigma+j\omega)(s+\sigma-j\omega)}$$ Poli complessi con zero

A)

$$G(s) = \frac{\mu}{(1 + s\tau_1)(1 + s\tau_2)}$$

$\tau_1 \neq \tau_2$

$\tau_1 > 0$

$\tau_2 > 0$

} Asintotica stabilità

Per esempio poniamo: $\tau_1 > \tau_2$

$\mu > 0$

$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{\mu}{s(1+s\tau_1)(1+s\tau_2)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{A}{s} + \frac{B}{1+s\tau_1} + \frac{C}{1+s\tau_2}\right]$$

$$A = \left.\frac{\mu}{(1+s\tau_1)(1+s\tau_2)}\right|_{s=0} = \mu$$

$$B = \left.\frac{\mu}{s(1+s\tau_2)}\right|_{s=-1/\tau_1} = \frac{\mu}{-\frac{1}{\tau_1}(1-\frac{\tau_2}{\tau_1})} = \frac{\mu\tau_1^2}{\tau_2-\tau_1}$$

$$C = \left.\frac{\mu}{s(1+s\tau_1)}\right|_{s=-1/\tau_2} = \frac{\mu}{-\frac{1}{\tau_2}(1-\frac{\tau_1}{\tau_2})} = \frac{\mu\tau_2^2}{\tau_1-\tau_2}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{\mu}{s} + \frac{\frac{\mu\tau_1^2}{\tau_2-\tau_1}}{1+s\tau_1} + \frac{\frac{\mu\tau_2^2}{\tau_1-\tau_2}}{1+s\tau_2}\right]$$

$$= \mu\left(1 - \frac{\tau_1}{\tau_1-\tau_2}e^{-t/\tau_1} + \frac{\tau_2}{\tau_1-\tau_2}e^{-t/\tau_2}\right), \quad t \geq 0$$

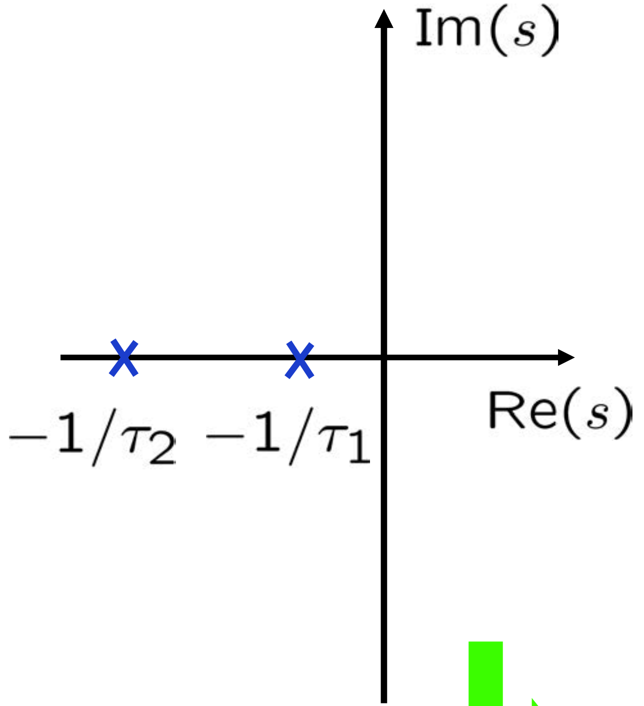
Alcune caratteristiche:

$$y(\infty) = \mu > 0$$

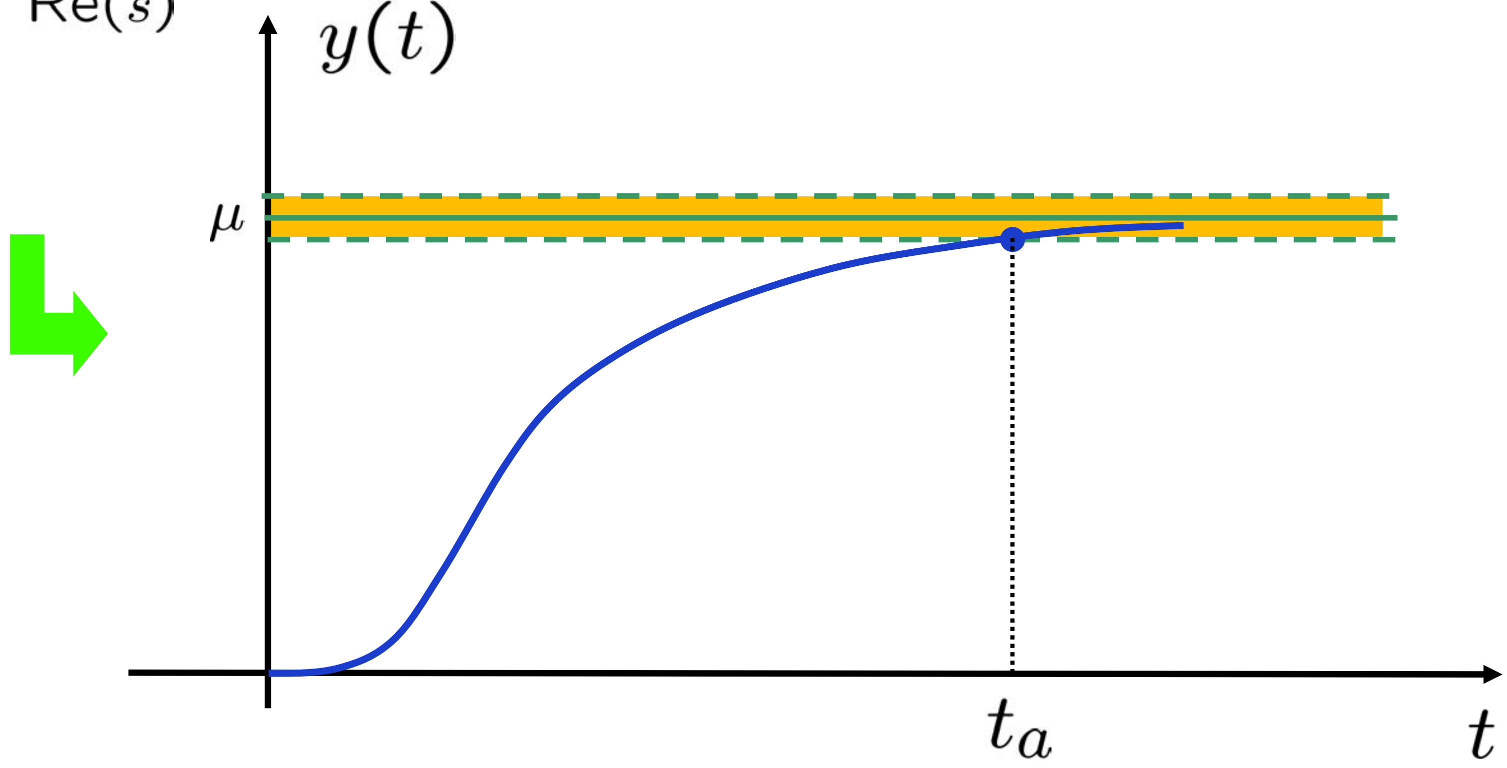
$$y(0) = 0$$

$$\dot{y}(0) = 0$$

$$\ddot{y}(0) = \frac{\mu}{\tau_1\tau_2} > 0$$



$t_a = ?$



Curva monotona crescente, con flesso

- Se $\tau_1 >> \tau_2$

$$y(t) = \mu \left(1 - \frac{\tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_2}{\tau_1 - \tau_2} e^{-t/\tau_2} \right) , \quad t \geq 0$$

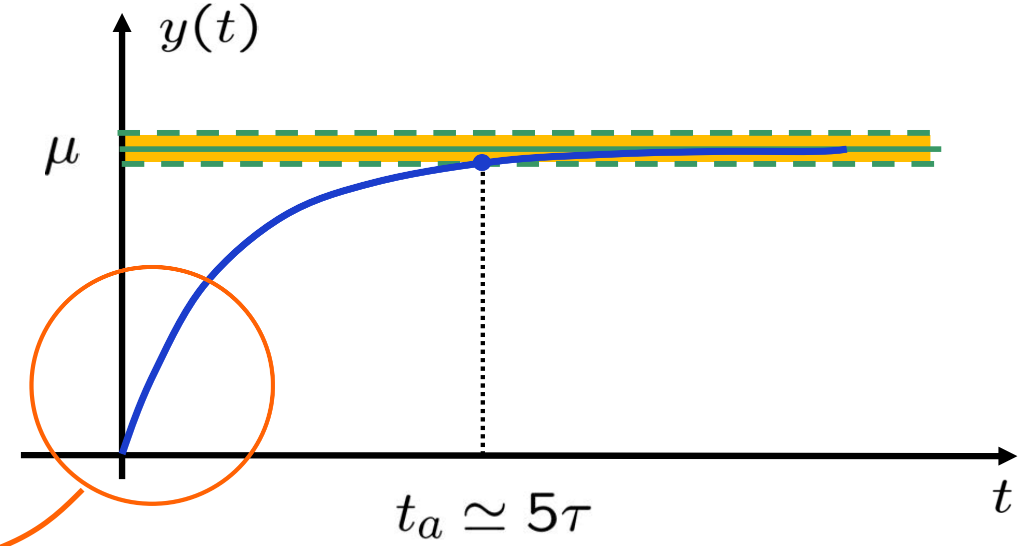
$$\simeq \mu \left(1 - e^{-t/\tau_1} \right) , \quad t \geq 0$$

In prima approx: $t_a \simeq 5\tau_1$

In generale, in assenza di zeri, i poli piu` influenti sull'andamento qualitativo della risposta sono quelli vicini all'asse immaginario

- Riepilogando:

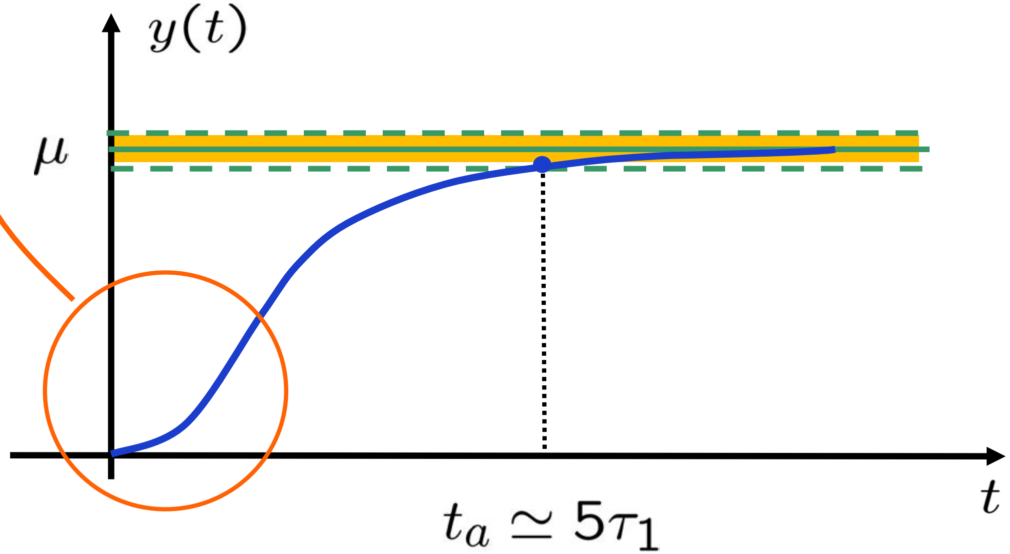
$$G(s) = \frac{\mu}{1 + s\tau}$$



A parita` di tempo di assestamento e` diverso il tempo di salita, ovvero la velocita` di risposta

$$G(s) = \frac{\mu}{(1 + s\tau_1)(1 + s\tau_2)}$$

$$\tau_1 >> \tau_2$$



B)

$$G(s) = \frac{\mu(1 + sT)}{(1 + s\tau_1)(1 + s\tau_2)}$$

$$\tau_1 \neq \tau_2$$

$$\left.\begin{array}{l}\tau_1 > 0 \\ \tau_2 > 0\end{array}\right\}$$ Asintotica stabilità

Per esempio poniamo: $\tau_1 > \tau_2$

$$\mu > 0$$

$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{\mu(1+sT)}{s(1+s\tau_1)(1+s\tau_2)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{A}{s} + \frac{B}{1+s\tau_1} + \frac{C}{1+s\tau_2}\right]$$

$$A = \left.\frac{\mu(1+sT)}{(1+s\tau_1)(1+s\tau_2)}\right|_{s=0} = \mu$$

$$B = \left.\frac{\mu(1+sT)}{s(1+s\tau_2)}\right|_{s=-1/\tau_1} = \frac{\mu(1-T/\tau_1)}{-\frac{1}{\tau_1}(1-\frac{\tau_2}{\tau_1})} = \frac{\mu\tau_1(\tau_1-T)}{\tau_2-\tau_1}$$

$$C = \left.\frac{\mu(1+sT)}{s(1+s\tau_1)}\right|_{s=-1/\tau_2} = \frac{\mu(1-T/\tau_2)}{-\frac{1}{\tau_2}(1-\frac{\tau_1}{\tau_2})} = \frac{\mu\tau_2(\tau_2-T)}{\tau_1-\tau_2}$$

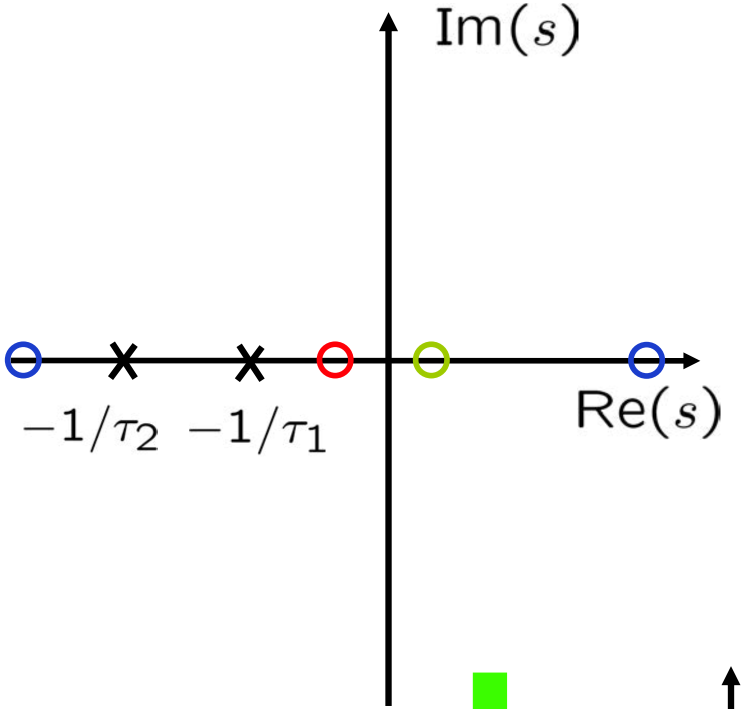
$$y(t) = \mu \left(1 - \frac{\tau_1 - T}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_2 - T}{\tau_1 - \tau_2} e^{-t/\tau_2} \right), \qquad t \geq 0$$

Alcune caratteristiche:

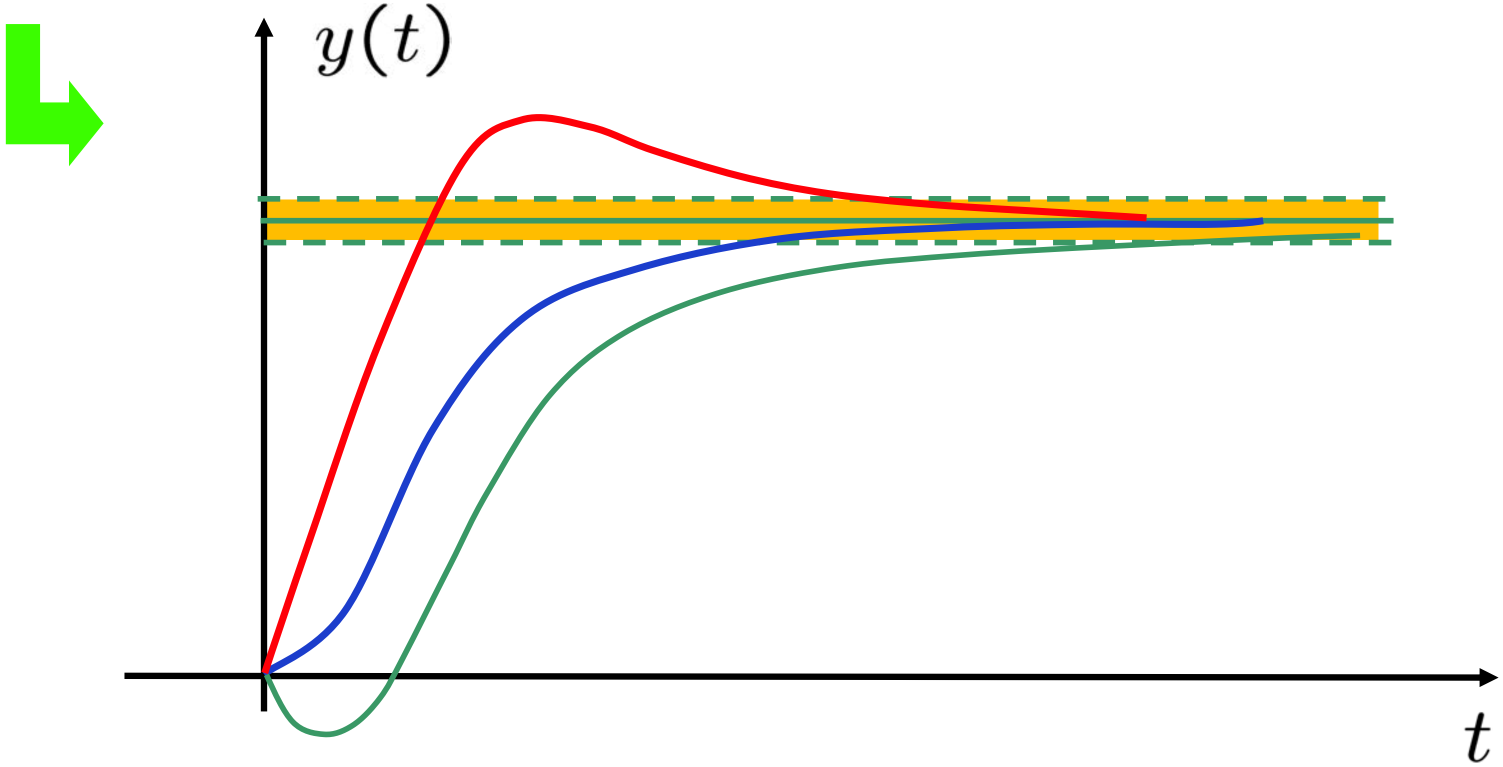
$$y(\infty) = \mu > 0$$

$$y(0) = 0$$

$$\dot{y}(0) = \frac{\mu T}{\tau_1 \tau_2} \quad \begin{array}{ll} > 0\,, & \text{se } T > 0 \\ < 0\,, & \text{se } T < 0 \end{array}$$



- zero approx irrilevante (blu)
- sovraelongazione (rosso)
- sottoelongazione (verde)



C)

$$G(s) = \frac{\varrho}{(s+\sigma+j\omega)(s+\sigma-j\omega)}$$

$$\mu = G(0) = \frac{\varrho}{\sigma^2+\omega^2}$$

Poli: $-\sigma \pm j\omega$

$\sigma > 0$ asintotica stabilità

$\omega > 0$

$\varrho > 0$

$$Y(s) = \frac{G(s)}{s} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\sigma s + \sigma^2 + \omega^2}$$

$$As^2 + 2A\sigma s + A\sigma^2 + A\omega^2 + Bs^2 + Cs = \varrho$$

$$\begin{cases} A + B = 0 \\ 2A\sigma + C = 0 \\ A(\sigma^2 + \omega^2) = \varrho \end{cases} \quad \rightarrow \quad \begin{cases} A = \dfrac{\varrho}{\sigma^2 + \omega^2} = \mu \\ B = -\mu \\ C = -2\sigma\mu \end{cases}$$

$$Y(s) = \mu \left[\frac{1}{s} - \frac{s + 2\sigma}{s^2 + 2\sigma s + \sigma^2 + \omega^2} \right] = \mu \left[\frac{1}{s} - \frac{s + \sigma + \sigma}{(s + \sigma)^2 + \omega^2} \right]$$

$$= \mu \left[\frac{1}{s} - \frac{s + \sigma}{(s + \sigma)^2 + \omega^2} - \frac{\sigma}{\omega} \frac{\omega}{(s + \sigma)^2 + \omega^2} \right]$$

$$y(t) = \mu\left[1 - e^{-\sigma t}\cos(\omega t) - \frac{\sigma}{\omega}e^{-\sigma t}\sin(\omega t)\right], \quad t \geq 0$$

$$= \mu\left[1 - \underbrace{e^{-\sigma t}\left(\cos(\omega t) + \frac{\sigma}{\omega}\sin(\omega t)\right)}_{\text{Oscillazioni smorzate}}\right], \quad t \geq 0$$

Oscillazioni smorzate

Alcune caratteristiche:

$$y(\infty) = \mu > 0$$

$$y(0) = 0$$

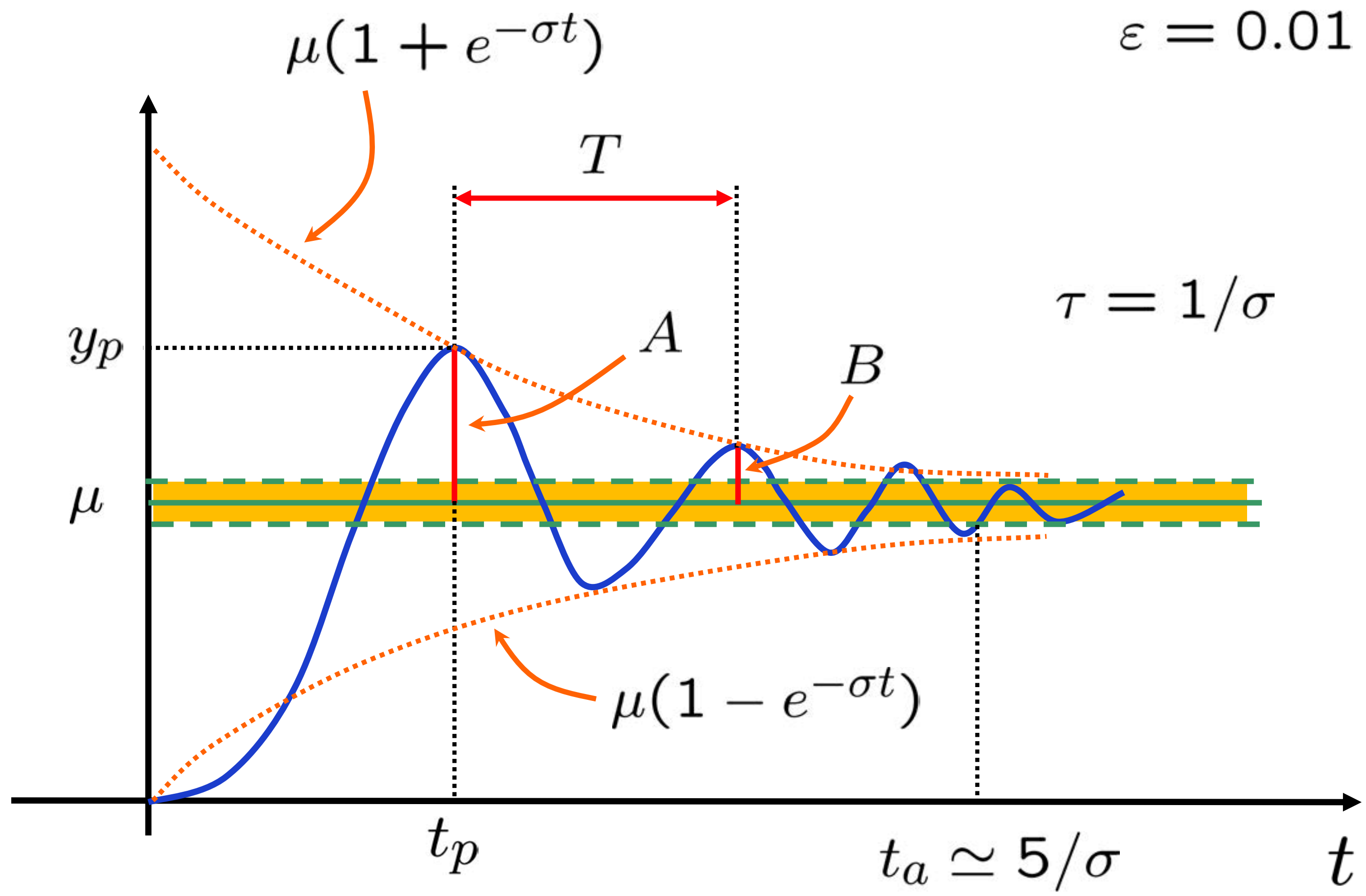
$$\dot{y}(0) = 0$$

$$\ddot{y}(0) = \varrho > 0$$

Per esercizio dimostrare che si puo` riscrivere la risposta cosi`:

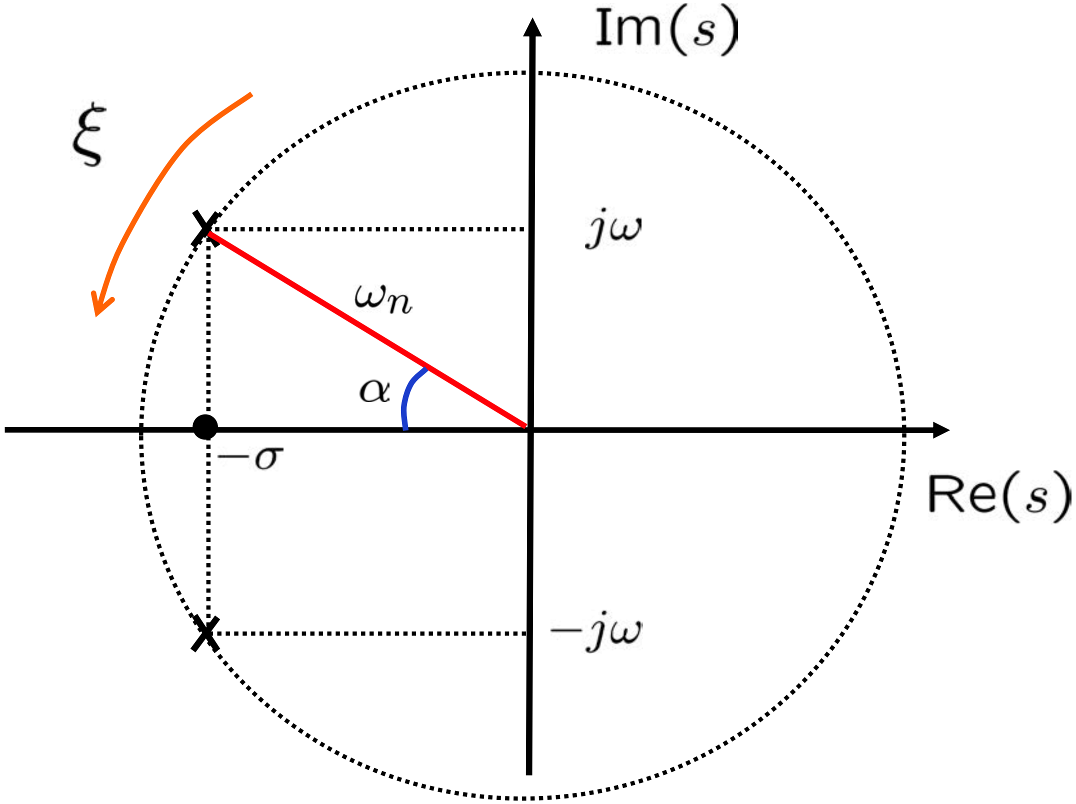
$$y(t) = \mu \left[1 - \frac{\sqrt{\sigma^2 + \omega^2}}{\omega} e^{-\sigma t} \sin(\omega t + \varphi) \right], \qquad t \geq 0$$

Dove: $$\varphi = \arccos\left(\frac{\sigma}{\sqrt{\sigma^2 + \omega^2}} \right)$$



$\mu(1 + e^{-\sigma t})$
$\varepsilon = 0.01$
T
$\tau = 1/\sigma$
y_p
A
B
μ
$\mu(1 - e^{-\sigma t})$
t_p
$t_a \simeq 5/\sigma$
t

- Diversa parametrizzazione



$$\omega_n^2 = \sigma^2 + \omega^2$$

$$\omega_n \xi = \sigma$$

$$\omega_n \sqrt{1-\xi^2} = \omega$$

ω_n Pulsazione naturale

$\cos\alpha = \xi$ Smorzamento $0 \leq \xi \leq 1$

$$G(s) = \frac{\varrho}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

- Espressione della FdT

$$G(s) = \frac{\varrho}{(s+\sigma+j\omega)(s+\sigma-j\omega)} = \frac{\varrho}{(s+\sigma)^2+\omega^2}$$

$$= \frac{\varrho}{s^2 + \underbrace{2\sigma}_{2\xi\omega_n} s + \underbrace{\sigma^2+\omega^2}_{\omega_n^2}} = \frac{\varrho}{s^2+2\xi\omega_n s+\omega_n^2}$$

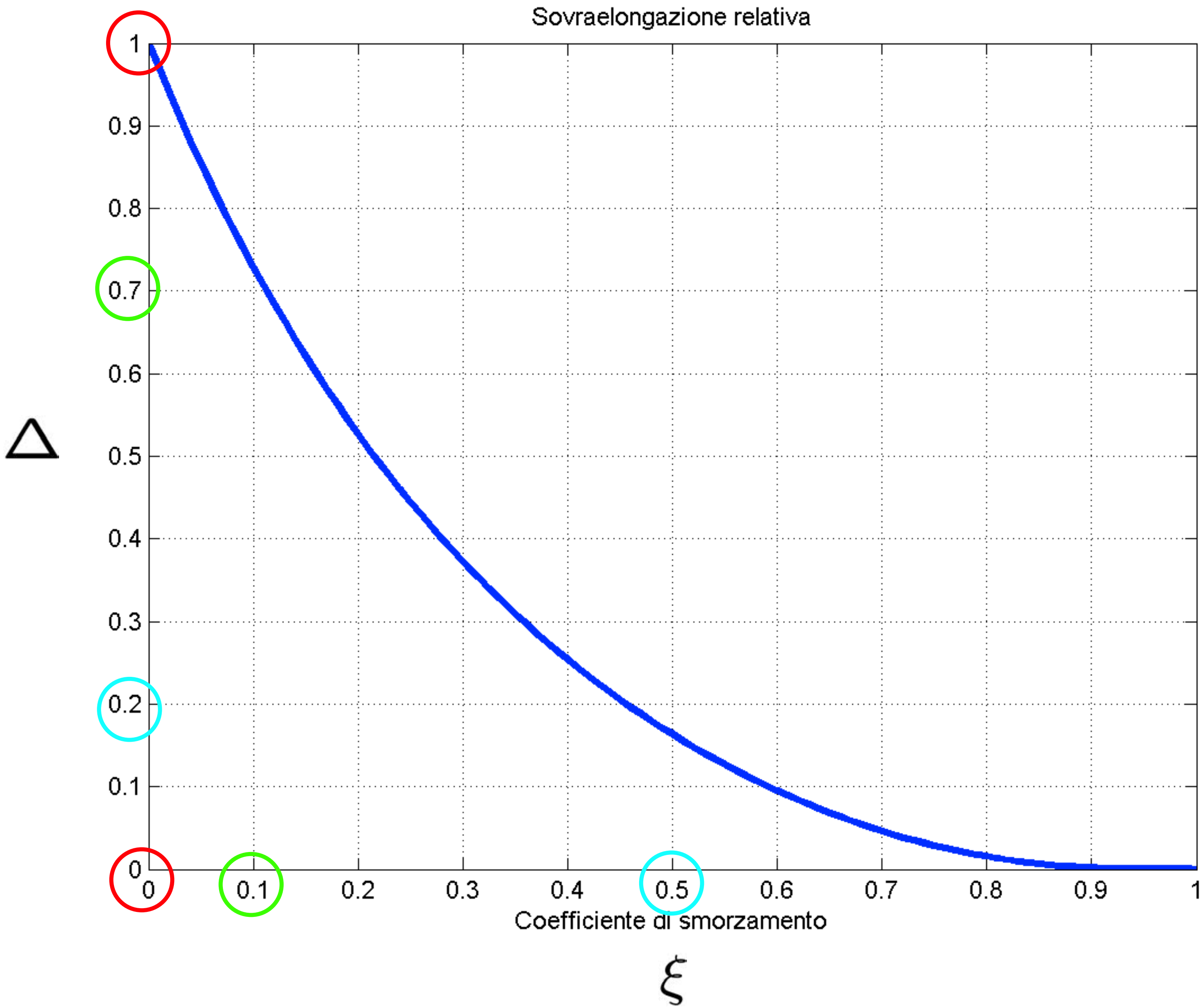
$$= \frac{\varrho/\omega_n^2}{1+\frac{2\xi}{\omega_n}s+\frac{1}{\omega_n^2}s^2} = \frac{\mu}{1+\frac{2\xi}{\omega_n}s+\frac{1}{\omega_n^2}s^2}$$

dove $\mu := \frac{\varrho}{\omega_n^2}$

- Parametri caratteristici

- $$t_a \simeq \frac{5}{\sigma} = \frac{5}{\xi\omega_n}$$
- $$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$$
- $$t_p = \frac{\pi}{\omega} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$
- $$y_p = \mu\left[1 + e^{-\frac{\sigma\pi}{\omega}}\right] = \mu\left[1 + e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}\right]$$
- $$\Delta = \frac{A}{\mu} = e^{-\sigma\pi/\omega} = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$$
- $$\frac{B}{A} = \cdots = \Delta^2 = e^{-2\sigma\pi/\omega} = e^{-\frac{2\xi\pi}{\sqrt{1-\xi^2}}}$$

Dipendono solo da ξ e non da ω_n


Sovraelongazione relativa
Δ
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Coefficiente di smorzamento
ξ

- Casi limite

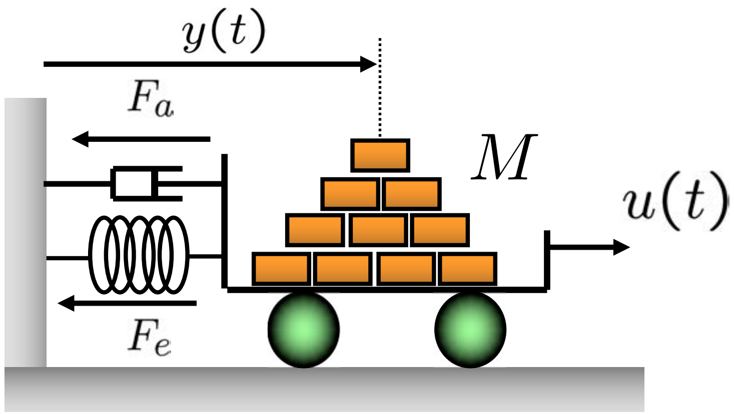
- $\xi = 0$ $$G(s) = \frac{\varrho}{s^2 + \omega_n^2}$$ Poli: $\pm j\omega_n$

  → Oscillazioni non smorzate

- $\xi = 1$ $$G(s) = \frac{\varrho}{(s + \omega_n)^2}$$ Poli: $-\omega_n, -\omega_n$

  → Assenza di oscillazioni

- Esempio 1



$$G(s) = \frac{1}{Ms^2 + hs + k}$$

$$= \frac{1/M}{s^2 + \frac{h}{M}s + \frac{k}{M}}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2$$

$$\mu = G(0) = \frac{1}{k}$$

$$2\xi\omega_n = \frac{h}{M} \qquad \omega_n^2 = \frac{k}{M}$$

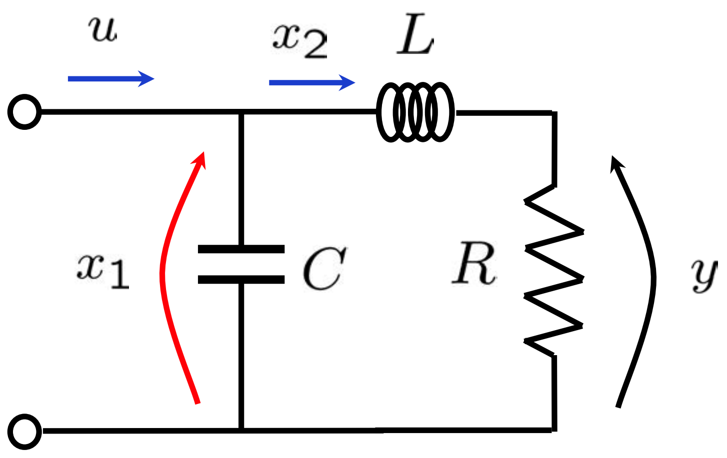
$$T = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$\omega_n = \sqrt{\frac{k}{M}}$$

$$\Delta = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$$

$$\xi = \frac{h}{2\sqrt{kM}}$$

- Esempio 2



$$\begin{cases} C\dot{x}_1 = u - x_2 \\ L\dot{x}_2 = x_1 - Rx_2 \\ y = Rx_2 \end{cases}$$

$$A = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix} \qquad B = \begin{bmatrix} 1/C \\ 0 \end{bmatrix} \qquad C = [0 \;\; R]$$

$$G(s) = [0 \;\; R] \begin{bmatrix} s & 1/C \\ -1/L & s + R/L \end{bmatrix}^{-1} \begin{bmatrix} 1/C \\ 0 \end{bmatrix} = \cdots = \frac{R/(LC)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\omega_n = \frac{1}{\sqrt{LC}} \qquad \xi = \frac{R}{2}\sqrt{\frac{C}{L}} \qquad \mu = R$$

D)

$$G(s) = \frac{\mu(1+sT)}{1 + \frac{2\xi}{\omega_n}s + \frac{1}{\omega_n^2}s^2}$$

$$0 < \xi < 1$$

$$\omega_n > 0$$

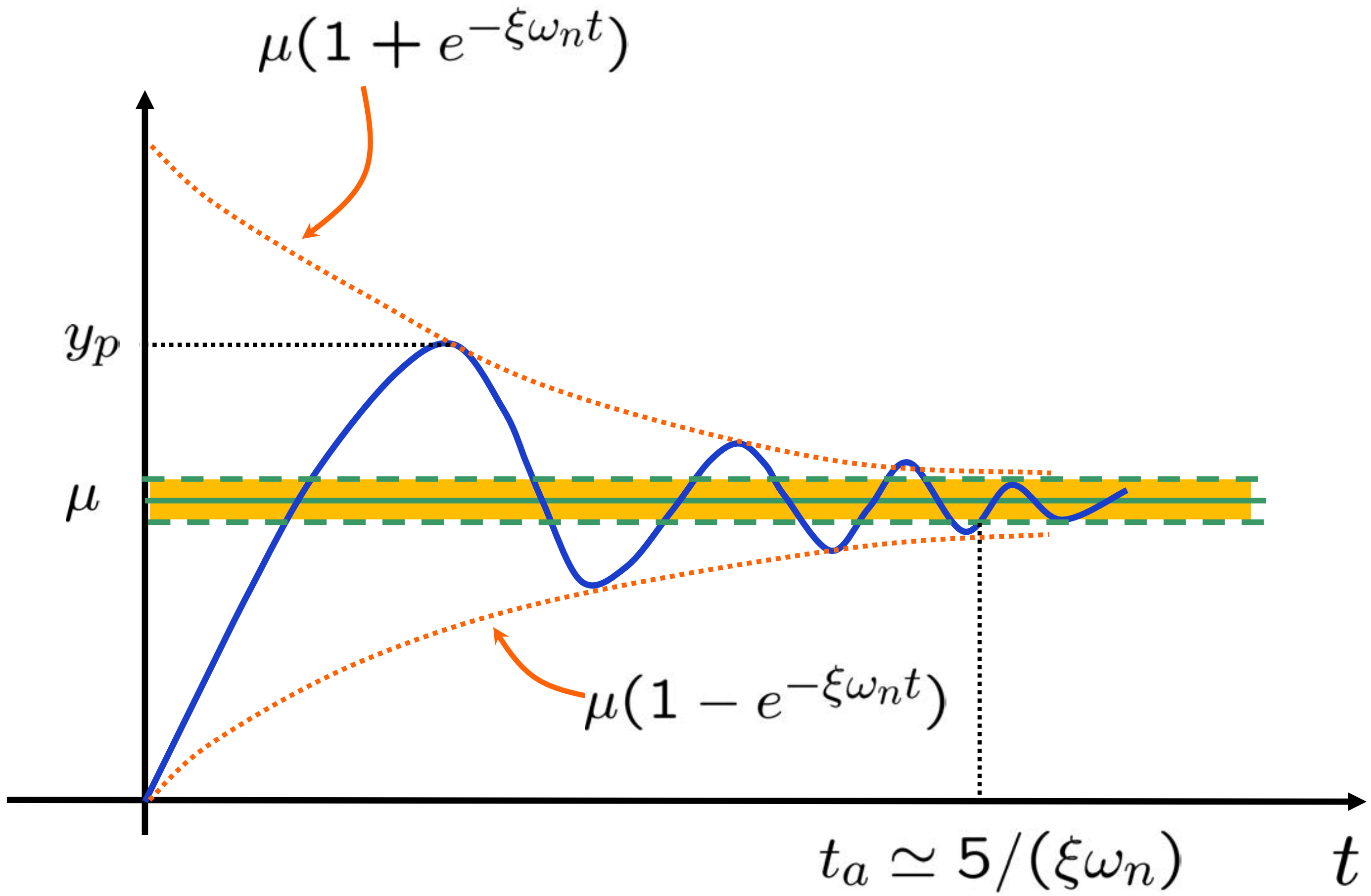
$$\mu > 0$$

Alcune caratteristiche della risposta:

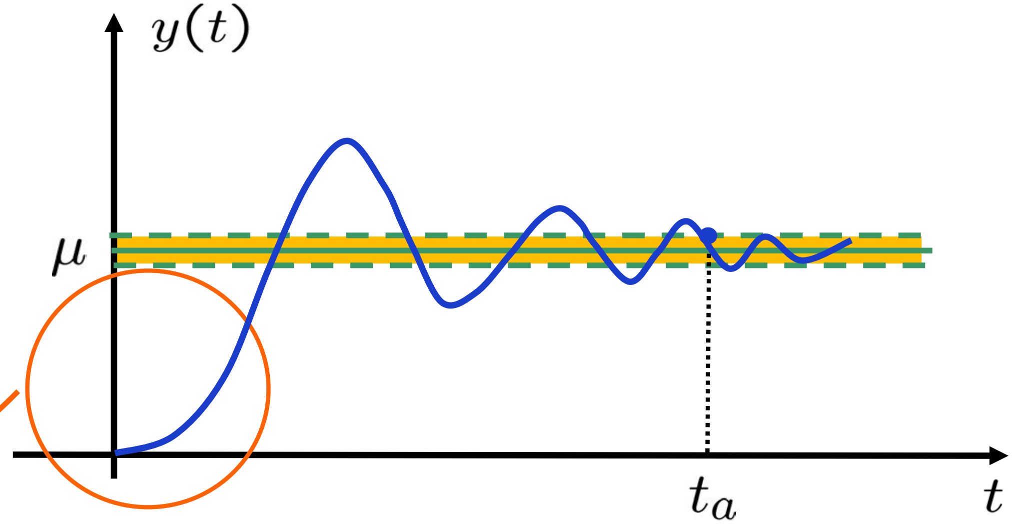
$$y(\infty) = \mu > 0$$

$$y(0) = 0$$

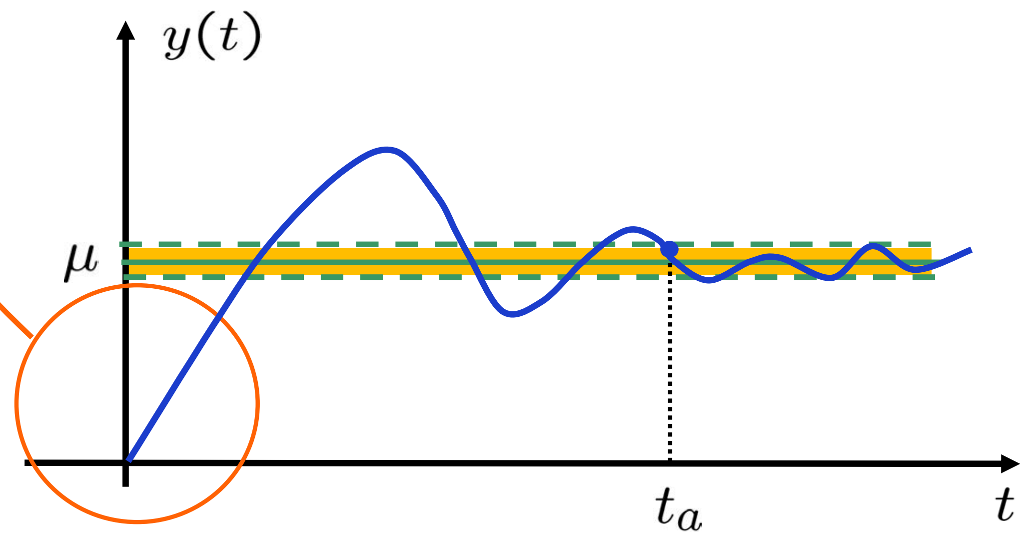
$$\dot{y}(0) = \mu T \omega_n^2 \quad \begin{array}{ll} > 0\,, & \text{se } T > 0 \\ < 0\,, & \text{se } T < 0 \end{array}$$


$\mu(1 + e^{-\xi\omega_n t})$
y_p
μ
$\mu(1 - e^{-\xi\omega_n t})$
$t_a \simeq 5/(\xi\omega_n)$
t

- Confrontando C) e D):



A parita` di tempo di assestamento e` diverso il tempo di salita, ovvero la velocita` di risposta



- Risposta allo scalino – Ordine > 2

$$G(s) = \frac{\mu}{s^g} \frac{\prod_{i=1}^{m} (1 + sT_i)}{\prod_{i=1}^{n} (1 + s\tau_i)}$$

As. stabilita` ⟺ $\text{Re}\,(\text{poli}) < 0$, $g \leq 0$

salvo cancellazioni

$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]$$

- Teo. valore iniziale

$$\lim_{t \to 0^+} y(t) = \lim_{s \to \infty} \cancel{s} \frac{1}{\cancel{s}} G(s) \quad \begin{array}{ll} = 0\,, & \text{se } m < n \\ \neq 0\,, & \text{se } m = n \end{array}$$

- Teo. valore finale

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} \cancel{s} \frac{1}{\cancel{s}} G(s) \quad \begin{array}{ll} = \mu\,, & \text{se } g = 0 \\ = 0\,, & \text{se } g < 0 \end{array}$$

- Approssimazione a poli dominanti

- Caso semplice – Poli reali distinti

$$Y(s) = G(s)\frac{1}{s} = \frac{\alpha_0}{s} + \frac{\alpha_1}{1+s\tau_1} + \cdots + \frac{\alpha_n}{1+s\tau_n}$$

$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]$$

$$= \alpha_0 + \frac{\alpha_1}{\tau_1}e^{-t/\tau_1} + \cdots + \frac{\alpha_n}{\tau_n}e^{-t/\tau_n}$$

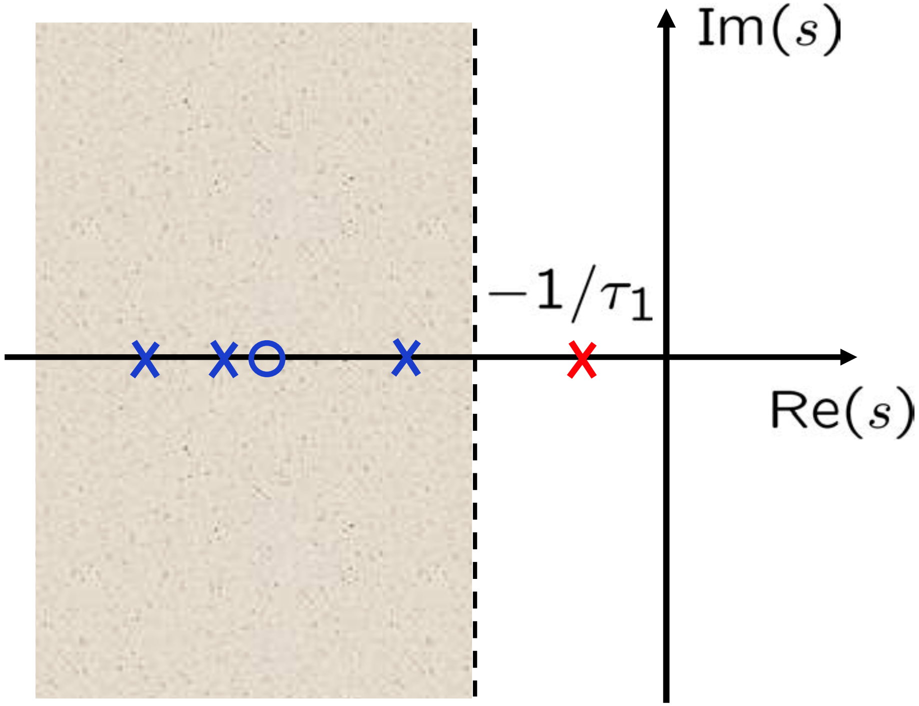
Se $\tau_1 > \tau_2 > \cdots > \tau_n$

$$y(t) = \alpha_0 + \frac{\alpha_1}{\tau_1}e^{-t/\tau_1} + \cdots + \frac{\alpha_n}{\tau_n}e^{-t/\tau_n}$$

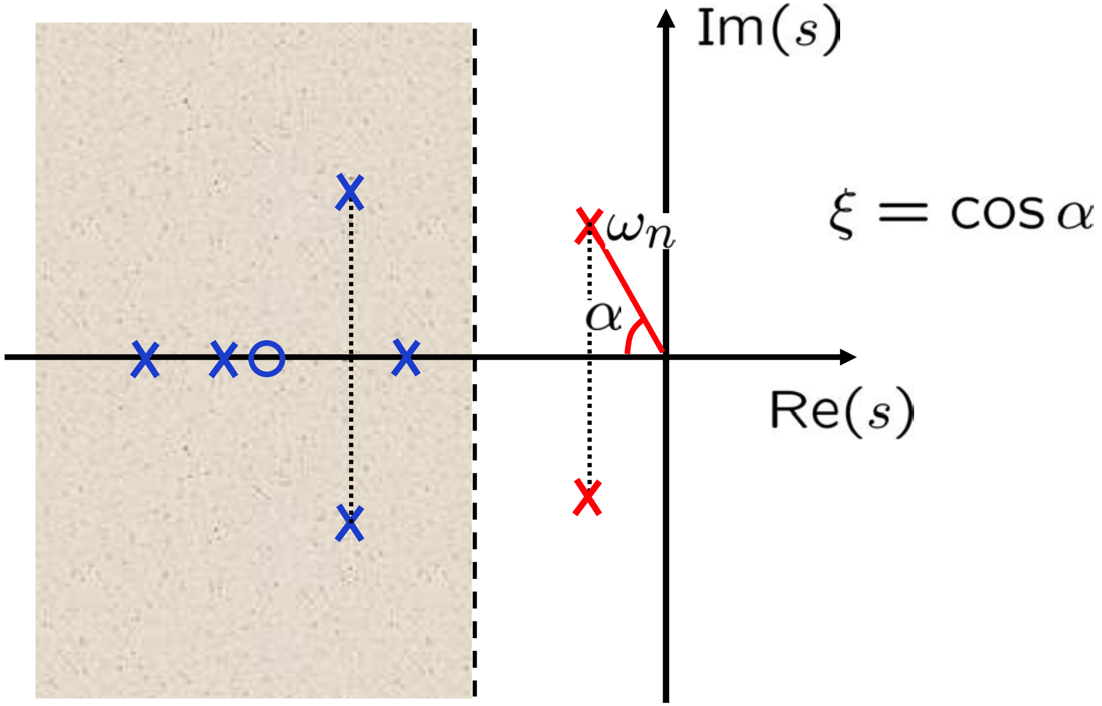
$$\simeq \alpha_0 + \frac{\alpha_1}{\tau_1}e^{-t/\tau_1}$$

componente dominante $t_a \simeq 5\tau_1$

- Polo dominante reale



- Poli dominanti complessi



- AVVERTENZA !

Nell'approssimazione a poli dominanti e` necessario

- Tener conto di zeri "vicini" all'asse immaginario
- Conservare il guadagno μ

- Esempio

$$G(s) = \frac{400(1+s)}{(1+0.2s)(1+0.1s)(s^2+2s+4)}$$

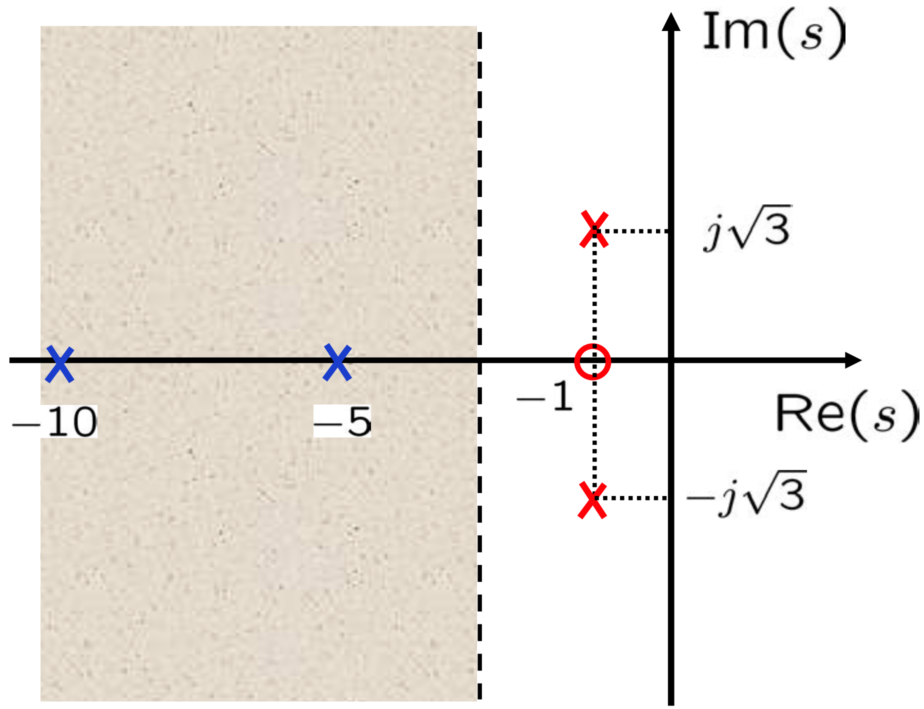
$\mu = G(0) = 100$

Poli: -5

-10

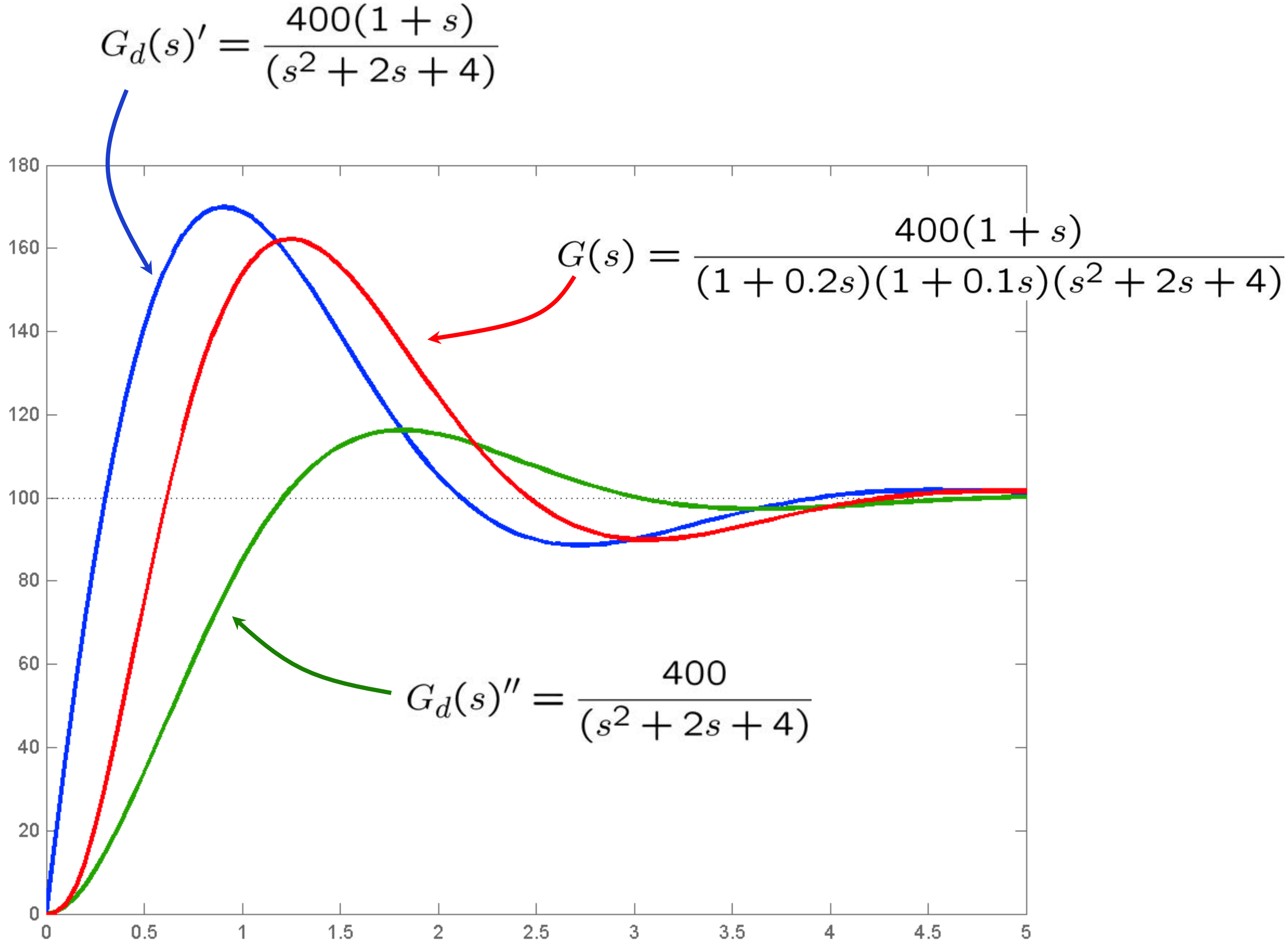
$-1 \pm j\sqrt{3}$

Zero: -1



$\omega_n = 2$

$\xi = 1/2$



$G_d(s)' = \frac{400(1+s)}{(s^2+2s+4)}$
$G(s) = \frac{400(1+s)}{(1+0.2s)(1+0.1s)(s^2+2s+4)}$
$G_d(s)'' = \frac{400}{(s^2+2s+4)}$
180
160
140
120
100
80
60
40
20
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

- Approssimazione a polo equivalente

–Hp: poli reali negativi

$$G(s) = \frac{\mu}{(1+s\tau_1)(1+s\tau_2)\cdots(1+s\tau_n)} \qquad \tau_i > 0$$

$$G_e(s) = \frac{\mu}{1+s\tau_e}, \qquad \tau_e := \sum_{i=1}^{n} \tau_i$$

Componente dominante $t_a \simeq 5\tau_e$

- Esempio

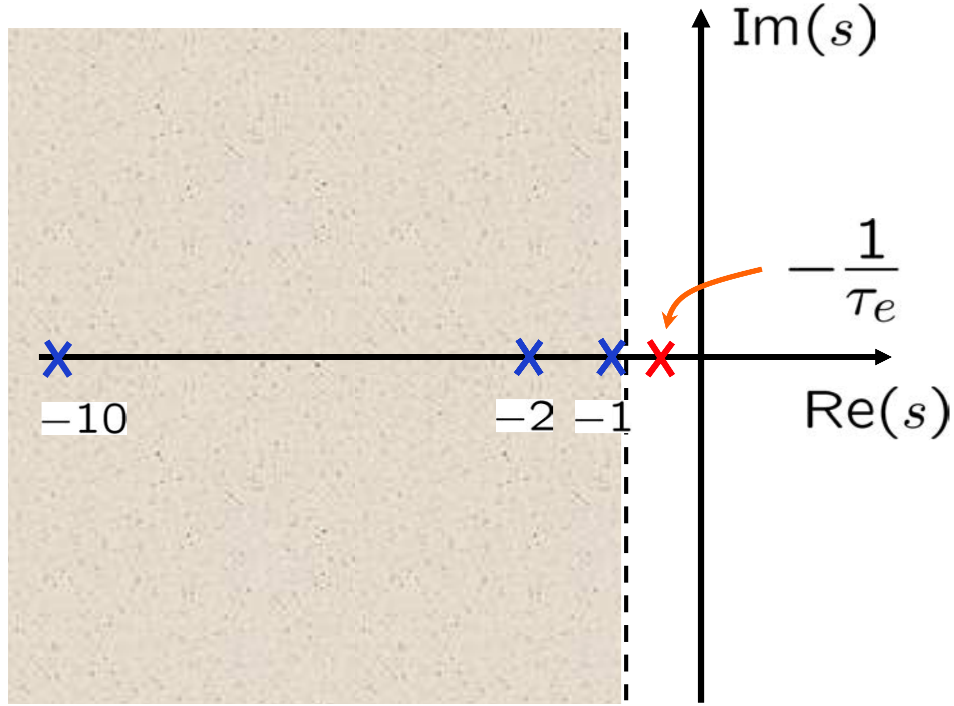
$$G(s) = \frac{1}{(1+s)(1+0.1s)(1+0.5s)}$$

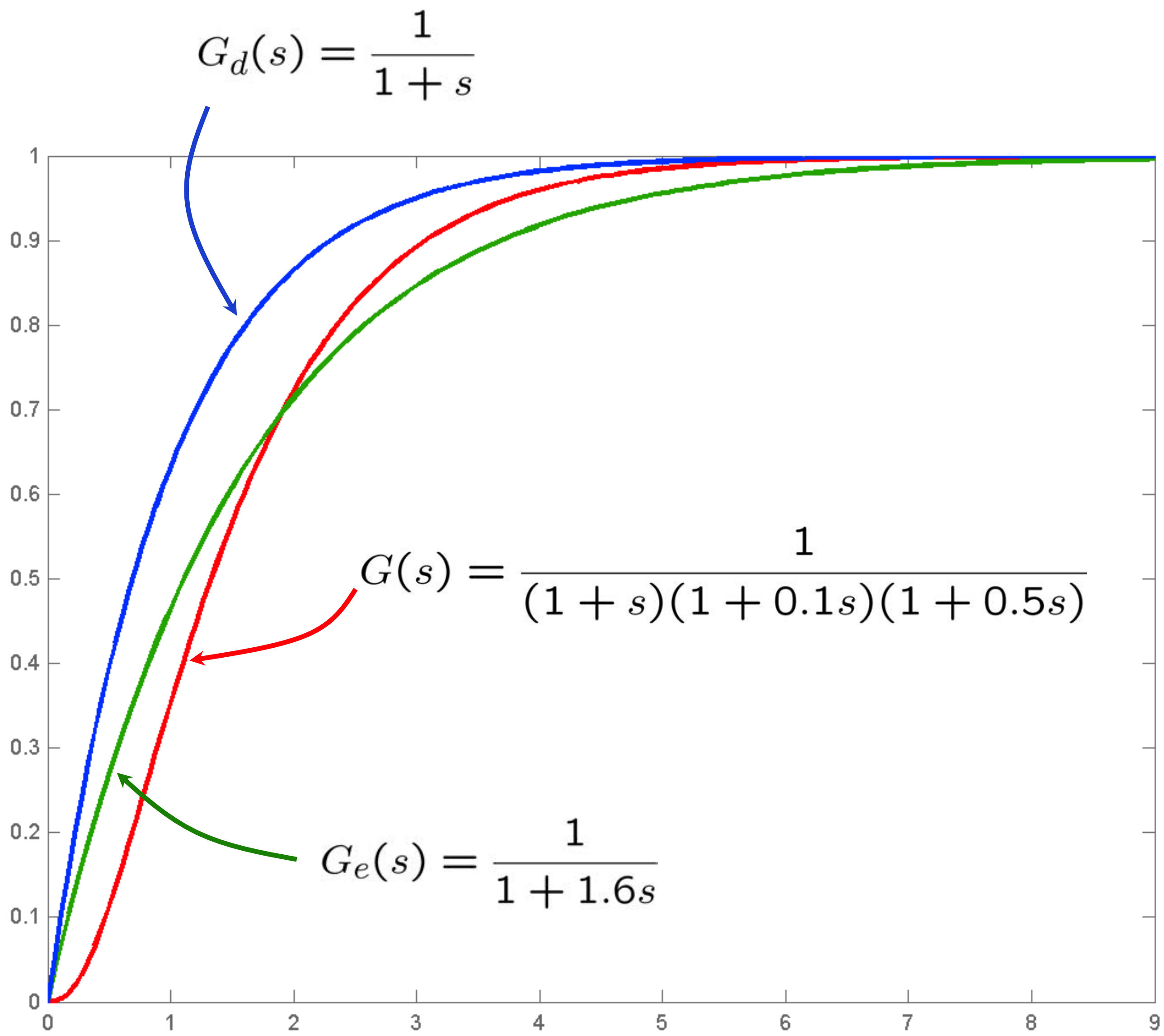
$$\mu = G(0) = 1$$

Poli: -1

-2

-10




$G_d(s) = \frac{1}{1+s}$
$G(s) = \frac{1}{(1+s)(1+0.1s)(1+0.5s)}$
$G_e(s) = \frac{1}{1+1.6s}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
1
2
3
4
5
6
7
8
9