Some more exercises on distributions

1) Let h be a function in $\mathcal{C}^1([0,1]\times\mathbb{R})$. Let $g:\mathbb{R}\to\mathbb{R}$ such that

$$g(x)=\int_0^1 h(s,x)\,ds.$$

Prove that g is in $\mathcal{C}^1(\mathbb{R})$ and

$$g'(x)=\int_0^1 \frac{\partial h}{\partial x}(s,x)\,ds.$$

2) Let h be a function in $\mathcal{C}^\infty(\mathbb{R})$. Let $g:\mathbb{R}\to\mathbb{R}$ such that

$$g(x)=\begin{cases}\dfrac{h(x)-h(0)}{x} & \text{if } x\neq 0,\\ h'(0) & \text{if } x=0.\end{cases}$$

Prove that g is in $\mathcal{C}^\infty(\mathbb{R})$.

3) Let $\tau_a\varphi$ the function $\tau_a\varphi(x)=\varphi(x-a)$ where $\varphi\in\mathcal{D}(\mathbb{R})$ and, similarly, $\tau_a T(\varphi)=T(\tau_{-a}\varphi)$ if $T\in\mathcal{D}'(\mathbb{R})$. Prove that, given $T\in\mathcal{D}'(\mathbb{R})$,

$$T'=\lim_n\, n(\tau_{-\frac{1}{n}}T-T),$$

where the limit is in the weak topology.

4) Prove that if f is an increasing function defined on $\mathbb{R}$ then its derivative in the sense of distribution is a positive Radon measure.

5) A distribution $T\in\mathcal{D}'(\mathbb{R}^n)$ is said to be positively homogeneous of degree k if, for all $\lambda>0$,

$$T(\frac{1}{\lambda^n}\varphi(\frac{\cdot}{\lambda}))=\lambda^k T(\varphi).$$

Prove that δ_0 is positively homogeneous of degree $-n$ (where $\delta_0\in\mathcal{D}'(\mathbb{R}^n)$) and $\mathrm{PV}\frac{1}{x}$ is positively homogeneous of degree -1.

6) Prove that, given $f\in\mathcal{C}^\infty(\Omega)$ and $a\in\Omega$

$$f\cdot\delta_a=f(a)\delta_a,$$

where $f\cdot\delta_a$ is the multiplication the distribution δ_a with the smooth function f, i. e.

$$f\cdot\delta_a(\varphi)=\delta_a(f\varphi).$$

7) Prove that, on $\mathbb{R}$,

$$\mathrm{Id}_{\mathbb{R}} \cdot \delta_0' = -\delta_0 \quad \text{and} \quad \mathrm{Id}_{\mathbb{R}} \cdot \mathrm{PV}_{\frac{1}{x}} = 1,$$

where $\mathrm{Id}_{\mathbb{R}}$ is the identity on $\mathbb{R}$, i. e. $\mathrm{Id}_{\mathbb{R}}(x) = x$ for all $x \in \mathbb{R}$.

8) Let $f \in \mathcal{C}^\infty(\mathbb{R})$. Prove that there exist $\alpha \in \mathbb{R}$ and $g \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$f \cdot \mathrm{PV}_{\frac{1}{x}} = \alpha \mathrm{PV}_{\frac{1}{x}} + g.$$

(Hint. May be exercise 2 is useful.)

9) Find all the distributions in $T \in \mathcal{D}'(\mathbb{R})$ such that $x \cdot T = 0$.

10) Find all the distributions in $T \in \mathcal{D}'(\mathbb{R})$ such that $x \cdot T = 1$.

11) Let $f\ :\ ]a, b[\to \mathbb{R}$ a piecewise $\mathcal{C}^1$. Compute T_f'.

12) Prove that, given I open interval of $\mathbb{R}$, the distributions on I such that $T' = 0$ are only the distributions associated to constant functions. Prove also that for all $T \in \mathcal{D}'(I)$ there exists $S \in \mathcal{D}'(I)$ such that $T' = S$.

13) Find all the distributions T in $\mathcal{D}'(\mathbb{R})$ such that

$$\mathrm{Id}_{\mathbb{R}} \cdot T' + T = 0.$$

14) Let f a function in $L^1_{loc}(\mathbb{R})$. Consider

$$F(x) = \int_0^x f(t)\, dt.$$

Compute the derivative of F in the sense of distributions.