Exercises 1

key words: Continuous functions, differentiable functions, continuous nowhere differentiable functions, Wieierstrass' example, Dini's derivatives, angle and cusp points, monotone functions, differentiability almost everywhere of monotone functions (Lebesgue's theorem), Vitali's covering lemma, right and left invisible points, Fubini's theorem on differentiability of a series.

1) Let

$$\mathcal{D} = \{f \in \mathcal{C}([0,1]) \,:\, (\exists x \in [0,1) \,:\, D^+ f(x), D_+ f(x) \in \mathbb{R})\}$$

and

$$\mathcal{C}_n = \{f \in \mathcal{C}([0,1]) \,:\, (\exists x \in [0, 1 - \frac{1}{n}] \,:\, \forall h \in (0, \frac{1}{n}] \,,\, |\frac{f(x+h) - f(x)}{h}| \le n)\}.$$

Prove that $\mathcal{D} = \bigcup_{n=1}^{+\infty} \mathcal{C}_n$.

2) Let $f : [a,b] \to \mathbb{R}$ a continuous function and let

$$I_r = \{x \in (a,b) \,:\, (\exists \xi \in (x,b) \,:\, f(x) < f(\xi))\}.$$

i) I_r is an open set and consequently $I_r = \cup_n (\alpha_n, \beta_n)$, where $((\alpha_n, \beta_n))_n$ is a finite or countable set (i.e a sequence) of open pairwise disjoint intervals.

ii) For all n, $f(\alpha_n) \le f(\beta_n)$.

3) Let $A \subseteq \mathbb{R}$ a measurable set and let $\rho \in [0,1)$. Suppose that, for all $(\alpha, \beta) \subseteq \mathbb{R}$,

$$\lambda(A \cap (\alpha, \beta)) \le \rho(\beta - \alpha),$$

where λ denotes the Lebesgue measure.

Prove that $\lambda(A) = 0$.

4) Let $x \in [0,1)$. We know that there exists a unique sequence $(a_j)_{j \in \mathbb{N}^*}$ such that

i) for all j, $a_j \in \{0,1,2\}$;

ii) $(a_j)_j$ is not definitively equal to 2 (i.e. $\forall \bar{j} \,, \exists j > \bar{j} \,:\, a_j \ne 2$);

iii) $x = \sum_{j=1}^{+\infty} \frac{a_j}{3^j}$.

Define $\varphi : [0,1] \to \mathbb{R}$ such that

$$\varphi(x) = \begin{cases} \dfrac{1}{2} & \text{if } a_1 = 1, \\ \displaystyle\sum_{j=1}^{k-1} \frac{a_j}{2^{j+1}} + \frac{1}{2^k} & \text{if } a_k = 1 \text{ and } a_j \neq 1 \text{ for all } j \leq k, \\ \displaystyle\sum_{j=1}^{+\infty} \frac{a_j}{2^{j+1}} & \text{if } a_j \neq 1 \text{ for all } j \\ 1 & \text{if } x = 1 \end{cases}$$

(φ is the so called *Cantor's function* or also *Lebesgue's singular function*).

i) Prove that φ is not decreasing and continuous and that $\varphi([0,1]) = [0,1]$.

ii) Compute $\varphi'(x)$ for all $x \in [0,1]$.