Exercises 2

key words: functions with bounded variation. Properties of BV functions, differentiability of BV functions. Integral function of a L^1 function. The integral function of a L^1 function is a BV function. Derivative of an integral function.

1) Let $f \in BV([a,b])$ and let $c \in]a,b[$. Prove that $f\big|_{[a,c]}$ and $f\big|_{[c,b]}$ are BV functions. Prove that $V_a^b(f) = V_a^c(f) + V_c^b(f)$.

2) Let $f \in BV([a,b])$ and let $x_0 \in]a,b[$. Prove that f is continuous at x_0 if and only if the function $x \mapsto V_a^x(f)$ is continuous at x_0.

3) Prove that Lipschitz continuous functions are BV functions. Estimate the total variation of f in terms of the Lipschitz constant of f.

4) Let $f \in C^1([a,b])$. Prove that $f \in BV([a,b])$ and $V_a^b(f) = \int_a^b |f'(t)|\, dt$.

5) Let $f \in C([a,b])$. Prove that $f \in BV([a,b])$ if and only if the curve $t \mapsto (t, f(t))$ is a rectifiable curve.

6) Let $f \in BV([a,b])$. Define $\|f\|_{BV} = |f(a)| + V_a^b(f)$. Prove that $\|\cdot\|_{BV}$ is a norm and that $BV([a,b])$ is a Banach space.

7) Consider, for α, $\beta > 0$,

$$f(x) = \begin{cases} x^\alpha \sin(x^{-\beta}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Find the values of α and β such that $f \in BV([0,1])$.

8) Let $\sum_n g_n$, $\sum_n h_n$ be two real absolutely convergent series. Let $(x_n)_n$ be a sequence with values in $[a,b]$. Define

$$\psi(x) = \sum_{x_n < x} g_n + \sum_{x_n < x} h_n.$$

The function ψ is said to be a *jump function* on $[a,b]$. Prove that $\psi \in BV([a,b])$. Find $V_b^a(f)$.

9) Prove that $f \in BV([a,b])$ if and only if f is the sum of a BV continuous function with a jump function.