Advanced Analysis - mod. B

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1

1.1 Continuous nowhere differentiable functions

Let $f : I \to \mathbb{R}$, where I is an open interval in $\mathbb{R}$. Let $x_0 \in I$. We recall two fundamental notions.

Definition 1. *f is continuous at x_0 if*

$$\lim_{x \to x_0} f(x) = f(x_0).$$

f is differentiable at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \in \mathbb{R}.$$

f continuous or differentiable (on I) means continuous or differentiable at every point of I.

It is well known that a differentiable function (at a point) is continuous (at the same point), but vice versa is not true. Anyhow immediate examples are available only in the case that the point in which a function is continuous and not differentiable is an isolated point in the domain of the function (e.g. the function $x \mapsto |x|$ at the point 0).

To Karl Weierstrass it is due the first example of *everywhere continuous nowhere differentiable function.* The proof of the following theorem can be found in [7, § 17]

Theorem 1 (Weierstrass' example)**.** *Let*

$$f(x) = \sum_{k=0}^{+\infty} b^k \cos(a^k \pi x), \qquad \text{with} \quad ab > 1 + \frac{3}{2}\pi \quad \text{and} \quad 0 < b < 1.$$

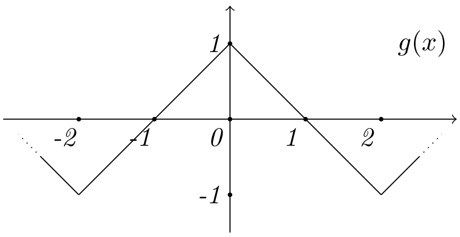
Then f is continuous and not differentiable at every point of $\mathbb{R}$.

We present here a similar example, may be easier from the point of view of computation, taken from [11].

Figure 1: Karl Theodor Wilhelm Weierstrass (1815–1897)

Example 1. *Let $g : \mathbb{R} \to \mathbb{R}$, periodic of period 4 and such that*

$$g(x) = \begin{cases} 1 - x & \text{if} \quad x \in [0, 2], \\ 1 + x & \text{if} \quad x \in [-2, 0]. \end{cases}$$



We define

$$f(x) = \sum_{n=1}^{+\infty} 2^{-n} g(2^{2^n} x).$$

f is continuous: in fact all the functions $x \mapsto 2^{-n} g(2^{2^n} x)$ are continuous and the series is totally and, consequently, uniformly convergent, so that the limit is continuous.

We want to prove that f is not differentiable at any point. Let $\bar{x} \in \mathbb{R}$, $k \in \mathbb{N}$. Consider $2^{2^k} \bar{x}$. We have $2^{2^k} \bar{x} \in [2m, 2m + 2]$ for some $m \in \mathbb{Z}$. We choose $h = 2^{-2^k}$ or $h = -2^{-2^k}$ depending on the fact that

$$2^{2^k} \bar{x}, \; 2^{2^k} \bar{x} + 1 \in [2m, 2m + 2],$$

or

$$2^{2^k} \bar{x}, \; 2^{2^k} \bar{x} - 1 \in [2m, 2m + 2].$$

Suppose we are in the first case. We consider

$$|g(2^{2^n}(\bar{x}+h))-g(2^{2^n}\bar{x})| = |g(2^{2^n}\bar{x}+2^{2^n-2^k}))-g(2^{2^n}\bar{x})| = \begin{cases} 0 & \text{if } n > k, \\ 1 & \text{if } n = k, \\ \leq 2^{2^n-2^k} & \text{if } n < k. \end{cases}$$

Consequently

$$f(\bar{x}+h) - f(\bar{x}) = \sum_{n=1}^{k} 2^{-n}(g(2^{2^n}(\bar{x}+h)) - g(2^{2^n}\bar{x})),$$

so that

$$\begin{aligned} |f(\bar{x}+h) - f(\bar{x})| &\geq 2^{-k} - \sum_{n=1}^{k-1} 2^{-n} \cdot 2^{2^n-2^k} \\ &\geq 2^{-k} - (k-1)2^{2^{k-1}-2^k} \\ &\geq 2^{-k} - (k-1)2^{-2^{k-1}}, \end{aligned}$$

and finally

$$|\frac{f(\bar{x}+h) - f(\bar{x})}{h}| \geq 2^{-k+2^k} - (k-1)2^{2^{k-1}}.$$

In this way, for all $\bar{x} \in \mathbb{R}$, we construct a sequence $(h_k)_k$ in $\mathbb{R}$, such that $\lim_k h_k = 0$ *and*

$$\lim_k |\frac{f(\bar{x}+h_k) - f(\bar{x})}{h_k}| = +\infty.$$

This implies that f is not differentiable at $\bar{x}$.

Remark 1. *The technique of considering series of functions which are rescaled in size and in the variable is typical in the construction of nowhere differentiable functions (see e.g. [9]). Like in the case of Von Koch curve, the graph of such functions is, in general, a fractal set.*

1.2 How many continuous nowhere differentiable functions are there?

The content of this paragraph can be found in [7, §17]. Nowhere differentiabilty is, in some sense, the normal situation for a continuous function. We will prove, in fact, that the set of continuous nowhere differentiable functions is the complement of a set which is contained in the countable union of closed sets with empty interior, in the metric space of continuous functions with the sup-distance.

Definition 2. *Let $\varphi : I \to \mathbb{R}$, with I open interval in $\mathbb{R}$, and let $x_0 \in I$. We define*

$$\liminf_{x\to x_0^+} \varphi(x) = \sup_{t>0}\{\inf_{x_0<x<x_0+t}\{\varphi(x)\}\},$$
$$\limsup_{x\to x_0^+} \varphi(x) = \inf_{t>0}\{\sup_{x_0<x<x_0+t}\{\varphi(x)\}\},$$
$$\liminf_{x\to x_0^-} \varphi(x) = \sup_{t>0}\{\inf_{x_0-t<x<x_0}\{\varphi(x)\}\},$$
$$\limsup_{x\to x_0^-} \varphi(x) = \inf_{t>0}\{\sup_{x_0-t<x<x_0}\{\varphi(x)\}\}.$$

Figure 2: The statue of Ulisse Dini (1845–1918) in Pisa

Definition 3. *Let $f : I \to \mathbb{R}$, with I open interval in $\mathbb{R}$, and let $x_0 \in I$. We define*

$$D^+ f(x_0) = \limsup_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}, \qquad D_+ f(x_0) = \liminf_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0},$$

$$D^- f(x_0) = \limsup_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}, \qquad D_- f(x_0) = \liminf_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

$D^+ f(x_0)$, $D_+ f(x_0)$, $D^- f(x_0)$, $D_- f(x_0)$ are the so called Dini's derivatives of the function f at the point x_0.

We have, for every function f and point x_0,

$$-\infty \le D_+ f(x_0) \le D^+ f(x_0) \le +\infty, \quad -\infty \le D_- f(x_0) \le D^- f(x_0) \le +\infty.$$

Remark that a function is differentiable at x_0 if and only if

$$D_- f(x_0) = D^- f(x_0) = D_+ f(x_0) = D^+ f(x_0) \in \mathbb{R}.$$

Theorem 2. *Denote by $C([0,1], \mathbb{R})$ the space of continuous functions on $[0,1]$ with the* sup*-distance. Let*

$$D = \{f \in C([0,1], \mathbb{R}) \mid \exists\, x \in [0,1[\, : \ D_+ f(x),\ D^+ f(x) \in \mathbb{R}\}.$$

Then D is contained in the union of a sequence of closed sets with empty interior.

Proof. First of all we notice that the set of functions which are differentiable (from the right) at least in one point, is contained in D so that the set of continuous nowhere differentiable functions contains the complement of D.

Let

$$C_n = \{f \in C([0,1],\mathbb{R}) \mid \exists\, x \in [0, 1 - \frac{1}{n}] \,:\, \forall\, h \in]0, \frac{1}{n}]\,,\ |\frac{f(x+h) - f(x)}{h}| \leq n\}.$$

Obviously $C_n \subseteq D$.

We prove that $D \subseteq \bigcup_n C_n$. Let $f \in D$. Then there exists $\bar{x} \in [0,1[$ and there exists C, $C' \in \mathbb{R}$, with $C' < C$, such that

$$\inf_{t>0}\{\sup_{\bar{x}<x<\bar{x}+t}\{\frac{f(x) - f(\bar{x})}{x - \bar{x}}\}\} < C$$

and

$$\sup_{s>0}\{\inf_{\bar{x}<x<\bar{x}+s}\{\frac{f(x) - f(\bar{x})}{x - \bar{x}}\}\} > C'.$$

In particular there exists $\bar{t} > 0$ such that

$$\sup_{\bar{x}<x<\bar{x}+\bar{t}}\{\frac{f(x) - f(\bar{x})}{x - \bar{x}}\} < C$$

and this means that for all $x \in]\bar{x}, \bar{x} + \bar{t}[$ we have

$$\frac{f(x) - f(\bar{x})}{x - \bar{x}} < C.$$

Similarly there exists $t^* > 0$ such that, for all $x \in]\bar{x}, \bar{x} + t^*[$, we have

$$\frac{f(x) - f(\bar{x})}{x - \bar{x}} > C'.$$

Consequently there exists α, $\delta > 0$ such that, for all $h \in]0, \delta]$,

$$|\frac{f(\bar{x} + h) - f(\bar{x})}{h}| \leq \alpha,$$

and this implies that $f \in C_n$ for some n.

We prove that C_n is closed. Let n fixed and let $f \in \bar{C_n}$ where $\bar{C_n}$ denotes the closure of C_n in the space $C([0,1],\mathbb{R})$ (remember that the distance in $C([0,1],\mathbb{R})$ is $d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)| = \|f - g\|_\infty$). There exists a sequence $(f_k)_k$ in C_n which converges uniformly to f. We have that, for each k, there exists a point x_k such that

$$x_k \in [0, 1 - \frac{1}{n}] \text{ and for all } h \in]0, \frac{1}{n}]\,,\ |\frac{f_k(x_k + h) - f_k(x_k)}{h}| \leq n.$$

Passing to a subsequence, we can suppose that there exists $\bar{x} \in [0, 1 - \frac{1}{n}]$ such that $x_k \to \bar{x}$. We fix now $h \in]0, \frac{1}{n}]$, we fix $\varepsilon > 0$ and we choose k in such a way that

$$\|f - f_k\|_\infty \leq \frac{\varepsilon h}{4}, \quad |f(x_k) - f(\bar{x})| \leq \frac{\varepsilon h}{4}, \quad |f(\bar{x} + h) - f(x_k + h)| \leq \frac{\varepsilon h}{4}.$$

Consequently

$$\begin{aligned}
&|f(\bar{x}+h)-f(\bar{x})| \\
&\quad \leq |f(\bar{x}+h)-f(x_k+h)|+|f(x_k+h)-f_k(x_k+h)| \\
&\qquad +|f_k(x_k+h)-f_k(x_k)|+|f_k(x_k)-f(x_k)|+|f(x_k)-f(\bar{x})| \\
&\qquad\quad \leq \frac{\varepsilon h}{4}+\frac{\varepsilon h}{4}+nh+\frac{\varepsilon h}{4}+\frac{\varepsilon h}{4} \\
&\qquad\quad \leq nh+\varepsilon h.
\end{aligned}$$

Since that last inequality holds for every $\varepsilon > 0$, we deduce that $f \in C_n$.

We prove finally that C_n has an empty interior. By contradiction suppose that there exists n, there exists $f \in C_n$ and there exists $\varepsilon > 0$ such that the ball $B(f,\varepsilon) = \{g \in C([0,1],\mathbb{R}) \mid \|g-f\|_\infty < \varepsilon\}$ is contained in C_n. Using the (Stone-)Weierstrass Theorem (see [13, Ch. 7]), there exists a polynomial p on $[0,1]$ such that $\|f-p\|_\infty < \varepsilon$. Let $\delta = \varepsilon - \|f-p\|_\infty$. As a consequence

$$B(p,\delta) \subseteq B(f,\varepsilon) \subseteq C_n.$$

We construct now a function $g \in C([0,1],\mathbb{R})$ such that $\|g\|_\infty < \delta$, g has a finite right derivative $g'_+(x)$ at each point x of $[0,1[$ and, for all $x \in \ [0,1[$,

$$|g'_+(x)| > n + \|p'\|_\infty,$$

(to find such a function g it is sufficient to take a suitable sawtooth function). Then we have $p+g \in C_n$, $(p+g)'_+ = p' + g'_+$ and, for all $x \in [0,1[$,

$$|(p+g)'_+(x)| \geq |g'_+(x)| - \|p'\|_\infty > n,$$

which is a contradiction. This completes the proof. □

1.3 How many angles or cusps are there?

The content of this paragraph can be found in [7, §17].

Definition 4. *Let $f : I \to \mathbb{R}$ a function, where I is an open interval in $\mathbb{R}$. Let $\bar{x} \in I$. Let f continuous at $\bar{x}$. $\bar{x}$ is called angle point (in italian: punto angoloso) if there exists*

$$\lim_{x\to\bar{x}^+} \frac{f(x)-f(\bar{x})}{x-\bar{x}} = f'_+(\bar{x}), \qquad \lim_{x\to\bar{x}^-} \frac{f(x)-f(\bar{x})}{x-\bar{x}} = f'_-(\bar{x}),$$

$f'_+(\bar{x})$ or $f'_-(\bar{x}) \in \mathbb{R}$ and $f'_+(\bar{x}) \neq f'_-(\bar{x})$.

$\bar{x}$ is called cusp point (in italian: punto di cuspide) if there exists $f'_+(\bar{x})$, $f'_-(\bar{x})$, they are different and both of them are infinite.

Take now a continuous nowhere differentiable function. We pose the following question: "how many angles or cusps are there?" We will see that there is only a finite or countable set of such points. This means that the real cause of nowhere differentiability is not the presence of angles or cusps, but the fact that for almost every point the limit of the difference quotient (from the left and from the right) does not exists. The real cause is oscillation.

Theorem 3. *Let $f : I =]a, b[\to \mathbb{R}$ a function. There exists at most a countable set of points in I in which f'_+ and f'_- exist and are different.*

Proof. Let

$$A = \{x \in I \mid f'_+(x),\ f'_-(x) \text{ exist and } f'_+(x) > f'_-(x)\},$$

and

$$B = \{x \in I \mid f'_+(x),\ f'_-(x) \text{ exist and } f'_+(x) < f'_-(x)\}.$$

We show that A is at most countable. The case of B will be similar. Let $x \in A$. We choose

$$r_x \in \mathbb{Q} \quad \text{such that} \quad f'_-(x) < r_x < f'_+(x).$$

We choose $t_x,\ s_x \in \mathbb{Q}$ such that $a < t_x < x < s_x < b$ and

$$\frac{f(y) - f(x)}{y - x} < r_x \quad \text{for all} \quad t_x < y < x,$$

$$\frac{f(y) - f(x)}{y - x} > r_x \quad \text{for all} \quad x < y < s_x.$$

Consequently

$$f(y) - f(x) > r_x(y - x) \quad \text{for all} \quad y \in]t_x,\ s_x[\ \text{ with } y \neq x.$$

Consider now

$$\Phi : A \to \mathbb{Q}^3, \quad x \mapsto \Phi(x) = (r_x,\ t_x,\ s_x).$$

We claim that Φ is injective and consequently A will be countable (since $\mathbb{Q}^3$ is countable). By contradiction, suppose there exists $\bar{x} \neq x$, e.g. $\bar{x} < x$ such that

$$(r_{\bar{x}},\ t_{\bar{x}},\ s_{\bar{x}}) = (r_x,\ t_x,\ s_x);$$

this means that

$$\text{for all } y \in]t_x,\ s_x[, \quad y \neq \bar{x} \quad \Rightarrow \quad f(y) - f(\bar{x}) > r_x(y - \bar{x}) \tag{1}$$

and

$$\text{for all } y \in]t_x,\ s_x[, \quad y \neq x \quad \Rightarrow \quad f(y) - f(x) > r_x(y - x). \tag{2}$$

Considering (1) with $y = x$ we have

$$f(x) - f(\bar{x}) > r_x(x - \bar{x}),$$

and considering (2) with $y = \bar{x}$ we have

$$f(\bar{x}) - f(x) > r_x(\bar{x} - x),$$

obtaining a contradiction. The proof is complete.

□

Figure 3: Henri Léon Lebesgue (1875–1941)

2

2.1 Differentiation of monotone functions

The content of this paragraph can be found in [10, Ch. 9] (see also the Ch. 6 of the italian edition of [10]) and in[7, §17].

As we have seen, differentiability can be destroyed by oscillations. Monotonicity may be considered the opposite of oscillations. This general idea is, in some sense, confirmed by the following fact: monotone functions are almost everywhere differentiable.

Theorem 4 (Lebesgue's differentiation theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a monotone function.*

Then f is almost everywhere differentiable.

Proof. We give here a proof in the case of *continuous* monotone functions. We need the following definition.

Definition 5. *Let $g : [a, b] \to \mathbb{R}$ be a continuous function. Let $x_0 \in [a, b[$. We say that x_0 is invisible from the right (or right-invisible) if there exists $\xi \in]x_0, b]$ such that $g(\xi) > g(x_0)$. Similarly $x_0 \in]a, b]$ is invisible from the left (or left-invisible) if there exists $\xi \in [a, x_0[$ such that $g(\xi) > g(x_0)$.*

The set of right-invisible points is described in the following lemma, the proof of which is let as an exercise.

Lemma 1. *Let $g : [a, b] \to \mathbb{R}$ be a continuous function. The set I_r of right-invisible points contained in $]a, b[$ is an open set. Moreover I_r is the union of a finite or countable set of pairwise disjoint open intervals $]\alpha_k, \beta_k[$ and, for all k, $g(\alpha_k) \le g(\beta_k)$.*

Let's go back to the proof of Lebesgue's differentiation theorem. Suppose that f is a continuous increasing function defined on $[a, b]$. Consider $\tilde{f} : [-b, -a] \to \mathbb{R}$, $\tilde{f}(x) = -f(-x)$. Also $\tilde{f}$ is a continuous increasing function. Let $x_0 \in\ ]a, b[$. We have

$$D^+ f(x_0) = D^- \tilde{f}(-x_0), \qquad D_+ f(x_0) = D_- \tilde{f}(-x_0),$$

$$D^- f(x_0) = D^+ \tilde{f}(-x_0), \qquad D_- f(x_0) = D_+ \tilde{f}(-x_0).$$

We know that

$$D_- f(x_0) \le D^- f(x_0), \qquad D_+ f(x_0) \le D^+ f(x_0).$$

If we are able to prove that, for a continuous increasing function, in a point x_0,

$$D^- f(x_0) \le D_+ f(x_0),$$

then the same will be true for $\tilde{f}$ in the point $-x_0$, so that

$$D^+ f(x_0) = D^- \tilde{f}(-x_0) \le D_+ \tilde{f}(-x_0) = D_- f(x_0)$$

and consequently

$$D^+ f(x_0) \le D_- f(x_0) \le D^- f(x_0) \le D_+ f(x_0) \le D^+ f(x_0).$$

As a conclusion we have that, for proving the theorem, it will be sufficient to prove that

i) $D^- f(x) \le D_+ f(x)$, for almost every $x \in\ ]a, b[$;

ii) $D^+ f(x) \in \mathbb{R}$, for almost every $x \in\ ]a, b[$.

Let's show ii). Suppose $x_0 \in\ ]a, b[$ such that $D^+ f(x_0) = +\infty$ (remember that f is increasing, so that all Dini's derivatives are non negative). We have

$$\limsup_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = +\infty,$$

so that, for all $t > 0$,

$$\sup_{x_0 < x < x_0 + t} \frac{f(x) - f(x_0)}{x - x_0} = +\infty.$$

Consequently, for all $C > 0$, there exists $\xi > x_0$ such that

$$\frac{f(\xi) - f(x_0)}{\xi - x_0} > C,$$

and then

$$f(\xi) - C\xi > f(x_0) - Cx_0.$$

This last inequality means that if $D^+f(x_0) = +\infty$ then x_0 is right-invisible with respect to the function $x \mapsto f(x) - Cx$. Let's denote by $I_{r,C}$ this set. We use Lemma 1 and we obtain that

$$I_{r,C} = \bigcup_n \,]\alpha_n, \beta_n[$$

and, for all n, $f(\alpha_n) - C\alpha_n \le f(\beta_n) - C\beta_n$. Consequently, for all n,

$$\beta_n - \alpha_n \le \frac{f(\beta_n) - f(\alpha_n)}{C},$$

and, since the intervals $]\alpha_n, \beta_n[$ are pairwise disjoint, we have

$$\lambda(I_{r,C}) = \sum_n \beta_n - \alpha_n \le \sum_n \frac{f(\beta_n) - f(\alpha_n)}{C} \le \frac{f(b) - f(a)}{C},$$

where $\lambda(I_{r,C})$ denotes the Lebesgue measure of the set $I_{r,C}$. Notice that in the last inequality it is essential that f is increasing. Finally denote by A the set of the points $x \in \,]a, b[$ such that $D^+f(x) = +\infty$. We have

$$A \subseteq I_{r,C}, \qquad \text{for all } C > 0.$$

Consequently, for all $C > 0$, A is contained in a set with Lebesgue measure less of equal than $\frac{f(b)-f(a)}{C}$. This implies that A is measurable and $\lambda(A) = 0$.

We prove now i). This condition is equivalent, for an increasing function f, to prove that

i') $D^+f(x) \le D_-f(x)$, for almost every $x \in \,]a, b[$;

Let

$$\tilde{E} = \{x \in \,]a, b[\; | \; D_-f(x) < D^+f(x)\}.$$

We have to prove that $\lambda(\tilde{E}) = 0$. Consider $c,\ C \in \mathbb{Q}$, with $0 < c < C$; denote by

$$E_{c,C} = \{x \in \,]a, b[\; | \; D_-f(x) < c < C < D^+f(x)\}.$$

If, for all such $c,\ C$, we have $\lambda(E_{c,C}) = 0$, then $\lambda(\tilde{E}) = 0$, in fact

$$\tilde{E} = \bigcup_{\substack{c,\ C\in\mathbb{Q} \\ 0<c<C}} E_{c,C}.$$

Let $\rho = \frac{c}{C}$. We will show that, for all $\,]\alpha, \beta[\, \subseteq \,]a, b[$,

$$\lambda^*(E_{c,C} \cap \,]\alpha, \beta[\,) \le \rho\,(\beta - \alpha), \tag{3}$$

where λ^* denotes the outer Lebesgue measure. From this we obtain the conclusion using the following lemma, the proof of which is let as an exercise.

Lemma 2. *Let $\rho \in \,]0, 1[$. Let A be a set in $]a, b[$ such that for all $\,]\alpha, \beta[\, \subseteq \,]a, b[$,*

$$\lambda^*(A \cap \,]\alpha, \beta[\,) \le \rho\,(\beta - \alpha).$$

Then $\lambda^*(A) = 0$. Consequently A is measurable and $\lambda(A) = 0$.

Let $x \in E_{c,C} \cap]\alpha, \beta[$. We have, in particular, that $D_- f(x) < c$ and consequently there exists $\xi \in]\alpha, x[$ such that

$$\frac{f(\xi) - f(x)}{\xi - x} < c.$$

This imply that

$$f(\xi) - c\,\xi > f(x) - c\,x,$$

i.e. x is left-invisible with respect to the function $x \mapsto f(x) - c\,x$, defined on $]\alpha, \beta[$. We denote by I_l the set of these points. We have

$$E_{c,C} \cap]\alpha, \beta[\subseteq I_l = \bigcup_k]\alpha_k, \beta_k[$$

with, for all k,

$$f(\alpha_k) - c\,\alpha_k \geq f(\beta_k) - c\,\beta_k,$$

or, equivalently, for all k,

$$c(\beta_k - \alpha_k) \geq f(\beta_k) - f(\alpha_k).$$

Suppose now $x \in E_{c,C} \cap]\alpha_k, \beta_k[$. We know that, in particular, $D^+ f(x) > C$. Consequently there exists $\eta \in]x, \beta_k[$ such that

$$\frac{f(\eta) - f(x)}{\eta - x} > C \qquad \text{i. e.} \qquad f(\eta) - C\,\eta > f(x) - C\,x.$$

This means that x is right-invisible with respect to the function $x \mapsto f(x) - C\,x$, defined on $]\alpha_k, \beta_k[$. We denote by $I_{r,k}$ the set of these points. We have

$$I_{r,k} = \bigcup_j]\alpha_{k,j}, \beta_{k,j}[,$$

where the intervals $]\alpha_{k,j}, \beta_{k,j}[$ are pairwise disjoint, with, for all j,

$$f(\alpha_{k,j}) - C\,\alpha_{k,j} \leq f(\beta_{k,j}) - C\,\beta_{k,j}$$

and, consequently,

$$(\beta_{k,j} - \alpha_{k,j}) \leq \frac{f(\beta_{k,j}) - f(\alpha_{k,j})}{C}.$$

We have

$$\lambda(I_{r,k}) = \sum_j (\beta_{k,j} - \alpha_{k,j}) \leq \sum_j \frac{f(\beta_{k,j}) - f(\alpha_{k,j})}{C} \leq \frac{f(\beta_k) - f(\alpha_k)}{C}.$$

Finally

$$\begin{aligned} \lambda^*(E_{c,C} \cap]\alpha, \beta[\,) \leq \sum_k \lambda(I_{r,k}) \quad &\leq \sum_k \frac{f(\beta_k) - f(\alpha_k)}{C} \\ &\leq \frac{c}{C} \sum_k (\beta_k - \alpha_k) \leq \frac{c}{C}(\beta - \alpha) \end{aligned}$$

and (3) follows. $\square$

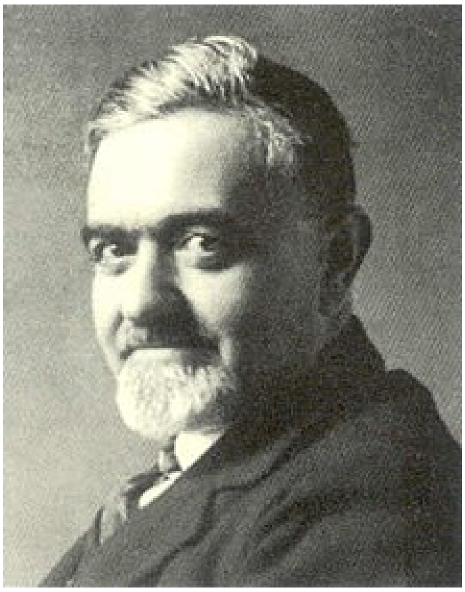


Figure 4: Giuseppe Vitali (1875–1932)

Remark 2. *The proof of Lebesgue's differentiation theorem in the case of non continuous functions can be obtained with a modification of the above proof. The key idea is that for monotone functions there exists, at each point $\bar{x}$, the limit from the right and from the left and, e.g. if f is increasing,*

$$\lim_{x\to\bar{x}^-} f(x) \leq f(\bar{x}) \leq \lim_{x\to\bar{x}^+} f(x).$$

The definition of right-invisible points has to be suitably modified, and so on.

In [7, §17] there is a proof (of Lebesgue's theorem for general monotone functions) with similar intricate reasoning as in the pages above. There the main point is the so called Vitali's covering lemma.

Definition 6. *Let A be a subset of $\mathbb{R}$. A family $\mathcal{F}$ of closed non degenerate (i. e. with strictly positive length) intervals of $\mathbb{R}$, is said Vitali's covering of A if for all $x \in A$ and for all $\varepsilon > 0$ there exists $I \in \mathcal{F}$ such that $x \in I$ and $\lambda(I) < \varepsilon$.*

Lemma 3 (Vitali's covering theorem)**.** *Let $A \subseteq \mathbb{R}$ and let $\mathcal{F}$ a (non empty) Vitali's covering of A.*

Then there exists a finite or countable set of pairwise disjoint elements of $\mathcal{F}$ (let's call it $\{I_n\}$) such that

$$\lambda(A \cap \mathcal{C}_{\mathbb{R}}(\bigcup_n I_n)) = 0$$

(here $\mathcal{C}_{\mathbb{R}}(\bigcup_n I_n)$ denotes the complement set of $\bigcup_n I_n$ in $\mathbb{R}$).

Suppose moreover that $\lambda(A) \leq +\infty$.

Then, for all $\varepsilon > 0$, there exists a finite set $\{I_1, \ldots, I_k\}$ of pairwise disjoint elements of $\mathcal{F}$ such that

$$\lambda(A \cap \mathcal{C}_{\mathbb{R}}(\bigcup_{i=1}^{k} I_i)) < \varepsilon.$$

Figure 5: Guido Fubini (1879–1943)

2.2 Fubini's theorem on the differentiation of a series

The content of this paragraph can be found in [7, §17]. An interesting consequence of Lebesgue's differentiation theorem is the following result on differentiation of functions defined as the sum of a series of monotone functions.

Theorem 5 (Fubini's theorem on the differentiation of a series)**.** *Let $(f_n)_n$ be a sequence of increasing functions defined on $[a,b]$, with value in $\mathbb{R}$. Note that for all n, f_n is differentiable a. e. in $[a,b]$, with derivative f'_n. Suppose that, for all $x \in [a,b]$, the series*

$$\sum_n f_n(x)$$

is convergent with sum $s(x)$. Then

i) the function $s : [a,b] \to \mathbb{R}$, $x \mapsto s(x)$, is almost everywhere differentiable;

ii) the series $\sum_n f'_n(x)$ is almost everywhere convergent;

iii) for almost every $x \in [a,b]$, $\sum_n f'_n(x) = s'(x)$.

Proof. Possibly considering $f_n(x) - f_n(a)$, it is not restrictive to suppose that $f_n(x) \geq 0$, for all $x \in [a,b]$. We set

$$s_n(x) = f_1(x) + \ldots + f_n(x),$$

$$r_n(x) = s(x) - s_n(x) = \sum_{j=n+1}^{+\infty} f_n(x).$$

Each function s_n is increasing and positive and, for all $x \in [a,b]$, the sequence $s_n(x)$ is convergent to $s(x)$. Then the function s is increasing and positive and consequently it is a. e. differentiable. We have

$$s'_n(x) = f'_1(x) + \ldots + f'_n(x)$$

and, for all n and a. e. $x \in [a,b]$, $f'_n(x) \geq 0$ (remember that f_{n+1} is increasing), so that, for a. e. $x \in [a,b]$,

$$s'_n(x) \leq s'_{n+1}(x).$$

Moreover, suppose $h > 0$ and x, $x+h \in [a,b]$, then

$$\frac{s(x+h)-s(x)}{h} = \frac{s_n(x+h)-s_n(x)}{h} + \frac{r_n(x+h)-r_n(x)}{h}$$

with $\frac{r_n(x+h)-r_n(x)}{h} \geq 0$, as r_n is increasing. Consequently

$$\frac{s(x+h)-s(x)}{h} \geq \frac{s_n(x+h)-s_n(x)}{h},$$

and, passing to the limit as h goes to 0, we have $s'_n(x) \leq s'(x)$. In conclusion, for a. e. $x \in [a,b]$,

$$0 \leq s'_n(x) \leq s'_{n+1}(x) \leq s'(x).$$

From the previous inequality we deduce immediately that the series $\sum_n f'_n(x)$ is a. e. convergent with sum less or equal than $s'(x)$. To conclude the proof it will be sufficient to show that the sequence $(s'_n)_n$ has a subsequence which converges a. e. to s'. Remark that the sequence $(s_n(b))_n$ is an increasing sequence converging to $s(b)$. As a consequence the sequence $(s(b)-s_n(b))_n$ is decreasing and converging to zero. It is then possible to choose a sequence $(n_k)_k$ such that the series

$$\sum_{k=1}^{+\infty}((s(b)-s_{n_k}(b))$$

is convergent. Since for all $x \in [a,b]$, $0 \leq s(x)-s_{n_k}(x) \leq s(b)-s_{n_k}(b)$, then the series

$$\sum_{k=1}^{+\infty}((s(x)-s_{n_k}(x)),$$

is convergent. We can then apply the already proved points i) and ii) of this theorem to the sequence of functions $(g_k)_k$ where $g_k(x) = s(x)-s_{n_k}(x)$. We have that, for almost every $x \in [a,b]$,

$$\sum_{k=1}^{+\infty} g'_k(x)$$

is convergent and then, for almost every $x \in [a,b]$,

$$\lim_k g'_k(x) = 0 \qquad \text{i. e.} \qquad s'_{n_k}(x) \to s'(x).$$

The proof is complete.

□

3

3.1 Bounded variation functions

The content of this paragraph can be found in [7, §17].

Definition 7. *Let* $f : [a, b] \to \mathbb{R}$ *(or* $\mathbb{C}$*). Consider*

$$\Delta = \{a = x_0 < x_1 < \ldots < x_n = b\},$$

a subdivision of $[a, b]$ *(a subdivision is a finite set of points in* $[a, b]$*, containing the points* a *and* b*). We define*

$$V(f, \Delta) = \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|$$

and

$$V_b^a(f) = \sup_{\Delta} V(f, \Delta).$$

The quantity $V_b^a(f)$ *is called total variation of the function* f*. If*

$$V_b^a(f) < +\infty$$

the function f *is a bounded variation function (in short: a* BV *function) on the interval* $[a, b]$*. The set of* BV *functions on* $[a, b]$ *is denoted by* $BV([a, b])$*.*

Remark 3. *If* $f : \mathbb{R} \to \mathbb{R}$ *or (or* $\mathbb{C}$*), we will consider*

$$V_{-\infty}^{+\infty}(f) = \sup_{a,\, b\, \in \mathbb{R},\ a<b} V_a^b(f).$$

We list here some properties of BV functions.

i) If $f \in BV([a, b])$, then f is bounded.

ii) $f \in BV([a, b])$ if and only if $\Re f \in BV([a, b])$ and $\Im f \in BV([a, b])$ (where $\Re z$ and $\Im z$ means real and imaginary part of z respectively).

iii) If $f \in BV([a, b])$ and $\alpha \in \mathbb{R}$ (or $\mathbb{C}$), then $\alpha f \in BV([a, b])$ and

$$V_a^b(\alpha f) = |\alpha|\, V_a^b(f).$$

iv) If $f,\, g \in BV([a, b])$, then $f + g \in BV([a, b])$ and

$$V_a^b(f + g) \le V_a^b(f) + V_a^b(g).$$

v) Let $f \in BV([a, b])$; if we set

$$\|f\|_{BV} = |f(a)| + V_a^b(f),$$

then $\|\cdot\|_{BV}$ is a norm and $BV([a, b])$ is a Banach space.

vi) Let $c \in]a, b[$. $f \in BV([a, b])$ if and only if $f_{|[a,c]} \in BV([a, c])$ and $f_{|[c,b]} \in BV([c, b])$; moreover

$$V_a^b(f) = V_a^c(f_{|[a,c]}) + V_c^b(f_{|[c,b]}) \quad \text{(we will write } V_a^b(f) = V_a^c(f) + V_c^b(f)\text{)}.$$

Figure 6: Camille Jordan (1838–1922)

vii) The function $x \mapsto V_a^x(f)$ is increasing. We denote this function by $V_a(f)$.

viii) If f is continuous in $\bar{x}$, then also $V_a(f)$ is continuous in $\bar{x}$.

We prove the point viii), while the other ones are let as an exercise. We show that $V_a(f)$ is continuous from the left, the continuity form the right being similar. We know that f is continuous from the left in $\bar{x} \in]a, b]$. In particular, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $x \in]\,\bar{x} - \delta,\ \bar{x}[$, then $|f(x) - f(\bar{x})| < \frac{\varepsilon}{2}$. Let

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = \bar{x}$$

such that

$$V_a^{\bar{x}}(f) - \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| < \frac{\varepsilon}{2}.$$

We can suppose, without loss of generality, that $x_{n-1} \in]\,\bar{x} - \delta,\ \bar{x}[$ (if it is not so, it is sufficient to add a point of the interval $]\,\bar{x} - \delta,\ \bar{x}[$ in the subdivision). Then

$$V_a^{x_{n-1}}(f) \geq \sum_{j=1}^{n-1} |f(x_j) - f(x_{j-1})|$$

or, equivalently,

$$-V_a^{x_{n-1}}(f) + \sum_{j=1}^{n-1} |f(x_j) - f(x_{j-1})| \leq 0.$$

Consequently

$$V_a^{\bar{x}}(f) - V_a^{x_{n-1}}(f) + \sum_{j=1}^{n-1} |f(x_j) - f(x_{j-1})| \leq V_a^{\bar{x}}(f)$$

and then

$$\begin{aligned}
&V_a^{\bar{x}}(f) - V_a^{x_{n-1}}(f) \\
&\quad \le V_a^{\bar{x}}(f) - \sum_{j=1}^{n-1} |f(x_j) - f(x_{j-1})| \\
&\qquad \le \underbrace{V_a^{\bar{x}}(f) - \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|}_{< \frac{\varepsilon}{2}} + \underbrace{|f(\bar{x}) - f(x_{n-1})|}_{< \frac{\varepsilon}{2}} < \varepsilon.
\end{aligned}$$

Since $x \mapsto V_a^x(f)$ is increasing, from the last inequality we deduce the left continuity of $V_a(f)$ at $\bar{x}$.

Theorem 6. *Let $f \in BV([a,b])$.*
Then f is the difference of two increasing functions.

Proof. We know that $V_a(f)$ is increasing. We show that $V_a(f) - f$ is increasing. In fact, for all $x,\ y \in [a,b]$, with $x < y$,

$$f(y) - f(x) \le |f(y) - f(x)| \le V_x^y(f) = V_a^y(f) - V_a^x(f)$$

and this implies

$$V_a^x(f) - f(x) \le V_a^y(f) - f(y).$$

□

Corollary 1 (differentiation of BV functions)**.** *Let $f \in BV([a,b])$.*
Then f is almost everywhere differentiable.

Proof. Since f is the difference of two increasing functions, it is sufficient to apply Lebesgue's theorem to these two functions. □

3.2 The integral function of a L^1 function is a BV function

The content of this paragraph can be find in [7, §18].

We prove that the integral function F of $f \in L^1$ is a BV function and the total variation of F is the L^1 norm of f.

Theorem 7. *Let $f \in L^1(a,b)$ (we denote by $L^1(a,b)$ the set of Lebesgue's integrable functions on the interval [a,b]). Let*

$$F(t) = \int_a^t f(x)\,dx = \int_{[a,t]} f$$

the so called integral function of f.
Then

i) F is uniformly continuous;

ii) F is a BV function and

$$V_a^b(F) = \int_{[a,b]} |f|.$$

Proof. Let $\bar{x} \in [a,b]$. Consider $(x_n)_n$ a sequence in $[a,b]$ such that $\lim_n x_n = \bar{x}$. Let

$$f_n(x) = \chi_{[a,x_n]}(x) f(x),$$

where $\chi_{[a,x_n]}$ is the characteristic function of the interval $[a, x_n]$. We have

$$f_n(x) \xrightarrow{n} \chi_{[a,\bar{x}]}(x) f(x) \quad \text{and} \quad |f_n(x)| \leq |f(x)|$$

for almost every $x \in [a,b]$. The dominated convergence theorem gives

$$\lim_n F(x_n) = \lim_n \int_{[a,b]} f_n = \int_{[a,\bar{x}]} f = F(\bar{x})$$

i. e. F is continuous. Since F is defined on $[a,b]$, F is uniformly continuous.

Let

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

We have

$$|F(x_j) - F(x_{j-1})| = |\int_{[x_{j-1},\, x_j]} f| \leq \int_{[x_{j-1},\, x_j]} |f|,$$

so that

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \leq \sum_{j=1}^{n} \int_{[x_{j-1},\, x_j]} |f| = \int_{[a,b]} |f|$$

and finally, passing to the supremum in all the subdivisions,

$$V_a^b(F) \leq \int_{[a,b]} |f|.$$

It remains to prove that

$$\int_{[a,b]} |f| \leq V_a^b(F).$$

We know that step functions are dense in $L^1(a,b)$ (see e. g. [7, §13, (13.23)]; step functions are functions such that there exists

$$a = x_0 < x_1 < \ldots < x_n = b \qquad \text{and} \qquad \alpha_1, \ldots, \alpha_n \in \mathbb{R} \text{ (or } \mathbb{C}\text{)},$$

such that

$$\sigma : [a,b[\to \mathbb{R} \text{ (or } \mathbb{C}), \qquad \sigma(x) = \sum_{j=1}^{n} \alpha_j \chi_{[x_{j-1},x_j[}(x);$$

the density can be deduced also approximating continuous function and using [3, Th. IV.6]). let $(\sigma_n)_n$ be a sequence of step functions in $[a,b]$ such that

$\lim_n \sigma_n = \operatorname{sgn} f$ in L^1 and almost everywhere in $[a,b]$. It not restrictive to suppose that $|\sigma_n(x)| \le 1$ for almost every $x \in [a,b]$. We have

$$|f(x)| = f(x) \operatorname{sgn} f(x) = \lim_n f(x)\sigma_n(x)$$

and

$$|f(x)\sigma_n(x)| \le |f(x)|$$

for almost every $x \in [a,b]$. Consequently, from the dominated convergence theorem,

$$\int_{[a,b]} |f| = \lim_n \int_{[a,b]} f\sigma_n.$$

Considering now that fact that

$$\sigma_n(x) = \sum_{j=1}^{k_n} \alpha_{n,j} \chi_{[x_{n,j-1},\, x_{n,j}[}(x) \quad \text{with} \quad |\alpha_{n,j}| \le 1,$$

and

$$\begin{aligned} |\int_{[a,b]} f\sigma_n| &\le \sum_{j=1}^{k_n} |\alpha_{n,j}|\, |\int_{[x_{n,j-1},\, x_{n,j}]} f| \le \sum_{j=1}^{k_n} |\alpha_{n,j}|\, |F(x_{n,j}) - F(x_{n,j-1})| \\ &\le \sum_{j=1}^{k_n} |F(x_{n,j}) - F(x_{n,j-1})| \le V_a^b(F), \end{aligned}$$

we finally obtain

$$\int_{[a,b]} |f| \le V_a^b(F)$$

and the conclusion follows.

□

Corollary 2. *Let $f \in L^1(a,b)$. Let F the integral function of f.*
Then F is a. e. differentiable.

3.3 Which is the derivative of an integral function?

The content of this paragraph can be found in [7, §18].

We have seen in the previous paragraph that an integral function of an L^1 function is almost everywhere differentiable. It is natural to pose the question about its derivative. Notice that we know how to answer, at least in the special case of the integral function of a continuous function: the fundamental theorem of calculus says that the derivative (which in this case exists in all the points of $[a,b]$) is exactly the continuous function we have considered. Here we want to give an answer in the general case.

Lemma 4. *Let $f : [a,b] \to \mathbb{R}$ be an increasing function (so that, from Lebesgue's theorem, f' exists a. e.).*
Then $f' \in L^1(a,b)$ and

$$\int_{[a,b]} f' \le f(b) - f(a).$$

Proof. We extend the values of f with $f(x) = f(b)$ for $x > b$. For all $n \geq 1$, we define $g_n : [a, b] \to \mathbb{R}$, setting

$$g_n(x) = n(f(x + \frac{1}{n}) - f(x)).$$

Since f is increasing, $g_n(x) \geq 0$ for all n and all x, and, since f is a. e. differentiable,

$$\lim_n g_n(x) = f'(x) \qquad \text{for almost every } x \in [a, b].$$

We apply Fatou's lemma (see [3, Lemma IV.1]) and we have

$$\begin{aligned}
\int_{[a,b]} f' &= \overbrace{\int_{[a,b]} \liminf_n g_n \leq \liminf_n \int_{[a,b]} g_n}^{\text{Fatou}} \\
&\leq \liminf_n \int_a^b n(f(x + \frac{1}{n}) - f(x))\, dx \\
&\leq \liminf_n \; n(\int_a^b f(x + \frac{1}{n})\, dx - \int_a^b f(x)\, dx) \\
&\leq \liminf_n \; n(\int_b^{b+\frac{1}{n}} f(x)\, dx - \int_a^{a+\frac{1}{n}} f(x)\, dx) \\
&\leq \liminf_n \; n(\frac{1}{n} f(b) - \int_a^{a+\frac{1}{n}} f(x)\, dx) \\
&\leq f(b) - \limsup_n \; n \int_a^{a+\frac{1}{n}} f(x)\, dx \\
&\leq f(b) - f(a).
\end{aligned}$$

□

Lemma 5. *Let $f \in L^1(a, b)$. Suppose that for all $x \in [a, b]$, $\int_{[a,x]} f = 0$. Then $f = 0$.*

Proof. Let α, $\beta \in [a, b]$, with $\alpha < \beta$. We have

$$\int_{[\alpha,\beta]} f = \int_{[a,\alpha]} f - \int_{[a,\beta]} f = 0.$$

Let A be an open set in $]a, b[$. We know A is a finite or countable union of pairwise disjoint intervals of $]a, b[$, i. e.

$$A = \bigcup_k \,]\alpha_k, \beta_k[\quad \text{with} \quad]\alpha_j, \beta_j[\cap]\alpha_h, \beta_h[= \emptyset \quad \text{if} \quad j \neq k.$$

Consequently, for all open set A in $]a, b[$,

$$\int_A f = \int_{\cup_k \,]\alpha_k,\beta_k[} f = \sum_k \int_{[\alpha_k,\beta_k]} f = 0.$$

Let now, for all $n > 0$,

$$E_n = \{x \in [a,b] \mid f(x) > \frac{1}{n}\} \quad \text{and} \quad F_n = \{x \in [a,b] \mid f(x) < -\frac{1}{n}\}.$$

E_n and F_n are measurable set. If we prove that $\lambda(E_n) = \lambda(F_n) = 0$, the conclusion easily follows remarking that

$$\{x \in [a,b] \mid f(x) \neq 0\} = \bigcup_n (E_n \cup F_n).$$

We show that $\lambda(E_n) = 0$. By contradiction suppose that $\lambda(E_n) > 0$. Then there exists a compact set C contained in E_n such that $\lambda(C) > \frac{\lambda(E_n)}{2} > 0$. As a consequence we would have

$$\int_C f \geq \frac{1}{n}\lambda(C) > 0.$$

But, denoting by A the open set $]a,b[\,\setminus C$, we have

$$\int_C f = \int_{[a,b]} f - \int_A f = 0 - 0 = 0.$$

□

Theorem 8. *Let $f \in L^1(a,b)$. Let $F(x) = \int_{[a,x]} f$.*

Then F is a. e. differentiable and $F'(x) = f(x)$ for almost every $x \in [a,b]$.

Proof. We can suppose without any restrictions that f is positive (if not we decompose $f = f^+ - f^-$ where f^+ and f^- are positive and negative part of f respectively).

We consider first that case of f bounded. Suppose that for all $x \in [a,b]$, $|f(x)| \leq M$. We set

$$g_n(x) = n(F(x + \frac{1}{n}) - F(x)) = n \int_{[x,x+\frac{1}{n}]} f.$$

Since F is a. e. differentiable, we know that $\lim_n g_n = F'$ a. e. in $[a,b]$ and, moreover, for all $n > 0$ and all $x \in [a,b]$, we have

$$|g_n(x)| \leq n|\int_{[x,x+\frac{1}{n}]} f| \leq n \int_{[x,x+\frac{1}{n}]} |f| \leq n \cdot \frac{1}{n} M \leq M,$$

consequently, the dominated convergence theorem implies that, for all $x \in [a,b]$,

$$\lim_n \int_{[a,x]} g_n = \int_{[a,x]} F'. \tag{4}$$

We remark that

$$\int_{[a,x]} g_n = n(\int_a^x (F(t + \frac{1}{n}) - F(t))\, dt = n \int_x^{x+\frac{1}{n}} F(t)\, dt - n \int_a^{a+\frac{1}{n}} F(t)\, dt.$$

Since F is continuous we can use the integral mean theorem and we have

$$\lim_n n \int_x^{x+\frac{1}{n}} F(t)\,dt = F(x) \qquad \lim_n n \int_a^{a+\frac{1}{n}} F(t)\,dt = F(a),$$

and finally

$$\lim_n \int_{[a,x]} g_n = F(x) - F(a) = \int_{[a,x]} f. \tag{5}$$

Putting together (4) and (5) we obtain, for all $x \in [a,b]$,

$$\int_{[a,x]} F' - f = 0.$$

Lemma 5 gives the wanted conclusion.

Consider now the case of f not bounded. We set

$$f_n(x) = \begin{cases} f(x) & \text{if} \quad f(x) \leq n, \\ n & \text{if} \quad f(x) > n, \end{cases}$$

(recall that $f \geq 0$). We have

$$F(x) = \int_{[a,x]} f - f_n + \int_{[a,x]} f_n. \tag{6}$$

Notice that the function $x \mapsto \int_{[a,x]} f - f_n$ is increasing, almost everywhere differentiable and its derivative is a. e. positive. Differentiating the identity (6) and using the first part of the proof for the derivative of $\int_{[a,x]} f_n$, we have

$$F'(x) \geq f_n(x) \quad \text{almost everywhere},$$

and passing to the limit in n we have

$$F'(x) \geq f(x) \quad \text{almost everywhere}, \tag{7}$$

We obtain

$$F(x) = \underbrace{\int_{[a,x]} f \leq \int_{[a,x]}}_{\text{from (7)}} \overbrace{F' \leq F(x) - F(a)}^{\text{from Lemma 4}} = F(x)$$

Finally, for all $x \in [a,b]$, $\int_{[a,x]} f - F' = 0$. Again the conclusion follows from Lemma 5. □

Corollary 3. *Let $f \in L^1(a,b)$.*

Then, for almost every $x \in [a,b]$,

$$f(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = \lim_{h \to 0} \frac{1}{h} \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t)\,dt. \tag{8}$$

Definition 8. *The points for which (8) is valid are called Lebesgue's points of the function f.*

Remark 4. *It can be shown that (see [7, §18]) if $f \in L^1(a,b)$, then, for almost every $x \in [a,b]$,*

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h |f(x+t)+f(x-t)-2f(x)|\,dt = 0$$

and

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h |f(x+t)-f(x)|\,dt = 0.$$

4

4.1 Absolutely continuous functions

The content of this paragraph can be found in [7, §18].

Definition 9. *Let $f : [a,b] \to \mathbb{R}$ (or $\mathbb{C}$). f is an absolutely continuous function if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all finite or countable family $(]\alpha_k,\ \beta_k[)_k$ of pairwise disjoint open intervals in $]a,b[$,*

$$\text{if} \quad \sum_k (\beta_k - \alpha_k) < \delta \quad \text{then} \quad \sum_k |f(\beta_k) - f(\alpha_k)| < \varepsilon.$$

The set of absolutely continuous function on $[a,b])$ is denoted by $AC([a,b])$.

We list here some properties of AC functions.

i) If $f,\ g \in AC([a,b])$ and $\alpha,\ \beta \in \mathbb{R}$ (or $\mathbb{C}$), then $\alpha f + \beta g \in AC([a,b])$.

ii) If $f \in AC([a,b])$, then $f \in C([a,b])$.

In fact, considering, in the definition of absolute continuity, only one interval $]\alpha,\ \beta[$, we have the definition of uniform continuity.

iii) If $f \in AC([a,b])$, then $f \in BV([a,b])$.

In fact, let, in the definition of absolute continuity, $\varepsilon = 1$. Consider a corresponding $\delta > 0$ and choose

$$\Delta = \{a = x_0 < x_1 < \ldots < x_n = b\} \quad \text{such that, for all } j, \quad x_j - x_{j-1} < \delta.$$

Take another subdivision

$$\tilde{\Delta} = \{a = y_0 < y_1 < \ldots < y_k = b\}$$

and consider

$$\Delta \cup \tilde{\Delta} = \{a = z_0 < z_1 < \ldots < z_m = b\}.$$

We have

$$\begin{aligned}\sum_{j=1}^{k} |f(y_j) - f(y_{j-1})| &\le \sum_{j=1}^{m} |f(z_j) - f(z_{j-1})| \\ &\le \sum_{h=1}^{n} \underbrace{\Big(\sum_{\substack{k \text{ such that} \\ x_{h-1} < z_k \le x_k}} |f(z_k) - f(z_{k-1})| \Big)}_{<1, \text{ since } f \in AC} \\ &\le n.\end{aligned}$$

Consequently $V_a^b(f) \le n$.

iv) If $f \in AC([a,b])$, then $V_a(f) \in AC([a,b])$.

In fact, fix $\varepsilon > 0$ and take a corresponding $\delta > 0$ from the definition of absolute continuity. Consider a finite or countable family $(]\alpha_k, \beta_k[)_k$ of pairwise disjoint open intervals in $]a,b[$ such that $\sum_k(\beta_k - \alpha_k) < \delta$. For all intervals $]\alpha_k, \beta_k[$ consider a subdivision

$$\Delta_k = \{\alpha_k = \alpha_{k,0} < \alpha_{k,1} < \ldots < \alpha_{k,n_k} = \beta_k\}$$

in such a way that

$$V_{\alpha_k}^{\beta_k}(f) - \sum_{j=1}^{n_k} |f(\alpha_{k,j}) - f(\alpha_{k,j-1})| < \frac{\varepsilon}{2^k}.$$

Then

$$\sum_k V_{\alpha_k}^{\beta_k}(f) \leq \underbrace{\sum_k (\sum_{j=1}^{n_k} |f(\alpha_{k,j}) - f(\alpha_{k,j-1})|)}_{<\varepsilon, \text{ since } f \in AC} + \underbrace{\sum_k \frac{\varepsilon}{2^k}}_{<\varepsilon} \leq 2\varepsilon.$$

It is sufficient to note that

$$\sum_k |V_a^{\alpha_k}(f) - V_a^{\beta_k}(f)| = \sum_k V_{\alpha_k}^{\beta_k}(f)$$

to have the conclusion.

v) If $f \in AC([a,b])$, then f is the difference of two AC increasing functions.

4.2 Absolute continuity of the integral

We recall a classical result from measure theory.

Theorem 9. *Let $(\Omega, \mathcal{A}, \lambda)$ a measure space (Ω is a set, $\mathcal{A}$ a σ-algebra of subsets of Ω, λ a positive measure on $\mathcal{A}$). Let $f \in L^1_\lambda(\Omega)$.*

Then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $A \in \mathcal{A}$,

$$\text{if} \quad \lambda(A) < \delta, \qquad \text{then} \qquad \int_A |f| \, d\lambda < \varepsilon.$$

Proof. Suppose first that $|f| \leq M$, for some $M > 0$. Then, taking $A \in \mathcal{A}$ with $\lambda(A) < \delta$,

$$\int_A |f| \, d\lambda \leq \lambda(A) \cdot M$$

and it sufficient to take $\delta < \frac{\varepsilon}{M}$ to have the wanted property.

Consider now the general case. Taking

$$f_n(x) = \begin{cases} f(x) & \text{if } \; |f(x)| \leq n, \\ n & \text{if } \; |f(x)| > n, \end{cases}$$

The dominated convergence theorem ensures that

$$\lim_n \int_\Omega |f - f_n| \, d\lambda = 0.$$

Let $\varepsilon > 0$; there exists $\bar{n} \in \mathbb{N}$ such that

$$\int_\Omega |f - f_{\bar{n}}| \, d\lambda < \frac{\varepsilon}{2}.$$

Since $f_{\bar{n}}$ is bounded, there exists $\delta > 0$ such that, for all $A \in \mathcal{A}$,

$$\text{if} \quad \lambda(A) < \delta, \qquad \text{then} \qquad \int_A |f_{\bar{n}}| \, d\lambda < \frac{\varepsilon}{2}.$$

Finally, for all $A \in \mathcal{A}$, if $\lambda(A) < \delta$,

$$\int_A |f| \, d\lambda \leq \int_A |f - f_{\bar{n}}| \, d\lambda + \int_A |f_{\bar{n}}| \, d\lambda \leq \underbrace{\int_\Omega |f - f_{\bar{n}}| \, d\lambda}_{< \frac{\varepsilon}{2}} + \underbrace{\int_A |f_{\bar{n}}| \, d\lambda}_{< \frac{\varepsilon}{2}} < \varepsilon.$$

□

Corollary 4. *Let $f \in L^1(a, b)$. Let $F(x) = \int_{[a,x]} f$.*
Then $F \in AC([a, b])$.

4.3 Characterization of AC functions

Let $f \in L^1(a, b)$ and let F its integral function, i.e. let $F(x) = \int_{[a,x]} f$. We have seen that F is $BV([a, b])$, F is differentiable almost everywhere in $[a, b]$, $F'(x) = f(x)$ for almost every $x \in [a, b]$ and finally F is $AC([a, b])$. In particular the integral function of an L^1 function is an absolutely continuous function. Is the converse true? Apart a constant, is an absolutely continuous function the integral function of an L^1 function?

Lemma 6. *Let F be a monotone, absolutely continuous function on $[a, b]$. Suppose that $F'(x) = 0$ for almost every $x \in [a, b]$.*
Then F is a constant function.

Proof. Let

$$E = \{x \in \,]a, b[\mid F'(x) \neq 0\}.$$

We know that $\lambda(E) = 0$. Consider the fact that $F \in AC([a, b])$. So that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that the definition of absolutely continuity is satisfied. Let now A be an open set in $]a, b[$ such that

$$\lambda(A) < \delta \qquad \text{and} \qquad E \subseteq A.$$

We have $A = \bigcup_k \,]\alpha_k, \, \beta_k[$, where $(]\alpha_k, \, \beta_k[)_k$ is a finite or countable set of pairwise disjoint intervals in $]a, b[$. Then, from the monotonicity of F (let's fix, e. g. F increasing),

$$F(E) \subseteq \bigcup_k \,]F(\alpha_k), \, F(\beta_k)[,$$

and, consequently,

$$\lambda(F(E)) \leq \sum_k (F(\beta_k) - F(\alpha_k)) < \varepsilon.$$

Since this last inequality holds for every $\varepsilon > 0$, we obtain that $\lambda(F(E)) = 0$.

Let now

$$G = \{x \in]a, b[\mid F'(x) = 0\}.$$

Let $\varepsilon > 0$ and consider $x_0 \in G$. From the fact F is increasing and $F'(x_0) = 0$ we deduce that there exists $r > 0$ such that

$$\text{for all} \quad x \in]x_0 - r, x_0 + r[\setminus \{x_0\}, \qquad 0 \le \frac{F(x) - F(x_0)}{x - x_0} < \frac{\varepsilon}{b-a}.$$

We obtain in particular that, for all $x \in]x_0, x_0 + r[$,

$$\frac{\varepsilon}{b-a} x_0 - F(x_0) < \frac{\varepsilon}{b-a} x - F(x),$$

i. e. x_0 is invisible from the right for the function $x \mapsto \frac{\varepsilon}{b-a}x - F(x)$. We denote by I_r this set and we have $G \subseteq I_r$. We know that there exists a finite or countable set of pairwise disjoint intervals $(]\alpha_k, \beta_k[)_k$ in $]a, b[$ such that

$$I_r = \bigcup_k]\alpha_k, \beta_k[\qquad \text{with} \qquad \frac{\varepsilon}{b-a}\alpha_k - F(\alpha_k) \le \frac{\varepsilon}{b-a}\beta_k - F(\beta_k),$$

in particular

$$F(\beta_k) - F(\alpha_k) < \frac{\varepsilon}{b-a}(\beta_k - \alpha_k).$$

We use the monotonicity of F and we have

$$F(G) \subseteq F(I_r) \subseteq \bigcup_k]F(\alpha_k), F(\beta_k)[$$

and then

$$\lambda(F(G)) \le \sum_k (F(\beta_k) - F(\alpha_k)) \le \frac{\varepsilon}{b-a} \sum_k (\beta_k - \alpha_k) < \varepsilon.$$

The last inequality is valid for every $\varepsilon > 0$ and consequently $\lambda(G) = 0$. We have

$$\lambda(F(]a, b[) = \lambda(F(E)) + \lambda(F(G)) = 0.$$

Since F is continuous, this implies that F is constant. □

Theorem 10. *Let $F \in AC([a, b])$.*

Then, for all $x \in [a, b]$, $F(x) = F(a) + \int_a^x F'(t)\, dt$.

Proof. Recalling that an absolutely continuous function is always difference of two increasing absolutely continuous functions, it is not restrictive to suppose that F is increasing. From Lemma 4 we have that, for all $x_1, x_2 \in [a, b]$ with $x_1 < x_2$,

$$\int_{x_1}^{x_2} F'(t)\, dt \le F(x_2) - F(x_1),$$

so that $x \mapsto F(x) - \int_a^x F'(t)\, dt$ is an increasing AC function. Recalling also Theorem 8, we have

$$(F(x) - \int_a^x F'(t)\, dt)' = F'(x) - F'(x) = 0$$

for almost every $x \in [a, b]$ and the conclusion of the proof is reached by using Lemma 6.

□

Corollary 5. *Let $F,\ G \in AC([a,b])$.*

Then $FG \in AC([a,b])$ and, for almost every $x \in [a,b]$,

$$(FG)'(x) = F(x)G'(x) + F'(x)G(x).$$

Proof. Remark that in each point in which F and G are differentiable, also FG is differentiable and Leibniz's formula holds. Then it is sufficient to prove that FG is AC. For this purpose we remark that, given a finite or countable family $(]\alpha_k,\ \beta_k[)_k$ of pairwise disjoint open intervals in $]a,b[$, we have

$$\sum_k |F(\beta_k)G(\beta_k) - F(\alpha_k)G(\alpha_k)| \\ \leq \max_{[a,b]} |F| \sum_k |G(\beta_k) - G(\alpha_k)| + \max_{[a,b]} |G| \sum_k |F(\beta_k) - F(\alpha_k)|.$$

The conclusion follows. □

Corollary 6. *Let $f,\ g \in L^1(a,b)$. Let $F(x) = \alpha + \int_{[a,x]} f$ and $G(x) = \beta + \int_{[a,x]} g$.*

Then

$$\int_{[a,b]} fG = F(b)G(b) - \alpha\beta - \int_{[a,b]} Fg.$$

Remark 5. *Consider $F \in AC([a,b])$. We know that $F' \in L^1$. Consider $\varphi \in C_0^1(]a,b[)$ (here $C_0^1(]a,b[)$ is the set of C^1 functions with compact support contained in $]a,b[$). Then*

$$\int_a^b F(t)\varphi'(t)\,dt = -\int_a^b F'(t)\varphi(t)\,dt.$$

If we introduce the set

$$W^{1,1}(a,b) \\ = \{u \in L^1(a,b) \mid \exists v \in L^1(a,b) :\ \forall \varphi \in C_0^1(]a,b[),\ \int_{[a,b]} u\varphi' = -\int_{[a,b]} v\varphi\},$$

then $AC([a,b]) \subseteq W^{1,1}(a,b)$. We will prove (when we will speak about Sobolev spaces) that, actually, $AC([a,b]) = W^{1,1}(a,b)$.

4.4 On the fundamental theorem of calculus

We want to make here a comparison between the fundamental theorem of calculus in Riemann's (integral) theory and the fundamental theorem of calculus in Lebesgue's theory. In Riemann theory the result is the following.

Theorem 11 (Riemann's fundamental theorem of calculus)**.** *Let $f \in C([a,b])$.*

Then f is Riemann-integrable and, denoted by F the integral function, F is differentiable in $[a,b]$ and, for all $x \in [a,b]$, $F'(x) = f(x)$.

It is possible to give also a slightly different version of this theorem.

Theorem 12. (see [13, Th. 6.21])**.** *Let f be an integrable function in the sense of Riemann. Suppose that there exists $G : [a,b] \to \mathbb{R}$, such that G is a primitive of f (i. e. G is differentiable on $[a,b]$ and, for all $x \in [a,b]$, $G'(x) = f(x)$).*

Then, for all $x \in [a,b]$,

$$\int_a^x f(t)\,dt = G(x) - G(a).$$

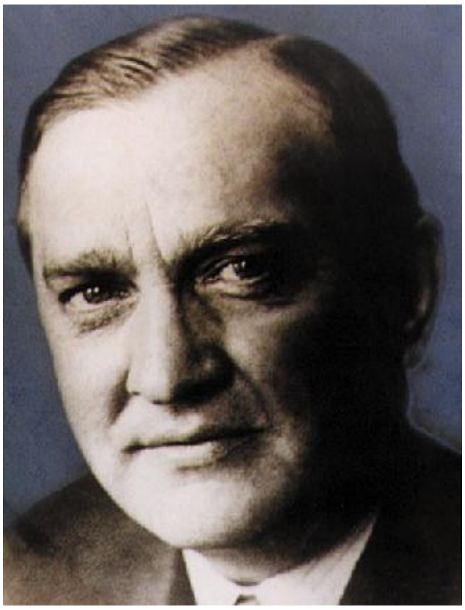


Figure 7: Stefan Banach (1892–1945)

In Lebesgue's theory we have the following result.

Theorem 13 (Lebesgue's fundamental theorem of calculus)**.** *Let $f \in L^1(a,b)$ and denote by F its integral function.*

Then F is differentiable almost everywhere in $[a,b]$ and for almost every $x \in [a,b]$, $F'(x) = f(x)$.

4.5 A theorem of Banach

Definition 10. *Let $g : [a,\ b] \to \mathbb{R}$. Let $g([a,\ b]) \subseteq [\alpha,\ \beta] \subseteq \mathbb{R}$. We say that g satisfies the condition (N) if*

$$E \subseteq [a,\ b] \quad \textit{and} \quad \lambda(E) = 0 \quad \textit{implies} \quad \lambda(g(E)) = 0.$$

Theorem 14 (S. Banach, 1925)**.** *Let $\varphi : [a,\ b] \to \mathbb{R}$ a continuous function with bounded variation.*

φ satisfies (N) if and only if φ is absolutely continuous.

4.6 The Cantor function

The Cantor function is an example of continuous increasing function (so that it is also of bounded variation) which is not absolutely continuous. We define this function as the uniform limit of a sequence of continuous increasing functions. Define, on the interval [0, 1],

$$f_(x) = x, \qquad f_2(x) = \begin{cases} \frac{3}{2}x & \text{if} \quad x \in [0,\ \frac{1}{3}], \\ \frac{1}{2} & \text{if} \quad x \in [\frac{1}{3},\ \frac{2}{3}] \\ \frac{3}{2}x - \frac{1}{2} & \text{if} \quad x \in [\frac{2}{3},\ 1]. \end{cases}$$

Figure 8: Georg Cantor (1845–1918)

$$f_3(x) = \begin{cases} \frac{9}{4}x & \text{if} \quad x \in [0,\ \frac{1}{9}], \\ \frac{1}{4} & \text{if} \quad x \in [\frac{1}{9},\ \frac{2}{9}] \\ \frac{9}{4}x - \frac{1}{4} & \text{if} \quad x \in [\frac{2}{9},\ \frac{1}{3}], \\ \frac{1}{2} & \text{if} \quad x \in [\frac{1}{3},\ \frac{2}{3}] \\ \frac{9}{4}x - 1 & \text{if} \quad x \in [\frac{2}{3},\ \frac{7}{9}] \\ \frac{3}{4} & \text{if} \quad x \in [\frac{7}{9},\ \frac{8}{9}] \\ \frac{9}{4}x - \frac{5}{4} & \text{if} \quad x \in [\frac{8}{9},\ 1], \end{cases}$$

and so on. To obtain f_{n+1} from f_n we subdivide in three equal parts the intervals in which f_n is not constant etcetera. It is possible to prove that this is a Cauchy sequence. The set in which the limit function is constant is a countable union of intervals, pairwise disjoint, such that the measure is 1. This means that the limit function is almost everywhere differentiable, with 0 derivative. But $f(1) = 1$ so that

$$\int_0^1 f'(x)\,dx = 0 < f(1) - f(0) = 1.$$

5

5.1 Signed and complex measures

The content of this paragraph can be found in [7, §19].

Definition 11. *Let $(\Omega, \mathcal{A})$ a measurable space (Ω is a set, $\mathcal{A}$ a σ-algebra of subset of Ω). Let $\nu : \mathcal{A} \to]-\infty, +\infty]$ (or $[-\infty, +\infty[$). ν is a signed measure if*

i) $\nu(\emptyset) = 0$;

ii) ν is countably additive.

Remark 6. *In the above definition, ν is countably additive means the following. Let $(A_n)_n$ be a sequence in $\mathcal{A}$ consisting of pairwise disjoint subsets of Ω. Let $A = \bigcup_n A_n$.*

i) If $\nu(A) < +\infty$, then $\sum_n |\nu(A_n)| < +\infty$ and $\sum_n \nu(A_n) = \nu(A)$.

ii) If $\nu(A) = +\infty$, then, denoting by

$$B_n = \begin{cases} A_n & \text{if } \nu(A_n) > 0, \\ \emptyset & \text{if } \nu(A_n) \leq 0, \end{cases} \qquad C_n = \begin{cases} \emptyset & \text{if } \nu(A_n) > 0, \\ A_n & \text{if } \nu(A_n) \leq 0, \end{cases}$$

we have $\sum_n \nu(B_n) = +\infty$ and $\sum_n -\nu(C_n) < +\infty$.

Definition 12. *Let $(\Omega, \mathcal{A})$ a measurable space. Let $\nu : \mathcal{A} \to \mathbb{C}$. ν is a complex measure if*

i) $\nu(\emptyset) = 0$;

ii) ν is countably additive.

Remark 7. *In the above definition, ν is countably additive means the following. Let $(A_n)_n$ be a sequence in $\mathcal{A}$ consisting of pairwise disjoint subsets of Ω. Let $A = \bigcup_n A_n$. Then $\sum_n |\nu(A_n)| < +\infty$ and $\sum_n \nu(A_n) = \nu(A)$.*

Theorem 15. *Let ν be a signed measure. We have*

i) if E, $F \in \mathcal{A}$, $F \subseteq E$ and $|\nu(E)| < +\infty$, then $|\nu(F)| < +\infty$;

ii) if $(A_n)_n$ is sequence in $\mathcal{A}$ with, for all n, $A_n \subseteq A_{n+1}$, then

$$\nu(\bigcup_n A_n) = \lim_n \nu(A_n);$$

iii) if $(A_n)_n$ is sequence in $\mathcal{A}$ with, for all n, $A_n \supseteq A_{n+1}$ and $|\nu(A_1)| < +\infty$, then

$$\nu(\bigcap_n A_n) = \lim_n \nu(A_n).$$

Remark 8. *A result similar to Theorem 15 is valid also for complex measures.*

5.2 The Hahn's decomposition theorem

The content of this paragraph can be find in [7, §19].

Let Ω be a set and $\mathcal{A}$ a σ-algebra on Ω. Let ν be a signed measure on the measurable space $(\Omega, \mathcal{A})$.

Definition 13. *Let $P,\ N \in \mathcal{A}$. The couple (P, N) is called a Hahn's decomposition of the measure ν if*

i) $P \cap N = \emptyset$ and $P \cup N = \Omega$;

ii) for all $A \in \mathcal{A}$, $\nu(A \cap P) \geq 0$, (we will say that P is non negative);

iii) for all $A \in \mathcal{A}$, $\nu(A \cap N) \leq 0$, (we will say that N is non positive).

Lemma 7. *Let $E \in \mathcal{A}$, with $-\infty < \nu(E) < +\infty$.*

Then, for all $\varepsilon > 0$, there exists $E_\varepsilon \in \mathcal{A}$ such that

$$E_\varepsilon \subseteq E,\ \nu(E_\varepsilon) \geq \nu(E) \text{ and, for all } A \in \mathcal{A}, \text{ if } A \subseteq E_\varepsilon, \text{ then } \nu(A) \geq -\varepsilon.$$

Proof. By contradiction, suppose that there exists $\varepsilon_0 > 0$ such that, for all $F \in \mathcal{A}$, the fact that $F \subseteq E$ and $\nu(F) \geq \nu(E)$ implies that there exists $A_0 \in \mathcal{A}$ such that $A_0 \subseteq F$ and $\nu(A_0) < -\varepsilon_0$.

Let's choose firstly $F = E$. We have that there exists $A_1 \in \mathcal{A}$ such that $A_1 \subseteq E$ and $\nu(A_1) < -\varepsilon_0$. Choose $F = E \setminus A_1$. Consequently there exists $A_2 \in \mathcal{A}$ such that $A_2 \subseteq E \setminus A_1$ and $\nu(A_2) < -\varepsilon_0$. Next choose $F = E \setminus (A_1 \cup A_2)$. We obtain $A_3 \in \mathcal{A}$ such that $A_3 \subseteq E \setminus (A_1 \cup A_2)$ and $\nu(A_3) < -\varepsilon_0$.

A sequence of pairwise disjoint sets $(A_n)_n$ is constructed in such a way that, for all n, $A_n \subseteq E$ and $\nu(A_n) < -\varepsilon_0$. We deduce that $\nu(\cup_n A_n) = \sum_n \nu(A_n) = -\infty$, obtaining a contradiction, since $\cup_n A_n \subseteq E$ and $\nu(E) > -\infty$. □

Lemma 8. *Let $E \in \mathcal{A}$, with $-\infty < \nu(E) < +\infty$.*

Then there exists $F \in \mathcal{A}$ such that

$$F \subseteq E,\ \nu(F) \geq \nu(E) \text{ and, for all } A \in \mathcal{A},\ \nu(A \cap F) \geq 0$$

(remark that the last point means that F is a non negative set).

Proof. We apply Lemma 7 to the set E, with $\varepsilon = 1$. We obtain that there exists $E_1 \in \mathcal{A}$, such that $E_1 \subseteq E$, $\nu(E_1) \geq \nu(E)$ and, for all $A \in \mathcal{A}$, if $A \subseteq E_1$ then $\nu(A) \geq -1$. Next we apply Lemma 7 to the set E_1 with $\varepsilon = -\frac{1}{2}$. We obtain that there exists $E_2 \in \mathcal{A}$, such that $E_2 \subseteq E_1$, $\nu(E_2) \geq \nu(E_1)$ and, for all $A \in \mathcal{A}$, if $A \subseteq E_2$ then $\nu(A) \geq -\frac{1}{2}$.

Applying successively this procedure, we construct a sequence $(E_n)_n$ such that, for all n, $E_n \subseteq \cdots \subseteq E_1 \subseteq E$, $\nu(E_n) \geq \cdots \geq \nu(E_1) \geq \nu(E)$ and, for all $A \in \mathcal{A}$, if $A \subseteq E_n$ then $\nu(A) \geq -\frac{1}{n}$.

To conclude the proof it is sufficient to take $F = \cap_n E_n$. It is easy to verify that $F \subseteq E$, $\nu(F) \geq \nu(E)$ and, for all $A \in \mathcal{A}$, if $A \subseteq F$ then $\nu(A) \geq 0$, since, for all n, $A \subseteq E_n$ and consequently, for all n, $\nu(A) \geq -\frac{1}{n}$. □

Theorem 16 (Hahn's decomposition theorem). *Let ν be a signed measure on the measurable space $(\Omega, \mathcal{A})$.*

Then there exists a Hahn's decomposition (P, N) of the measure ν. If (P', N') is another Hahn decomposition, then the sets $P \setminus P'$, $P' \setminus P$, $N \setminus N'$ and $N' \setminus N$ are negligeable.

Figure 9: Hans Hahn (1879–1934)

Proof. We suppose that, for all $E \in \mathcal{A}$, $-\infty \leq \nu(E) < +\infty$. We set $\alpha = \sup\{\nu(E)\,,\, E \in \mathcal{A}\}$. We take $(E_n)_n$ a sequence in $\mathcal{A}$ such that $\lim_n \nu(E_n) = \alpha$. It is not restrictive to suppose that, for all n, $-\infty < \nu(E_n) < +\infty$. We apply Lemma 8 to each set E_n, obtaining a sequence $(F_n)_n$ such that, for all n,

$$F_n \subseteq E_n,\ \nu(F_n) \geq \nu(E_n) \text{ and, for all } A \in \mathcal{A},\ \nu(A \cap F_n) \geq 0.$$

We set now $G_n = \cup_{j=1}^n F_j$. We have that $\nu(G_n) \geq \nu(F_n)$ since $G_n = \cup_{k=1}^n \tilde{F}_k$, where $\tilde{F}_n = F_n$ and $\tilde{F}_k = F_k \setminus (\cup_{j=k+1}^n F_j)$ for $k = 1, \ldots, n-1$. The sets $\tilde{F}_k$ are pairwise disjoint and have non negative measure. Remark that, for all n, G_n is a non negative set (in fact is the union of non negative sets) and the sequence $(G_n)_n$ is increasing. We set $P = \cup_{n=1}^{+\infty} F_n = \cup_{n=1}^{+\infty} G_n$. We have that

$$\nu(P) = \lim_n \nu(G_n) = \lim_n \nu(F_n) = \lim_n \nu(E_n) = \alpha \text{ and consequently } \alpha \in \mathbb{R}.$$

It is immediate to see that P is non negative, as it is union of non negative sets.

It remains to prove that $\Omega \setminus P$ is non positive. Suppose by contradiction there exists A contained in $\Omega \setminus P$ such that $\nu(A) > 0$. Then $\nu(P \cup A) = \alpha + \nu(A) > \alpha$, and this is impossible.

Suppose now that (P, N) and (P', N') are two Hahn's decomposition. Considering that $P \setminus P' = P \cap N'$, using the fact that P si non negative we have $\nu(P \setminus P') \geq 0$ and using the fact that N' is non positive we have $\nu(P \setminus P') \leq 0$ and the conclusion follows. The other cases are similar. $\square$

Exercise 1. *Let ν be a signed measure on a measurable space $(\Omega, \mathcal{A})$. Suppose that for all $E \in \mathcal{A}$, $-\infty \leq \nu(E) < +\infty$, i.e. $\nu(\mathcal{A}) \in [-\infty, +\infty[$.*

Prove that if $\alpha = \sup\{\nu(E)\,,\, E \in \mathcal{A}\}$, then $\alpha < +\infty$.

5.3 Total variation of a measure

The content of this paragraph can be find in [7, §19].

Definition 14. *Let ν be a signed measure on $(\Omega, \mathcal{A})$. Let (P, N) a Hahn's decomposition. We set, for all $E \in \mathcal{A}$,*

$$\begin{aligned}
\nu^+(E) &= \nu(E \cap P),\\
\nu^-(E) &= -\nu(E \cap N),\\
|\nu|(E) &= \nu^+(E) + \nu^-(E).
\end{aligned}$$

ν^+, ν^- and $|\nu|$ are positive measures and they are called positive variation, negative variation and total variation of ν, respectively.

We give a characterization of the total variation of a measure.

Theorem 17. *Let ν be a signed measure on $(\Omega, \mathcal{A})$. We set, for all $E \in \mathcal{A}$,*

$$\mu(E) = \sup\{\sum_{j=1}^{k} |\nu(E_j)| \mid E = \cup_{j=1}^{k} E_j, \quad E_j \cap E_h = \emptyset \quad \text{if} \quad j \neq h\}. \tag{9}$$

Then, for all $E \in \mathcal{A}$,

$$\mu(E) = |\nu|(E).$$

Proof. Let (P, N) a Hahn's decomposition of ν. We have that

$$|\nu(E_j)| = |\nu(E_j \cap P) + \nu(E_j \cap N)| \leq |\nu(E_j \cap P)| + |\nu(E_j \cap N)| = |\nu|(E_j).$$

Consequently

$$\sum_{j=1}^{k} |\nu(E_j)| \leq \sum_{j=1}^{k} |\nu|(E_j) = |\nu|(E),$$

and we obtain that

$$\mu(E) \leq |\nu|(E).$$

Converserly, if we write $E = (E \cap P) \cup (E \cap N)$, since $(E \cap P) \cap (E \cap N) = \emptyset$, we have

$$\mu(E) = \sup\{\sum_{j=1}^{k} |\nu(E_j)| \ldots\} \geq |\nu(E \cap P)| + |\nu(E \cap N)| = |\nu|(E).$$

□

Remark 9. *Suppose not having proved Hahn's decomposition theorem. It is still possible to prove that μ, defined by (9), is a positive measure. To prove that $\mu(\emptyset) = 0$ is immediate. To prove that μ is countably additive we proceed in the following way. Let $(A_n)_n$ be a sequence in $\mathcal{A}$ consisting of pairwise disjoint subsets of Ω. Let $A = \bigcup_n A_n$. Let*

$$\beta < \sup\{\sum_{j=1}^{k} |\nu(E_j)| \mid A = \cup_{j=1}^{k} E_j, \quad E_j \cap E_h = \emptyset \quad \text{if} \quad j \neq h\} = \mu(A).$$

Then there exists $E_1, \ldots, E_k$ *in* $\mathcal{A}$ *such that* $A = \cup_{j=1}^k E_j$, $E_j \cap E_h = \emptyset$ *if* $j \neq h$ *and*

$$\begin{aligned}
\beta &< \sum_{j=1}^{k} |\nu(E_j)| = \sum_{j=1}^{k} |\sum_{h=1}^{+\infty} \nu(E_j \cap A_h)| \\
&\leq \sum_{j=1}^{k}\sum_{h=1}^{+\infty} |\nu(E_j \cap A_h)| = \sum_{h=1}^{+\infty}\sum_{j=1}^{k} |\nu(E_j \cap A_h)| \\
&\leq \sum_{h=1}^{+\infty} \sup\{\sum_{j=1}^{k} |\nu(F_j \cap A_h)| \mid A_h = \cup_{j=1}^k F_j, \ \ldots\} \\
&\leq \sum_{h=1}^{+\infty} \mu(A_h).
\end{aligned}$$

We deduce that $\mu(A) \leq \sum_{h=1}^{+\infty} \mu(A_h)$.

It remains to prove the converse inequality, i. e. $\sum_{h=1}^{+\infty} \mu(A_h) \leq \mu(A)$. *The interesting case is when* $\mu(A) < +\infty$. *For all* $h_0 \in \mathbb{N}$ *and for all partition* $B_1, \ldots, B_m$ *of* A_{h_0}, *we have*

$$\sum_k |\nu(B_k)| \leq |\nu(\bigcup_{h \neq h_0} A_h)| + \sum_k |\nu(B_k)| \leq \mu(A),$$

so that $\mu(A_{h_0}) < +\infty$. *Let now* $\varepsilon > 0$ *and, for all* $h \in \mathbb{N}$, *choose* $E_{h,1}, \ldots, E_{h,n_h}$ *such that*

$$\mu(A_h) \leq \sum_k |\nu(E_{h,k}| + \frac{\varepsilon}{2^h}.$$

Then, for all $m \in \mathbb{N}$,

$$\begin{aligned}
\sum_{h=1}^{m} \mu(A_h) &\leq \sum_{h=1}^{m} (\sum_k |\nu(E_{h,k}| + \frac{\varepsilon}{2^h}) \\
&\leq \sum_{h=1}^{m} (\sum_k |\nu(E_{h,k}|) + |\nu(\bigcup_{h \geq m+1} A_h)| + \varepsilon \\
&\leq \mu(A) + \varepsilon.
\end{aligned}$$

Since this is true for all $m \in \mathbb{N}$ *and for all* $\varepsilon > 0$, *the conclusion follows.*

let's now consider the case of complex measures.

Definition 15. *Let* ν *be a complex measure on* $(\Omega, \mathcal{A})$. *We set, for all* $E \in \mathcal{A}$,

$$|\nu|(E) = \sup\{\sum_{j=1}^{k} |\nu(E_j)| \mid E = \cup_{j=1}^k E_j, \quad E_j \cap E_h = \emptyset \quad \text{if} \quad j \neq h\}. \tag{10}$$

Theorem 18. *Let* ν *be a complex measure on* $(\Omega, \mathcal{A})$.

Then $|\nu|$, *defined in (10), is a positive finite measure.*

Proof. The fact that $|\nu|$ is a positive measure it is proved in analogy of what we have seen in the previous Remark 9. It remains to prove that $|\nu|$ is finite (see [14, Th 6.4]). □

Theorem 19. *Let* ν *be a complex measure on* $(\Omega, \mathcal{A})$. *Let* ν_1 *and* ν_2 *the real and the imaginary part of* ν.

Then

i) ν_1 *and* ν_2 *are signed measures on* $(\Omega, \mathcal{A})$*;*

ii) ν_1^+, ν_1^-, ν_2^+ *and* ν_2^- *are positive finite measures on* $(\Omega, \mathcal{A})$.

For all $E \in \mathcal{A}$*, we have*

iii) $\nu(E) = \nu_1^+(E) - \nu_1^-(E) + i(\nu_2^+(E) - \nu_2^-(E))$ *(this is the so called Jordan's decomposition of* ν*);*

iv) $|\nu|(E) \leq \nu_1^+(E) + \nu_1^-(E) + \nu_2^+(E) + \nu_2^-(E)$*;*

v) $\nu_1^+(E)$, $\nu_1^-(E)$, $\nu_2^+(E)$, $\nu_2^-(E) \leq |\nu|(E)$.

Proof. (See [7, §19, (19.13)]). The first and the second point are immediate. In fact, $\nu(\emptyset) = 0$ implies $\Re\nu(\emptyset) = \nu_1(\emptyset) = 0$, $\Im\nu(\emptyset) = \nu_2(\emptyset) = 0$ and the countably additivity of ν implies the same property for ν_1 and ν_2. The fact that ν_1^+, ν_1^-, ν_2^+ and ν_2^- are positive finite measures comes from the fact that $|\nu|(\Omega) < +\infty$. In fact, let (P_1, N_1) a Hahn's decomposition for ν_1, we have,

$$\begin{aligned} |\nu|(\Omega) &\geq |\nu(P_1)| + |\nu(N_1)| \\ &\geq |\nu_1(P_1) + i\nu_2(P_1)| + |\nu_1(N_1) + i\nu_2(N_1)| \\ &\geq \nu_1(P_1) + \nu_1(N_1) \\ &\geq \nu_1(\Omega). \end{aligned}$$

The other points are let as an exercise. □

6

6.1 The Radon-Nikodym theorem

The content of this paragraph can be found in [7, §19].

Definition 16. *Let* $(\Omega, \mathcal{A})$ *a measurable space. Let* μ *and* λ *be a positive measure and a signed or complex measure on* $(\Omega, \mathcal{A})$ *respectively. We say that* λ *is absolutely continuous with respect to* μ*, and we write* $\lambda \ll \mu$*, if, for all* $E \in \mathcal{A}$*,*

$$\mu(E) = 0 \qquad \textit{implies} \qquad \lambda(E) = 0.$$

Theorem 20. *Let* μ *and* λ *as in previous definition.*
$\lambda \ll \mu$ *if and only if* $|\lambda| \ll \mu$.

Proof. We prove first that $|\lambda| \ll \mu$ implies $\lambda \ll \mu$. In fact, let $E \in \mathcal{A}$ with $\mu(E) = 0$. Then $|\lambda|(E) = 0$ and then $\lambda(E) = 0$ (remember that $|\lambda(E)| \leq |\lambda|(E)$).

Conversely consider $E \in \mathcal{A}$ with $\mu(E) = 0$. Take $E = \cup_{j=1}^k E_j$, where $E_j \in \mathcal{A}$ and $E_j \cap E_k = \emptyset$ if $j \neq k$. We have $\mu(E_j) = 0$ and consequently $\lambda(E_j) = 0$ for all j. We infer that $\sum_j |\lambda(E_j)| = 0$. Then

$$|\lambda|(E) = \sup\{\sum_j |\lambda(E_j)| \,\big|\, E = \cup_{j=1}^k E_j \quad \text{with} \quad \dots \quad \} = 0.$$

□

Theorem 21. *Let μ and λ be a positive and a complex measure, on the same measurable space $(\Omega, \mathcal{A})$, respectively.*

λ is absolutely continuous with respect to μ if and only if

($*$) *for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $A \in \mathcal{A}$, if $\mu(A) < \delta$, then $|\lambda(A)| < \varepsilon$.*

Proof. Suppose that the property ($*$) holds. Take $E \in \mathcal{A}$ such that $\mu(E) = 0$. Then $|\lambda(E)| < \varepsilon$, for all $\varepsilon > 0$. This means that $\lambda(E) = 0$. Consequently $\lambda \ll \mu$.

Conversely we prove that $\lambda \ll \mu$ implies

($**$) for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $A \in \mathcal{A}$, if $\mu(A) < \delta$, then $|\lambda|(A) < \varepsilon$.

The fact that ($**$) implies ($*$) is clear (remember that $|\lambda(E)| \leq |\lambda|(E)$). Suppose by contradiction that ($**$) is not true. Then there exists $\varepsilon_0 > 0$ such that, for all $\delta > 0$, there exists $A_\delta \in \mathcal{A}$ such that $\mu(A_\delta) < \delta$ and $|\lambda|(A_\delta) > \varepsilon_0$. In particular, If $\delta = \frac{1}{2^n}$, there exists $A_n \in \mathcal{A}$ such that $\mu(A_n) < \frac{1}{2^n}$ and $|\lambda|(A_n) > \varepsilon_0$. Let us define

$$B_n = \bigcup_{k=n}^{+\infty} A_k \qquad \text{and} \qquad C = \bigcap_{n=1}^{+\infty} B_n.$$

We have, for all $n \in \mathbb{N}$,

$$\mu(B_n) \leq \sum_{k=n}^{+\infty} \mu(A_k) \leq \sum_{k=n}^{+\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}} \qquad \text{and} \qquad B_n \supseteq B_{n+1}.$$

Consequently

$$\mu(C) = \lim_n \mu(B_n) \leq \lim_n \frac{1}{2^{n-1}} = 0.$$

Since $\lambda \ll \mu$, from Theorem 20 we have $|\lambda| \ll \mu$ and then $|\lambda|(C) = 0$, but

$$C = \bigcap_{n=1}^{+\infty} B_n, \quad \text{for all } n \in \mathbb{N} \quad B_n \supseteq B_{n+1} \quad \text{and} \quad |\lambda|(B_n) > \varepsilon_0,$$

so that $\lim_n |\lambda|(B_n) = |\lambda|(C)$ cannot be equal to 0. □

Remark 10. *The proof of Theorem 21 is valid also in the case when λ is a signed measure with the extra hypothesis that*

$$\textit{for all } A \in \mathcal{A}, \quad \mu(A) < +\infty \quad \textit{implies} \quad |\lambda(A)| < +\infty.$$

Consider μ a positive measure on the measurable space $(\Omega, \mathcal{A})$. Let's $f \in L^1_\mu(\Omega)$. If we define, for all $A \in \mathcal{A}$,

$$\lambda(A) = \int_\Omega \chi_A \cdot f \, d\mu = \int_A f \, d\mu,$$

we have that λ is a complex measure on $(\Omega, \mathcal{A})$, which is absolutely continuous with respect to μ (the fact that if $\mu(A) = 0$ then $\int_A f \, d\mu = 0$ is immediate, but we have also proved directly that for all $\varepsilon > 0$ there exists $\delta > 0$ such that, if $\mu(A) < \delta$, then $|\int_A f \, d\mu| \leq \int_A |f| \, d\mu < \varepsilon$). The Radon-Nikodym theorem says that this is the general case, at least if μ is σ-finite.

Figure 10: Otto Nikodym and Stefan Banach Memorial Bench, Kraków

Theorem 22 (Radon-Nikodym). *Let μ and λ be measures on the measurable space $(\Omega, \mathcal{A})$. Let μ be a positive σ-finite measure. Let λ be a signed or complex measure. Suppose that λ is absolutely continuous with respect to μ.*

Then there exists a measurable function f_0 such that, for all $A \in \mathcal{A}$, if $|\lambda(A)| < +\infty$, then $\chi_A \cdot f_0 \in L^1_\mu$ and

$$\lambda(A) = \int_\Omega \chi_A \cdot f_0 d\mu = \int_A f_0 \, d\mu. \tag{11}$$

In particular, if λ is a complex measure (or a signed measure with $|\lambda(\Omega)| < +\infty$), $f_0 \in L^1_\mu$ and (11) is valid for all $A \in \mathcal{A}$.

Proof. We present here the proof only in the case of μ and λ positive finite measures. Let

$$C = \{f \in L^1_\mu \mid f \geq 0 \text{ and, for all } A \in \mathcal{A}, \int_A f \, d\mu \leq \lambda(A)\}.$$

Remark that C is not empty, since the function $f = 0$ is an element of C. Consider

$$\alpha = \sup_{f \in C} \int_\Omega f \, d\mu.$$

We have $\alpha \leq \lambda(\Omega)$. Let $(f_n)_n$ a sequence in C such that $\lim_n \int_\Omega f_n \, d\mu = \alpha$. We set

$$g_n(x) = \max\{f_1(x), \; f_2(x), \ldots, f_n(x)\}.$$

We have $g_n \in C$ and $\lim_n \int_\Omega g_n \, d\mu = \alpha$. Since $(g_n)_n$ is an increasing sequence, using Beppo Levi's theorem we have that, setting

$$g(x) = \lim_n g_n(x),$$

we have that $g \in C$ (in particular $g \geq 0$ and $g \in L^1_\mu$), $\int_\Omega g \, d\mu = \alpha$ and, for all $A \in \mathcal{A}$,

$$\int_A g \, d\mu \leq \lambda(A).$$

Consequently, if we set

$$\nu(A) = \lambda(A) - \int_A g \, d\mu,$$

then ν is positive measure on $\mathcal{A}$.

Our goal is to prove that $\nu \equiv 0$. Suppose by contradiction that $\nu(\Omega) > 0$. Then there exists $k > 0$ such that

$$\mu(\Omega) - k\nu(\Omega) < 0.$$

We denote by ν_1 the signed measure $\mu - k\nu$. Let (P, N) a Hahn's decomposition for ν_1. We remark that $\mu(N) > 0$. In fact, if $\mu(N) = 0$, then by the absolute continuity of λ with respect to μ, $\lambda(N) = 0$ and consequently $\nu(N) = 0$ and $\nu_1(N) = 0$. Then

$$0 \le \mu(P) - k\nu(P) = (\mu(P) - k\nu(P)) + (\mu(N) - k\nu(N)) = \mu(\Omega) - k\nu(\Omega) < 0,$$

which is a contradiction. Consider now

$$h(x) = \begin{cases} \dfrac{1}{k} & \text{if} \quad x \in N, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for all $A \in \mathcal{A}$,

$$\int_A h\, d\mu = \frac{1}{k}\mu(A \cap N).$$

Now

$$\nu_1(A \cap N) = \mu(A \cap N) - k\nu(A \cap N) \le 0,$$

consequently

$$\nu(A \cap N) \ge \frac{1}{k}\mu(A \cap N) = \int_A h\, d\mu,$$

and hence

$$\nu(A) \ge \nu(A \cap N) \ge \int_A h\, d\mu.$$

Finally, for all $A \in \mathcal{A}$,

$$\lambda(A) = \nu(A) + \int_A g\, d\mu \ge \int_A (g + h)\, d\mu$$

i. e. $(g + h) \in C$. But

$$\int_\Omega (g + h)\, d\mu = \alpha + \frac{1}{k}\mu(N) > \alpha,$$

which is impossible, since $\alpha = \sup_{f \in C} \int_\Omega f\, d\mu$. The proof is complete.

□

Remark 11. *The hypothesis that μ is a positive σ-finite measure cannot be neglected. An example is given here below. Let μ be the measure on $[0, 1]$ such that μ counts the points of each set (let this measure, e. g., on the Borelian sets $\mathcal{B}$). μ is not σ-finite. Let λ be the Lebesgue's measure. It is immediate to verify that $\lambda \ll \mu$. Suppose there exists $f_0 \in L^1_\mu$ such that, for all $A \in \mathcal{B}$,*

$$\lambda(A) = \int_A f_0\, d\mu.$$

Figure 11: Johann Radon (1887–1956)

Let $x_0 \in [0,1]$ and $A = \{x_0\}$. We have, on one hand, $\lambda(A) = 0$, and on the other hand,

$$\int_{\{x_0\}} f_0 \, d\mu = f_0(x_0).$$

This implies that $f_0(x_0) = 0$ and this is valid for all $x_0 \in [0,1]$. Then $f_0 = 0$, which is a contradiction.

Corollary 7. *Let μ and λ be measures on the measurable space $(\Omega, \mathcal{A})$. Let μ be a positive σ-finite measure. Let λ be a signed or complex measure. Suppose that λ is absolutely continuous with respect to μ.*

Then there exists a measurable function f_0 such that, for all $f \in L^1_\lambda$, $f \cdot f_0 \in L^1_\mu$ and

$$\int_\Omega f \, d\lambda = \int_\Omega f \cdot f_0 \, d\mu.$$

Remark 12. *Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing right-continuous function. It is possible to show that there exists one, and only one, positive measure μ_F on the Borelian sets of $\mathbb{R}$ (denoted by $\mathcal{B}$) such that, for all a, $b \in \mathbb{R}$, with $a < b$,*

$$\mu_F(\,]a, b]) = F(b) - F(a). \tag{12}$$

μ_F is called the Lebesgue-Stieltjes (positive) measure associated to F.

Similarly, let $F \in BV(\mathbb{R})$ (it is sufficient that, for all $M > 0$, $F \in BV([-M, M])$ and or the positive or the negative variation is finite) and suppose that F is right-continuous. It is possible to show that there exists one, and only one, signed measure μ_F on $\mathcal{B}$, such that, for all a, $b \in \mathbb{R}$, with $a < b$, (12) is verified. Also in this case we will say that μ_F is the Lebesgue-Stieltjes (signed) measure associated to F.

It is possible to show that $F \in AC(\mathbb{R})$ (this means that, for all $M > 0$, $F \in AC([-M, M])$) if and only if $\mu_F \ll \lambda$, where λ is the Lebesgue measure.

Remark 13. *Let $F : \mathbb{R} \to [0, 1]$ a right-continuous increasing function. Suppose that $\lim_{t\to-\infty} F(t) = 0$ and $\lim_{t\to+\infty} F(t) = 1$. We define*

$$j_F : \mathbb{R} \to \mathbb{R}, \qquad j_F(t) = F(t) - \lim_{z\to t^-} F(z).$$

The function j_X is different from 0 *only on a finite or countable set of points $(x_n)_n$. We define*

$$\chi(t) = \sum_{x_n \le t} j_F(x_n).$$

We define $F_1 = F - \chi$. The function $F_1 : \mathbb{R} \to [0\,1]$ is continuous and increasing. We know that F_1' is a L^1 (Lebesgue integrable) function and, for all $t_1 < t_2$,

$$\int_{t_1}^{t_2} F_1'(s)\,ds \le F_1(t_2) - F_1(t_1).$$

In particular, for all $t_1 < t_2$,

$$F_1(t_1) - \int_0^{t_1} F_1'(s)\,ds \le F_1(t_2) - \int_0^{t_2} F_1'(s)\,ds.$$

We set

$$G(t) = F_1(t) - \int_0^t F_1'(s)\,ds \qquad \text{and} \qquad F_2(t) = \int_0^t F_1'(s)\,ds.$$

In conclusion

$$F(t) = F_2(t) + G(t) + \chi(t),$$

where F_2 is an absolutely continuous increasing function, G is a continuous increasing function such that $G'(x) = 0$ for almost every $x \in \mathbb{R}$ and χ is a jump-function.

F can be thought as the distribution function (in Italian: funzione di ripartizione) of a random variable X. We have

$$P(X \le t) = F(t),$$

where P is the probability measure associate to X. This random variable has an absolutely continuous density (the function F_2') and a discrete density (the function j_X) but has also a "singular part" (linked to the function G) which cannot be described in term of Lebesgue measure nor in term of discrete random variables.

7

7.1 The Hardy-Littlewood maximal function

The content of this paragraph can be (partially) found in [17, Ch. 1] and [14, Ch. 8].

Let us denote by $\mathcal{B}$ the σ-algebra of Borel sets of $\mathbb{R}^d$. Let λ be the Lebesgue measure on $\mathbb{R}^d$ and let ν be a complex measure defined on $\mathcal{B}$. For any ball $B(x,r) = \{y \in \mathbb{R}^d \mid |y - x| < r\}$, we set

$$Q_r\nu(x) = \frac{\nu(B(x,r))}{\lambda(B(x,r))}.$$

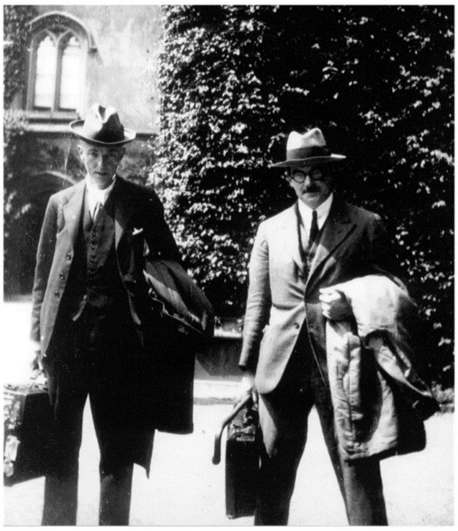


Figure 12: G. H. Hardy and J. E. Littlewood in 1924

Definition 17. *Let $x \in \mathbb{R}^d$. If the limit*

$$\lim_{r\to 0^+} Q_r\nu(x) = \lim_{r\to 0^+} \frac{\nu(B(x,r))}{\lambda(B(x,r))}$$

exists, we call this limit symmetric derivative of ν with respect to λ at the point x and we denote it with $\frac{d\nu}{d\lambda}(x)$.

Remark 14. *Let $f \in L^1_\lambda(\mathbb{R}^n)$ and ν_f such that $\nu_f(A) = \int_A f\, d\lambda$, then*

$$Q_r\nu_f(x) = \frac{\int_{B(x,r)} f\, d\lambda}{\lambda(B(x,r))} \qquad \textit{and} \qquad \frac{d\nu_f}{d\lambda}(x_0) = \lim_{r\to 0^+} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f\, d\lambda.$$

We are interested in conditions guaranteeing the existence of $\frac{d\nu}{d\lambda}$ and also to the value of this quantity.

Definition 18. *Let ν and $|\nu|$ be a complex measure and its total variation, respectively. Let $x \in \mathbb{R}^d$. We define*

$$M_\nu(x) = \sup_{r>0} \frac{|\nu|(B(x,r))}{\lambda(B(x,r))}.$$

The function $M_\nu : \mathbb{R}^d \to [0, +\infty]$ is called Hardy-Littlewood maximal function of ν.

Theorem 23. *The function M_ν is lower semicontinuous.*

Proof. It is not restrictive to suppose that ν is a positive measure. Proving that M_ν is lower semicontinuous means to show that for all $\alpha \geq 0$, the set

$$E = \{x \in \mathbb{R}^n \mid M_\nu(x) > \alpha\}$$

is an open set. Let $x \in E$. Then $M_\nu(x) > \alpha$ and consequently

$$\sup_{r>0} \frac{\nu(B(x,r))}{\lambda(B(x,r))} > \alpha.$$

Hence there exist $r > 0$ and $t > \alpha' > \alpha$ such that

$$\frac{\nu(B(x,r))}{\lambda(B(x,r))} > t > \alpha' > \alpha.$$

Take now $\delta > 0$ such that

$$(r+\delta)^n < r^n \frac{t}{\alpha'},$$

so that, if $|x-y| < \delta$, then $B(y, r+\delta) \supseteq B(x,r)$ and consequently

$$\begin{aligned} \nu(B(y,r+\delta)) &\geq \nu(B(x,r)) > t\lambda(B(x,r)) \\ &> \alpha' \frac{(r+\delta)^n}{r^n} \lambda(B(x,r)) = \alpha' \lambda(B(x,r+\delta)) = \alpha' \lambda(B(y,r+\delta)). \end{aligned}$$

Finally

$$\frac{\nu(B(y,r+\delta))}{\lambda(B(y,r+\delta))} > \alpha' > \alpha,$$

i. e. we have proved that if $|x-y| < \delta$, then $y \in E$, and consequently E is an open set. □

Corollary 8. *The function M_ν is Lebesgue measurable.*

Lemma 9 (Wiener)**.** *Let W be the union of a finite number of balls $B(x_1, r_1)$, $B(x_2, r_2)$, ..., $B(x_k, r_k)$.*

Then there exists $S \subseteq \{1, 2, \ldots, k\}$ such that

i) if $i, j \in S$, with $i \neq j$, then $B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$;

ii) $W \subseteq \bigcup_{i \in S} B(x_i, 3r_i)$;

iii) $\lambda(W) \leq 3^n \sum_{i \in S} \lambda(B(x_i, r_i))$.

Proof. The fact that ii) implies iii) is a consequence of the homogeneity property of Lebesgue measure. Let's show i) and ii). It is not restrictive to suppose that

$$r_1 \geq r_2 \geq \ldots \geq r_k.$$

Let $n_1 = 1$. We define

$$A_2 = \{j \in \{n_1+1, \ldots, k\} \mid B(x_{n_1}, r_{n_1}) \cap B(x_j, r_j) = \emptyset\}.$$

If $A_2 = \emptyset$, we take $S = \{n_1\}$. If $A_2 \neq \emptyset$, we define $n_2 = \min A_2$. We consider

$$A_3 = \{j \in A_2 \mid B(x_{n_2}, r_{n_2}) \cap B(x_j, r_j) = \emptyset\}.$$

If $A_3 = \emptyset$, we take $S = \{n_1, n_2\}$. If $A_3 \neq \emptyset$, we define $n_3 = \min A_3$ and we go on with this procedure up to obtaining

$$S = \{n_1, n_2, \ldots, n_h\} \quad \text{with} \quad 1 = n_1 < n_2 < \ldots < n_h \leq k.$$

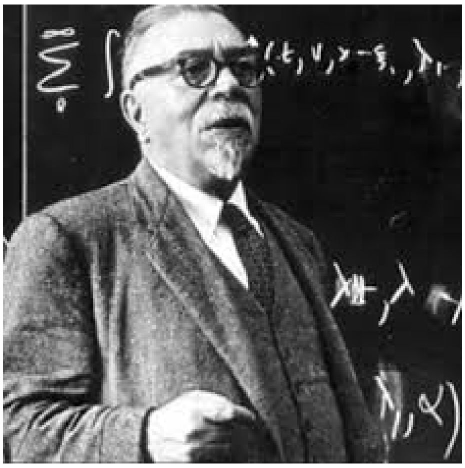


Figure 13: Norbert Wiener (1894–1964)

With such a construction, condition i) is verified. In fact, let $n_i < j < n_{i+1}$. Since n_{i+1} is the minimum index h grater than n_i such that $B(x_h, r_h) \cap B(x_{n_m}, r_{n_m}) = \emptyset$ for every $m = 1, \ldots, i$, we have $B(x_j, r_j) \cap B(x_{n_h}, r_{n_h}) \neq \emptyset$ for at least one h in $\{1, \ldots, i\}$, so that

$$B(x_j, r_j) \subseteq B(x_{n_h}, 3r_{n_h})$$

and condition ii) follows. $\square$

We are now ready to show the main property of Hardy-Littlewood maximal function.

Theorem 24 (Hardy-Littlewood)**.** *Let ν be a complex measure. Let $\alpha > 0$. Then*

$$\lambda(\{x \in \mathbb{R}^d \mid M_\nu(x) > \alpha\}) \leq 3^d \cdot \frac{1}{\alpha} \cdot |\nu|(\mathbb{R}^d).$$

Proof. Let K be a compact set contained in $E = \{x \in \mathbb{R}^d \mid M_\nu(x) > \alpha\}$ (remember that the set E is measurable). Let $x \in K \subseteq E$. We know that $M_\nu(x) > \alpha$. Then there exists $r_x > 0$ such that

$$\frac{|\nu|(B(x, r_x))}{\lambda(B(x, r_x))} > \alpha.$$

The set $\{B(x, r_x) \mid x \in K\}$ is an open covering of the compact set K. Let

$$B(x_1, r_1),\ B(x_2, r_2), \ldots, B(x_n, r_n),$$

a finite subcovering and let S be the set of indexes given from Wiener's lemma. We have

$$K \subseteq \bigcup_{i=1}^{n} B(x_i, r_i) \subseteq \bigcup_{j \in S} B(x_j, 3r_j).$$

Consequently

$$\begin{aligned}\lambda(K) &\leq \sum_{j\in S} \lambda(B(x_j, 3r_j)) \leq 3^d \cdot \sum_{j\in S} \lambda(B(x_j, r_j)) \\ &\leq 3^d \cdot \frac{1}{\alpha} \cdot \sum_{j\in S} |\nu|(B(x_j, r_j)) \leq 3^d \cdot \frac{1}{\alpha} \cdot |\nu|(\mathbb{R}^d).\end{aligned}$$

Since this last inequality holds true for all the compact sets contained in E, the conclusion follows. □

Remark 15. *Usually the Hardy-Littlewood theorem is stated in a slightly different way. In particular let $f \in L^1(\mathbb{R}^d)$. Denote by M_f the function*

$$M_f(x) = \sup_{r>0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f| \, d\lambda.$$

The result shown here above implies that, for all $\alpha > 0$,

$$\lambda(\{x \in \mathbb{R}^d \mid M_f(x) > \alpha\}) \leq 3^d \cdot \frac{1}{\alpha} \cdot \|f\|_{L^1}.$$

It is possible to prove also that, if $f \in L^p(\mathbb{R}^d)$, with $1 < p \leq +\infty$, then $M_f \in L^p(\mathbb{R}^d)$ and

$$\|M_f\|_{L^p} \leq A_p \|f\|_{L^p},$$

where A_p depends only on p and d (see [17, Ch. 1]).

7.2 Lebesgue's points

Definition 19. *Let $f \in L^1_{loc}(\mathbb{R}^d)$ (this means that, for all K compact sets in $\mathbb{R}^d$, $\chi_K \cdot f \in L^1(\mathbb{R}^d)$). Let $x \in \mathbb{R}^d$. x is said to be a Lebesgue's point for f if*

$$f(x) = \lim_{r\to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \, dy,$$

(from now on, given $A \in \mathcal{B}$, $|A|$ will denote the Lebesgue measure of A).

Theorem 25. *Let $f \in L^1(\mathbb{R}^d)$.*
Then, for almost all $x \in \mathbb{R}^d$,

$$\lim_{r\to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)| \, dy = 0.$$

Proof. Let $g \in C^0_0(\mathbb{R}^d)$. Then

$$\begin{aligned}&\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)| \, dy \\ &\quad \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - g(y)| \, dy + \frac{1}{|B(x,r)|} \int_{B(x,r)} |g(y) - f(x)| \, dy.\end{aligned}$$

Denoting by M_{f-g} the maximal function of $f - g$, i. e.

$$M_{f-g}(x) = \sup_{r>0} \frac{1}{|(B(x,r)|} \int_{B(x,r)} |f(y) - g(y)| \, dy,$$

we have

$$\begin{aligned}\frac{1}{|B(x,r)|}\int_{B(x,r)}&|f(y)-f(x)|dy\\&\leq M_{f-g}(x)+\frac{1}{|B(x,r)|}\int_{B(x,r)}|g(y)-f(x)|dy.\end{aligned}\tag{13}$$

We consider, on both sides of (13), the $\limsup_{r\to 0^+}$. We have

$$\begin{aligned}T(x)&=\limsup_{r\to 0^+}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)-f(x)|\,dy\\&\leq M_{f-g}(x)+\limsup_{r\to 0^+}\frac{1}{|B(x,r)|}\int_{B(x,r)}|g(y)-f(x)|\,dy.\end{aligned}$$

Remarking that the function $y\mapsto|g(y)-f(x)|$ is a continuous function, we obtain that

$$\limsup_{r\to 0^+}\frac{1}{|B(x,r)|}\int_{B(x,r)}|g(y)-f(x)|\,dy=|g(x)-f(x)|,$$

and, finally,

$$T(x)\leq M_{f-g}(x)+|g(x)-f(x)|.$$

Take now $\varepsilon>0$ and consider

$$\{x\in\mathbb{R}^d \mid T(x)>2\varepsilon\}\subseteq\{x\in\mathbb{R}^d \mid M_{f-g}(x)>\varepsilon\}\cup\{x\in\mathbb{R}^d \mid |f(x)-g(x)|>\varepsilon\}.$$

From the Hardy-Littlewood theorem we know that

$$\lambda(\{x\in\mathbb{R}^d \mid M_{f-g}(x)>\varepsilon\})\leq 3^d\cdot\frac{1}{\varepsilon}\cdot\|f-g\|_{L^1}$$

and, from a direct calculation,

$$\lambda(\{x\in\mathbb{R}^d \mid |f(x)-g(x)|>\varepsilon\})\leq\frac{1}{\varepsilon}\cdot\|f-g\|_{L^1}.$$

Hence

$$\lambda(\{x\in\mathbb{R}^d \mid T(x)>2\varepsilon\})\leq(3^d+1)\frac{1}{\varepsilon}\|f-g\|_{L^1}.\tag{14}$$

It is sufficient to consider a sequence $(g_n)_n$ in C_0^0 such that, for all n, $\|f-g_n\|_{L^1}<1/n$ (see Theorem 26 in Lesson 9), obtaining, from (14), that, for all $\varepsilon>0$, we have

$$\lambda(\{x\in\mathbb{R}^d \mid T(x)>2\varepsilon\})=0.$$

In conclusion, for almost every $x\in\mathbb{R}^d$,

$$\lim_{r\to 0^+}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)-f(x)|\,dy=0.$$

□

Corollary 9. *Let $f \in L^1_{loc}(\mathbb{R}^d)$.*

Then, for almost every $x \in \mathbb{R}^d$,

$$\lim_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy = f(x).$$

Remark 16. *Let ν be a complex measure on $\mathcal{B}(\mathbb{R}^d)$. Suppose that $\nu \ll \lambda$, where λ is the Lebesgue measure. From the Radon-Nikodym theorem we know that there exists $f_0 \in L^1_\lambda(\mathbb{R}^d)$ such that, for all $E \in \mathcal{B}$,*

$$\nu(E) = \int_E f_0 \, d\lambda.$$

From Corollary 9 we have also that, for almost every $x \in \mathbb{R}^d$,

$$f_0(x) = \lim_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} f_0(y)\, dy = \lim_{r \to 0^+} \frac{\nu(B(x,r))}{\lambda(B(x,r))} = \frac{d\nu}{d\lambda}(x).$$

Hence, if $\nu \ll \lambda$, then, for almost every $x \in \mathbb{R}^d$, ν possesses a finite symmetric derivative with respect to λ and the value of the symmetric derivative is exactly the value of the Radon-Nikodym density function, i. e., for all $E \in \mathcal{B}$,

$$\nu(E) = \int_E \frac{d\nu}{d\lambda}(x)\, d\lambda.$$

8

8.1 Preliminary results (to distribution theory)

8.1.1 $C_0(\Omega)$ is dense in $L^1(\Omega)$

The following density result is considered (by H. Brezis) "un résultat d'intégration qu'il faut absolument connaître".

Theorem 26 (Th. IV.3 in [3])**.** *Let Ω be an open set in $\mathbb{R}^n$. Let $f \in L^1(\Omega)$. Let $\varepsilon > 0$.*

Then there exists $\varphi \in C_0(\Omega)$ such that

$$\|f - \varphi\|_{L^1(\Omega)} < \varepsilon,$$

i. e. $C_0(\Omega)$ is dense in $L^1(\Omega)$, where $C_0(\Omega)$ denotes the space of continuous functions φ such that the closure of the set $\{x \in \Omega \mid \varphi(x) \neq 0\}$, i. e. the support of φ, is a compact set in Ω.

8.1.2 $C_0(\Omega)$ is dense in $L^p(\Omega)$, for all $1 \le p < +\infty$

The density result of the previous paragraph can be extended to L^p, for all $1 \le p < +\infty$.

Lemma 10 (Lemma IV.2 in [3])**.** *Let $f \in L^1_{loc}(\Omega)$. Suppose that, for all $\varphi \in C_0(\Omega)$,*

$$\int_\Omega f\varphi = 0.$$

Then $f = 0$.

Proof. Let us suppose that $f \in L^1(\Omega)$ and $|\Omega| < +\infty$.

Since $C_0(\Omega)$ is dense in $L^1(\Omega)$, then, for all $\varepsilon > 0$, there exists $f_\varepsilon \in C_0(\Omega)$ such that

$$\|f - f_\varepsilon\|_{L^1(\Omega)} < \varepsilon.$$

Consequently, for all $\varphi \in C_0(\Omega)$,

$$|\int_\Omega f_\varepsilon \varphi| = |\int_\Omega (f_\varepsilon - f)\varphi| \leq \|f - f_\varepsilon\|_{L^1(\Omega)} \|\varphi\|_{L^\infty(\Omega)} < \varepsilon \|\varphi\|_{L^\infty(\Omega)}. \qquad (15)$$

Consider

$$K_1 = \{x \in \Omega \mid f_\varepsilon(x) \geq \varepsilon\}, \qquad K_2 = \{x \in \Omega \mid f_\varepsilon(x) \leq -\varepsilon\}$$

and $K = K_1 \cup K_2$. K_1, K_2 and K are compact sets in Ω. We use Uryshon's Lemma to construct $u_\varepsilon \in C_0(\Omega)$ such that

$$|u_\varepsilon(x)| \leq 1 \quad \text{for all} \quad x \in \Omega \qquad \text{and} \qquad \begin{cases} u_\varepsilon(x) = 1 & \text{on} \quad K_1, \\ u_\varepsilon(x) = -1 & \text{on} \quad K_2. \end{cases}$$

We have

$$\int_\Omega |f| \leq \underbrace{\int_\Omega |f - f_\varepsilon|}_{\leq \varepsilon} + \int_\Omega |f_\varepsilon| \leq \varepsilon + \int_{\Omega \setminus K} |f_\varepsilon| + \int_K |f_\varepsilon|.$$

Remark now that

$$\int_K |f_\varepsilon| = \int_K f_\varepsilon u_\varepsilon = \int_\Omega f_\varepsilon u_\varepsilon - \int_{\Omega \setminus K} f_\varepsilon u_\varepsilon$$

and

$$|\int_\Omega f_\varepsilon u_\varepsilon| \leq \varepsilon \|u_\varepsilon\|_{L^\infty} \leq \varepsilon, \qquad \text{as a consequence of (15)},$$

$$|\int_{\Omega \setminus K} f_\varepsilon u_\varepsilon| \leq \int_{\Omega \setminus K} |f_\varepsilon| \leq \varepsilon \cdot |\Omega \setminus K|, \qquad \text{since, on } \Omega \setminus K, \text{ we have } |f_\varepsilon| \leq \varepsilon,$$

so that

$$\int_K |f_\varepsilon| \leq \underbrace{|\int_\Omega f_\varepsilon u_\varepsilon|}_{\leq \varepsilon} + \underbrace{|\int_{\Omega \setminus K} f_\varepsilon u_\varepsilon|}_{\leq \varepsilon \cdot |\Omega \setminus K|} \leq \varepsilon(1 + |\Omega \setminus K|).$$

Finally

$$\int_\Omega |f| \leq \underbrace{\int_\Omega |f - f_\varepsilon|}_{\leq \varepsilon} + \int_\Omega |f_\varepsilon| \leq \varepsilon + \underbrace{\int_{\Omega \setminus K} |f_\varepsilon|}_{\varepsilon \cdot |\Omega \setminus K|} + \underbrace{\int_K |f_\varepsilon|}_{\varepsilon(1 + |\Omega \setminus K|)} \leq 2\varepsilon(1 + |\Omega|).$$

This last inequality implies that $\int_\Omega |f| = 0$ and consequently $f = 0$.

Suppose now $f \in L^1_{loc}$ and Ω open in $\mathbb{R}^n$. Consider

$$\Omega_n = B(0, n) \cap \{x \in \Omega \mid \operatorname{dist}(x, \mathcal{C}\Omega) > \frac{1}{n}\}.$$

From what we have already proved, we deduce that, for all n,

$$f \cdot \chi_{\Omega_n} = 0,$$

and this conclude the proof. $\square$

Theorem 27 (Th IV.12 in [3]). *$C_0(\Omega)$ is dense in $L^p(\Omega)$, for all $1 \le p < +\infty$.*

Proof. This result, in the case $p = 1$, is already known. Let $1 < p < +\infty$. We know that a consequence of the Hahn-Banach theorem is the following: let W a subspace of a normed space $V$and suppose that, for all $\Phi \in V'$, $\Phi(W) = 0$ implies $\Phi = 0$, then W is a dense subspace of V. Consider $\Phi \in (L^p(\Omega))'$. From Riesz's theorem we have that there exists $g \in L^{p'}$, with $\frac{1}{p} + \frac{1}{p'} = 1$, such that

$$\Phi(\varphi) = \int_\Omega g\varphi$$

Suppose that $\Phi(\varphi) = 0$ for all $\varphi \in C_0(\Omega)$, i. e. $\int_\Omega g\varphi = 0$ for all $\varphi \in C_0(\Omega)$. From the previous lemma we have that $g = 0$, i. e. $\Phi = 0$. As a consequence $C_0(\Omega)$ is dense in $L^p(\Omega)$. $\square$

8.1.3 Convolution of functions

We collect here some (supposed) known results on convolution (see [3, Ch. IV.4]).

Theorem 28 (Th. IV.15 in [3]). *Let $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, with $1 \le p \le +\infty$.*

Then, for almost every $x \in \mathbb{R}^n$, the function

$$y \mapsto f(x-y)g(y) \qquad \text{is in} \qquad L^1(\mathbb{R}^n)$$

and setting

$$f * g\,(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy$$

we have $f * g \in L^p(\mathbb{R}^n)$ and

$$\|f * g\|_{L^p} \le \|f\|_{L^1}\|g\|_{L^p}.$$

More generally, let $1 \le p,\ q,\ r \le +\infty$, with $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$. Let $f \in L^r(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$.

Then

$$f * g \in L^q(\mathbb{R}^n) \qquad \text{and} \qquad \|f * g\|_{L^q} \le \|f\|_{L^r}\|g\|_{L^p} \qquad \text{(Young inequality).}$$

Definition 20. *Let f be a continuous function defined on Ω, open set of $\mathbb{R}^n$. We call support of f the closure, in Ω, of the set $\{x \in \Omega \mid f(x) \neq 0\}$.*

Let f be a $L^1_{loc}(\Omega)$ function. Consider W, the set of points of Ω, having an open neighborhood U in Ω, such that f is identically equal to 0 on U. We call support of f the complementary set of W in Ω.

The support of f in Ω is the smallest relatively closed set in Ω outside of which f is identically equal to 0.

Theorem 29 (Prop. IV.18 in [3]). *Let $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, with $1 \le p \le +\infty$. Then*

$$\mathrm{Supp}\,(f * g) \subseteq \overline{\mathrm{Supp}\, f + \mathrm{Supp}\, g}$$

Remark 17. *Let $f \in L^1(\mathbb{R}^n)$ with compact support (i. e. f is identically equal to 0 outside a compact). Let $g \in L^p_{loc}(\mathbb{R}^n)$. Then it is possible to define $f * g$ in the usual way and we have that $f * g \in L^p(\mathbb{R}^n)$.*

Theorem 30 (Prop. IV.20 in [3]). *Let $f \in C_0(\mathbb{R}^n)$ and $g \in L^1_{loc}(\mathbb{R}^n)$.*
Then $f * g \in C(\mathbb{R}^n)$.

Let $f \in C_0^m(\mathbb{R}^n)$, with $m \ge 1$, and $g \in L^1_{loc}(\mathbb{R}^n)$.
Then

$$f * g \in C^m(\mathbb{R}^n) \qquad \text{and} \qquad \frac{\partial}{\partial x_j}(f * g) = \frac{\partial f}{\partial x_j} * g.$$

8.1.4 Test functions and mollifiers

We collect here some notions on test functions and mollifiers (see [8, Ch. 1.2]).

Definition 21. *We set*

$$C_0(\Omega) = \{\text{continuous functions with compact support contained in } \Omega\},$$

for $m \in \mathbb{N}$,

$$C_0^m(\Omega) = C_0(\Omega) \cap C^m(\Omega),$$

and, finally,

$$\mathcal{D}(\Omega) = C_0^\infty(\Omega) = \bigcap_m C_0^m(\Omega).$$

The elements of $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ are called test functions.

Example 2. *Let*

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(t) = \begin{cases} e^{-\frac{1}{t}} & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t \le 0. \end{cases}$$

It is possible to prove that $f \in C^\infty(\mathbb{R})$ and $f^{(j)}(t) = 0$ for all j and for all $t \le 0$. The function

$$u : \mathbb{R}^n \to \mathbb{R}, \qquad u(x) = f(1 - |x|^2),$$

is a test function, with $\mathrm{Supp}\, u = \overline{B(0,1)}$.

Definition 22. *Let $\rho \in \mathcal{D}(\mathbb{R}^d)$, $\rho \ge 0$, $\mathrm{Supp}\,\rho \subseteq \overline{B(0,1)}$ and $\int_{\mathbb{R}^d} \rho(x)\, dx = 1$. The set*

$$\{\rho_\varepsilon, \mid \varepsilon \in]0,1], \quad \rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\, \rho(\frac{x}{\varepsilon})\} = (\rho_\varepsilon)_{\varepsilon \in]0,1]},$$

is called mollifier (or also family of mollifiers). Similarly we will call mollifier (or family of mollifiers) the sequence

$$(\rho_n)_n \qquad \text{with} \qquad \rho_n(x) = n^d \rho(nx).$$

Theorem 31 (Th. 1.2.1 in [8]). *Let $(\rho_\varepsilon)_\varepsilon$ be a mollifier.*

i) Let $u \in L^1(\Omega)$, with $u = 0$ outside a compact set of Ω.
Then there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, $\rho_\varepsilon * u \in C_0^\infty(\Omega)$.

ii) Let $u \in C_0(\Omega)$.
Then, for ε going to 0^+, $\rho_\varepsilon * u$ converges uniformly to u.

iii) Let $u \in L^p(\Omega)$, with $1 \le p < +\infty$. Let

$$\bar{u}(x) = \begin{cases} u(x) & \text{for} \quad x \in \Omega, \\ 0 & \text{for} \quad x \notin \Omega. \end{cases}$$

Then, for ε going to 0^+, $\rho_\varepsilon * \bar{u}$ converges to u in $L^p(\Omega)$.

Proof. i) Denote by K the compact set of Ω outside of which the function u is identically 0. Take $\varepsilon_0 > 0$ less than the distance between K and the border of Ω. Theorem 4 and Theorem 5 give the conclusion.

ii) Let $\varepsilon_0 > 0$ as in the previous point, and let $0 < \varepsilon < \varepsilon_0$. Then

$$\rho_\varepsilon * u\,(x) - u(x) = \int_{|y| \le \varepsilon} \rho_\varepsilon(y)(u(x-y) - u(x))\, dy.$$

Consider now that u is uniformly continuous, so that for all $r > 0$ there exists $\delta > 0$ such that, if $|x_1 - x_2| < \delta$ then $|u(x_1) - u(x_2)| < r$. Consequently, if $\varepsilon < \delta$, for all $x \in \Omega$,

$$|\rho_\varepsilon * u\,(x) - u(x)| \le \int_{|y| \le \varepsilon} \rho_\varepsilon(y)|u(x-y) - u(x)|\, dy \le \int_{|y| \le \varepsilon} \rho_\varepsilon(y) r\, dy = r$$

and the conclusion follows.

iii) We know that $C_0(\Omega)$ is dense in $L^p(\Omega)$ (recall that $1 \le p < +\infty$). Fix $\delta > 0$ and consider $w \in C_0(\Omega)$ such that $\|u - w\|_{L^p(\Omega)} < \delta$. We have

$$\begin{aligned} &\|(\rho_\varepsilon * \bar{u}) - u\|_{L^p(\Omega)} \\ &\quad \le \|(\rho_\varepsilon * \bar{u}) - \bar{u}\|_{L^p(\mathbb{R}^n)} \\ &\quad \le \|(\rho_\varepsilon * \bar{u}) - (\rho_\varepsilon * w)\|_{L^p(\mathbb{R}^n)} + \|(\rho_\varepsilon * w) - w\|_{L^p(\mathbb{R}^n)} + \|w - u\|_{L^p(\Omega)}. \end{aligned}$$

We consider now the fact that

$$\|(\rho_\varepsilon * \bar{u}) - (\rho_\varepsilon * w)\|_{L^p(\mathbb{R}^n)} = \|\rho_\varepsilon * (\bar{u} - w)\|_{L^p(\mathbb{R}^n)} \le \|\rho_\varepsilon\|_{L^1} \|u - w\|_{L^p(\Omega)} \le \delta,$$

and

$$\|w - u\|_{L^p(\Omega)} \le \delta.$$

Consequently

$$\|(\rho_\varepsilon * \bar{u}) - u\|_{L^p(\Omega)} \le \|(\rho_\varepsilon * w) - w\|_{L^p(\mathbb{R}^n)} + 2\delta.$$

From the point ii) we know that $\rho_\varepsilon * w$ is converging uniformly on Ω to w and both $\rho_\varepsilon * w$ and w are $C_0(\Omega)$ functions, so that $\rho_\varepsilon * w$ is converging to w also in $L^p(\mathbb{R}^n)$. This means that, if ε is sufficiently small,

$$\|(\rho_\varepsilon * \bar{u}) - u\|_{L^p(\Omega)} \le 3\delta,$$

and the proof is complete. □

Remark 18. *Convolution with a mollifier is a good way to construct a* C_0^∞ *function which value is 1 in a neighborhood of a certain compact* K*. Let's show how to do it.*

Let K *be a compact set in* $\mathbb{R}^n$*. Consider the covering* $\{B(x, \varepsilon_0) \mid x \in K\}$ *and extract a finite subcovering*

$$B(x_1, \varepsilon_0),\ B(x_2, \varepsilon_0), \ldots, B(x_N, \varepsilon_0).$$

Define

$$K_1 = \bigcup_{j=1}^{N} \overline{B(x_j, 2\varepsilon_0)}$$

and finally consider $\rho_\varepsilon * \chi_{K_1}$*, with* $\varepsilon < \varepsilon_0$*. We let as an exercise to verify that* $\rho_\varepsilon * \chi_{K_1}$ *is a* C_0^∞ *and that its value is* 1 *inside each ball* $B(x_j, \varepsilon_0)$*.*

We end this paragraph with a refinement of the previous density results.

Lemma 11. *Let* $f \in L^1_{loc}(\Omega)$*. Suppose that for all* $\varphi \in C_0^\infty(\Omega)$*,* $\int_\Omega f\varphi = 0$*.*
Then $f = 0$*.*

Proof. Suppose first that $f \in L^1(\Omega)$. Let $\psi \in C_0(\Omega)$. Let $(\rho_n)_n$ be a mollifier. Consider $\varphi_n = \rho_n * \psi$. We have that, for all n, $\varphi_n \in C_0^\infty$ and φ_n converges uniformly to ψ. Remark that

$$|\varphi_n(x)| = |\int_{\mathbb{R}^n} \rho_n(y)\psi(x-y)\, dy| \leq \max|\psi| \int_{\mathbb{R}^n} |\rho_n(y)|\, dy \leq \max|\psi|.$$

Then

$$f(x)\varphi_n(x) \xrightarrow{n} f(x)\psi(x) \qquad \text{almost everywhere,}$$

and

$$|f(x)\varphi_n(x)| \leq \max|\psi||f(x)|.$$

We can apply the dominated convergence theorem and we have

$$\int_\Omega f(x)\varphi_n(x)\, dx \xrightarrow{n} \int_\Omega f(x)\psi(x)\, dx,$$

but we know that, for all n, $\int_\Omega f(x)\varphi_n(x)\, dx = 0$, so that $\int_\Omega f(x)\psi(x)\, dx$. The conclusion is a consequence of Lemma 1.

Let now f be in $L^1_{loc}(\Omega)$. The above part of the proof guarantees that, for all compact set K, the function $f \cdot \chi_K$ is identically equal to 0 and this implies that $f = 0$.

□

Corollary 10 (Cor. IV.23 in [3]). $C_0^\infty(\Omega)$ *is dense in* $L^p(\Omega)$*, for all* $1 \leq p < +\infty$*.*

8.1.5 Partition of unity

We conclude the list of preliminary results with a partition of unity theorem. We need, before, a property that we let as an exercise.

Exercise 2. *Let K be a compact set in $\mathbb{R}^n$. Let Ω_1 and Ω_2 be two open sets in $\mathbb{R}^n$, with $K \subseteq \Omega_1 \cup \Omega_2$ and $K_j \cap \Omega_j \neq \emptyset$, for $j = 1, 2$. Show that there exists two compact sets $K_1 \subseteq \Omega_1$ and $K_2 \subseteq \Omega_2$ such that $K = K_1 \cup K_2$.*

Hint. First of all, if $\Omega_1 \cap \Omega_2 = \emptyset$ then it is sufficient to take $K_j = K \cap \Omega_j$, for $j = 1, 2$. If $\Omega_1 \cap \Omega_2 \neq \emptyset$, for every $x \in K$, consider an open ball $B(x, r_x)$ such that,

$$\text{if } x \in K \setminus \Omega_j, \text{ then } B(x, 2r_x) \subseteq \Omega_j, \text{ for } j = 1, 2,$$

$$\text{if } x \in K \cap \Omega_1 \cap \Omega_2, \text{ then } B(x, 2r_x) \subseteq \Omega_1 \cap \Omega_2.$$

$\{B(x, r_x) \mid x \in K\}$ is an open covering of K. Take a finite subcovering

$$B_1(x_1, r_1), \ldots, B_1(x_N, r_N)$$

Define

$$K_1 = K \cap \Big(\bigcup_{x_i \in \Omega_1} \overline{B_1(x_i, r_i)} \Big) \qquad \text{and} \qquad K_2 = K \cap \Big(\bigcup_{x_i \in \Omega_2} \overline{B_1(x_i, r_i)} \Big).$$

Theorem 32 (Th. 1.2.3 in [8])**.** *Let K be a compact set in $\mathbb{R}^n$. Let $\Omega_1, \ldots, \Omega_N$ be open sets in $\mathbb{R}^n$, with $K \subseteq \bigcup_{j=1}^N \Omega_j$.*

Then there exist $\varphi_1, \ldots, \varphi_N$ with, for all j, $\varphi_j \in C_0^\infty(\Omega_j)$ such that,

$$\sum_{j=1}^N \varphi_j(x) = 1, \qquad \text{for all} \quad x \in K.$$

Proof. Using the exercise we can find $K_1, \ldots, K_N$ compact sets, with, for all j, $K_j \subseteq \Omega_j$ and $\cup_j K_j = K$. We consider, for all j, $\psi_j \in C_0^\infty(\Omega_j)$, such that $\psi_j = 1$ in a neighborhood of K_j. We set

$$\begin{aligned}
&\varphi_1 = \psi_1, \\
&\varphi_2 = \psi_2(1 - \psi_1), \\
&\varphi_3 = \psi_3(1 - \psi_2)(1 - \psi_1), \\
&\vdots \\
&\varphi_N = \psi_N(1 - \psi_{N-1})(1 - \psi_{N-2}) \cdot \ldots \cdot (1 - \psi_1).
\end{aligned}$$

By induction, it is possible to prove that

$$\varphi_1 + \varphi_2 + \ldots + \varphi_N = 1 - (1 - \psi_1) \cdot \ldots \cdot (1 - \psi_N),$$

and the conclusion follows. □

9

9.1 Distributions

The content of this paragraph can be found in [8, Ch. 1.3] (see also [15]).

9.1.1 Notations

Let α be a multi-index of lenght n, i. e.

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \qquad \text{with} \quad \alpha_j \in \mathbb{N};$$

we set

$$|\alpha| = \alpha_1 + \dots + \alpha_n, \qquad\qquad \alpha! = \alpha_1! \cdot \dots \cdot \alpha_n!.$$

If α, β are two multi-indexes of lenght n, $\alpha \leq \beta$ means

$$\alpha_1 \leq \beta_1, \ \dots \ , \ \alpha_n \leq \beta_n,$$

and, in this case,

$$\binom{\beta}{\alpha} = \frac{\beta_1!}{\alpha_1!(\beta_1 - \alpha_1)!} \cdot \dots \cdot \frac{\beta_n!}{\alpha_n!(\beta_n - \alpha_n)!}.$$

Let $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$, we set

$$x^\alpha = x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n},$$

and finally

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}} \quad \text{and} \quad D_x^\alpha = (-i)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}} \quad \text{(Hörmander's notation)}.$$

9.1.2 Definition of distribution

Definition 23. *Let Ω be an open set in $\mathbb{R}^n$. Let $T : \mathcal{D}(\Omega) \to \mathbb{R}$ (or $\mathbb{C}$). Suppose that*

i) T is linear, i. e. $T(\lambda\varphi + \mu\psi) = \lambda T(\varphi) + \mu T(\psi)$, for all φ, $\psi \in \mathcal{D}(\Omega)$ and λ, $\mu \in \mathbb{R}$ (or $\mathbb{C}$);

ii) for all K, compact set in Ω, there exist $C_K > 0$, $m_K \in \mathbb{N}$ such that

$$|T(\varphi)| \leq C_K \sum_{|\alpha| \leq m_K} \sup_{x \in \Omega} |D^\alpha \varphi(x)|$$

for all $\varphi \in \mathcal{D}(\Omega)$ such that $\operatorname{Supp} \varphi \subseteq K$.

We call T distribution on the open set Ω. The set of distributions on Ω is denoted by $\mathcal{D}'(\Omega)$.

Definition 24. *Let $T \in \mathcal{D}'(\Omega)$. If the constant $m_K \in \mathbb{N}$ in condition ii) can be chosen independently of K, we say that T is a distribution of finite order, and the minimal m for which this is valid is the order of T. The set of finite order distributions is denoted by $\mathcal{D}'_F(\Omega)$*

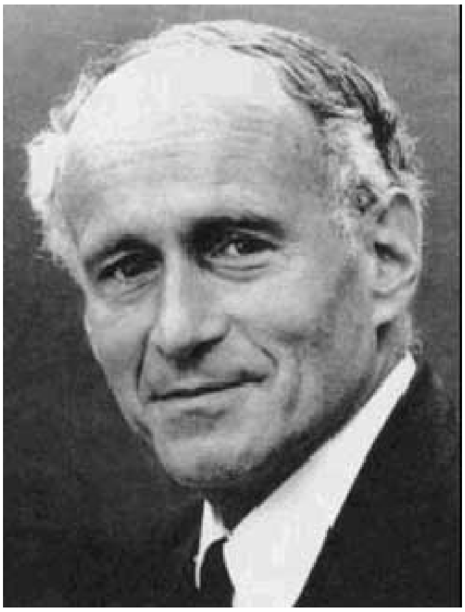


Figure 14: Laurent Schwartz (1915–2002)

Example 3. *Let* $f \in L^1_{loc}(\Omega)$. *We define*

$$T_f : \mathcal{D}(\Omega) \to \mathbb{R} \ (\textit{or } \mathbb{C}), \qquad \varphi \mapsto T_f(\varphi) = \int_\Omega f\varphi.$$

We have, for all compact set K,

$$|T_f(\varphi)| \leq \Big(\int_K |f|\Big) \sup_\Omega |\varphi| \quad \textit{for all } \varphi \in \mathcal{D}(\Omega) \textit{ with } \operatorname{Supp}\varphi \subseteq K.$$

Setting $\int_K |f| = C_K$, *we have that* T_f *is a distribution of order* 0. *Remark that if* f_1, $f_2 \in L^1_{loc}(\Omega)$ *and*

$$T_{f_1}(\varphi) = T_{f_2}(\varphi) \quad \textit{for all } \varphi \in \mathcal{D}(\Omega),$$

then $f_1 = f_2$ *(as functions of* $L^1_{loc}(\Omega)$*, remember Lemma 11 in Lesson 9). Consequently*

$$L^1_{loc}(\Omega) \to \mathcal{D}'(\Omega), \qquad f \mapsto T_f$$

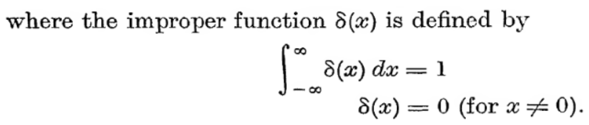
is an injective functional. We can think that $L^1_{loc}(\Omega)$ *is a subset of* $\mathcal{D}'(\Omega)$ *or, conversely,* $\mathcal{D}'(\Omega)$ *is an extension of* $L^1_{loc}(\Omega)$ *(in early Soviet Union mathematical tradition, distributions are called "generalized functions").*

Example 4. *Let* $x_0 \in \Omega$. *Consider*

$$\delta_{x_0} : \mathcal{D}(\Omega) \to \mathbb{R} \ (\textit{or } \mathbb{C}), \qquad \varphi \mapsto \delta_{x_0}(\varphi) = \varphi(x_0).$$

Since

$$|\delta_{x_0}(\varphi)| = |\varphi(x_0)| \leq 1 \cdot \sup_\Omega |\varphi| \quad \textit{for all } \varphi \in \mathcal{D}(\Omega),$$


where the improper function $\delta(x)$ is defined by

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$
$$\delta(x) = 0 \text{ (for } x \neq 0).$$

Figure 15: From page 64 of Dirac's book [5]

δ_{x_0} is a distribution of order 0. We call it Dirac's delta at the point x_0. We show now that δ_{x_0} is not a distribution obtained from a function in $L^1_{loc}(\Omega)$. Suppose, by contradiction, that there exists $f \in L^1_{loc}(\Omega)$ such that

$$\delta_{x_0}(\varphi) = \varphi(x_0) = \int_\Omega f\varphi = T_f(\varphi) \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Consider $\psi \in \mathcal{D}(\Omega \setminus \{x_0\})$, then

$$0 = \psi(x_0) = \int_{\Omega\setminus\{x_0\}} f\psi = T_f(\psi)$$

and consequently $f = 0$ almost everywhere in $\Omega \setminus \{x_0\}$. This means that $f = 0$ almost everywhere in Ω i. e. $T_f = 0$ and this a contradiction.

Example 5. *Consider $\Omega =]0, 1[\subseteq \mathbb{R}$ and $x_0 \in \Omega$. We set, for $\varphi \in \mathcal{D}(\Omega)$,*

$$\mathrm{dip}_{x_0}(\varphi) = \varphi'(x_0).$$

We have that $\mathrm{dip}_{x_0} \in \mathcal{D}'(\Omega)$ with order equal to 1.

Example 6. *Consider $\Omega =]0, 2[\subseteq \mathbb{R}$. We set, for $\varphi \in \mathcal{D}(\Omega)$,*

$$T(\varphi) = \sum_{j=0}^{+\infty} \varphi^{(j)}(\frac{1}{j+1}) = \varphi(1) + \varphi'(1/2) + \varphi''(1/3) + \ldots + \varphi^{(n-1)}(1/n) + \ldots$$

T is a distribution of infinite order. In fact let K be a compact set in $]0, 2[$. There exists $\bar{n} \in \mathbb{N}$ such that $K \subseteq [\frac{1}{\bar{n}}, 2 - \frac{1}{\bar{n}}]$. If $\varphi \in \mathcal{D}(\Omega)$ with $\mathrm{Supp}\,\varphi \subseteq K$, then

$$T(\varphi) = \sum_{j=0}^{\bar{n}-1} \varphi^{(j)}(\frac{1}{j+1})$$

and consequently

$$|T(\varphi)| \leq \sum_{j=0}^{\bar{n}-1} \sup_\Omega |\varphi^{(j)}| \qquad \text{for all } \varphi \in \mathcal{D}(\Omega) \text{ with } \mathrm{Supp}\,\varphi \subseteq K.$$

Obviously the index $\bar{n}$ depends on K and cannot be chosen independently of it.

Remark 19. *The set $\mathcal{D}'(\Omega)$ is a vector space with the addition and multiplication by scalars given by*

$$(\lambda T + \mu S)(\varphi) = \lambda \cdot T(\varphi) + \mu \cdot S(\varphi).$$

On the space $\mathcal{D}'(\Omega)$ we shall always use the weak topology, i. e., given a sequence $(T_j)_j$ in $\mathcal{D}'(\Omega)$,

$$T_j \xrightarrow{j} T \qquad \textit{will means} \qquad T_j(\varphi) \xrightarrow{j} T(\varphi) \ \ \textit{for all} \ \ \varphi \in \mathcal{D}(\Omega).$$

A consequence of the Banach-Steinhaus theorem is the following: given a sequence $(T_j)_j$ in $\mathcal{D}'(\Omega)$ such that, for all $\varphi \in \mathcal{D}(\Omega)$, there exists, in $\mathbb{R}$ or $\mathbb{C}$, $\lim_j T_j(\varphi)$, then the functional

$$\varphi \mapsto \lim_j T_j(\varphi)$$

is a distribution T and $\lim_j T_j = T$ in the weak topology.

Example 7. *Let $(\rho_n)_n$ be a family of mollifiers on $\mathbb{R}$ (remember: $\rho \in \mathcal{D}(\mathbb{R})$, $\rho \geq 0$, $\operatorname{Supp}\rho \subseteq [-1,\, 1]$, $\int_{\mathbb{R}} \rho(x)\, dx = 1$ and $\rho_n(x) = n\rho(nx)$). Consider the sequence of the distributions associated to the functions ρ_n, i. e. $(T_{\rho_n})_n$. We have*

$$T_{\rho_n} \xrightarrow{n} \delta_0.$$

In fact, for $\varphi \in \mathcal{D}(\mathbb{R})$,

$$T_{\rho_n}(\varphi) = \int_{-\infty}^{+\infty} \rho_n(t)\varphi(t)\, dt = \int_{-\infty}^{+\infty} n\rho(nt)\varphi(t)\, dt = \int_{-\infty}^{+\infty} \rho(s)\varphi(\frac{s}{n})\, ds.$$

We have

$$\rho(s)\varphi(\frac{s}{n}) \xrightarrow{n} \rho(s)\varphi(0) \qquad \textit{for all } s \in \mathbb{R},$$

and

$$|\rho(s)\varphi(\frac{s}{n})| \leq |\rho(s)|\|\varphi\|_{L^\infty}.$$

The dominated convergence theorem gives

$$\lim_n T_{\rho_n}(\varphi) = \lim_n \int_{-\infty}^{+\infty} \rho(s)\varphi(\frac{s}{n})\, ds = \int_{-\infty}^{+\infty} \rho(s)\varphi(0)\, ds = \varphi(0) = \delta_0(\varphi).$$

Looking at the behavior of the sequence $(\rho_n)_n$, i. e. functions with support in $[-\frac{1}{n},\, \frac{1}{n}]$, with value in 0 equal to $n\rho(0)$ which goes to $+\infty$ and with integral equal to 1, this convergence is the reason why, very naively, some one says that Dirac's delta is a function with value 0 outside of 0, with value $+\infty$ in 0 and with integral equal to 1.

We state now a theorem with a characterization of the distributions.

Theorem 33. *Let $T : \mathcal{D}(\Omega) \to \mathbb{R}$ (or $\mathbb{C}$). Let T be a linear functional. The following two conditions are equivalent*

i) T is a distribution.

ii) For every sequence $(\varphi_n)_n$ in $\mathcal{D}(\Omega)$ such that

a) there exits a compact set $K \subseteq \Omega$ such that, for all n, $\operatorname{Supp}\varphi_n \subseteq K$,

b) for all $\alpha \in \mathbb{N}^n$, $D^\alpha \varphi_n \xrightarrow{n} 0$ uniformly,

we have that $T(\varphi_n) \xrightarrow{n} 0$.

Proof. *i)* ⇒ *ii)*. Let T be a distribution. Consider a sequence with the properties *a)* and *b)*. Since T is a distribution, there exist $C_K > 0$, $m_K \in \mathbb{N}$ such that

$$|T(\varphi_n)| \leq C_K \sum_{|\alpha| \leq m_K} \sup_{x \in \Omega} |D^\alpha \varphi_n(x)|.$$

Then *b)* implies that $\sum_{|\alpha| \leq m_K} \sup_\Omega |D^\alpha \varphi_n| \xrightarrow{n} 0$ and consequently $T(\varphi_n) \xrightarrow{n} 0$.

ii) ⇒ *i)*. Suppose by contradiction that T is not a distribution. Then there exists a compact set K such that, for every $C > 0$ and $m \in \mathbb{N}$ there exists a test function $\varphi \in \mathcal{D}(\Omega)$ such that

$$\operatorname{Supp} \varphi \subseteq K \qquad \text{and} \qquad |T(\varphi)| > C \sum_{|\alpha| \leq m} \sup_\Omega |D^\alpha \varphi|.$$

Choose $C = m = j$. There exists $\varphi_j \in \mathcal{D}(\Omega)$ such that

$$\operatorname{Supp} \varphi_j \subseteq K \qquad \text{and} \qquad |T(\varphi_j)| > j \sum_{|\alpha| \leq j} \sup_\Omega |D^\alpha \varphi_j|.$$

Consider

$$\psi_j(x) = \frac{\varphi_j(x)}{j \sum_{|\alpha| \leq j} \sup_\Omega |D^\alpha \varphi_j|}.$$

We have, for all j, $\operatorname{Supp} \psi_j \subseteq K$ and, if $|\beta| \leq j$,

$$\sup_{x \in \Omega} |D^\beta \psi_j(x)| = \frac{\sup_{x \in \Omega} |D^\beta \varphi_j(x)|}{j \sum_{|\alpha| \leq j} \sup_\Omega |D^\alpha \varphi_j|} \leq \frac{1}{j}.$$

Hence, for all β,

$$\sup_\Omega |D^\beta \psi_j| \xrightarrow{j} 0.$$

We have proved the sequence $(\psi_j)_j$ satisfies *a)* and *b)* but, since, for all j, $|T(\psi_j)| > 1$, the condition *ii)* is not verified, and this is impossible. □

Definition 25. *Given a sequence $(\varphi_n)_n$ in $\mathcal{D}(\Omega)$ such that*

a) there exits a compact set $K \subseteq \Omega$ such that, for all n, $\operatorname{Supp} \varphi_n \subseteq K$,

b) for all $\alpha \in \mathbb{N}^n$, $D^\alpha \varphi_n \xrightarrow{n} 0$ uniformly,

we will say that $(\varphi_n)_n$ is converging to 0 in the sense of $\mathcal{D}$.

9.1.3 Topology of $\mathcal{D}(\Omega)$ (see [18])

The idea is the following: we would like to put a topology on $\mathcal{D}(\Omega)$ in such a way that $\mathcal{D}'(\Omega)$ is the dual space, i. e. the space of linear functionals defined on $\mathcal{D}(\Omega)$ which are continuous with respect to this topology. This is possible but not easy, and, for our purposes, not so useful. Actually in applications, it will be much more useful the definition we have given at the beginning or the characterization given in Theorem 1.

Let Ω be an open set in $\mathbb{R}^n$. Let $m \in \mathbb{N}$. Consider the space $C^m(\Omega)$ consisting of all the continuous differentiable functions on Ω, with continuous derivatives up to the order m. On $C^m(\Omega)$ we put the topology of uniform convergence on compact subsets of Ω, for the functions and for all the derivatives up to the order m. Consider a sequence of open relatively compact sets $(\Omega_j)_j$ contained in Ω, such that

$$\overline{\Omega}_j \subseteq \Omega_{j+1} \qquad \text{and} \qquad \bigcup_j \Omega_j = \Omega.$$

The cited topology is generated by the countable set of seminorms

$$p_j(f) = \sum_{|\alpha| \leq m} \sup_{\Omega_j} |D^\alpha f|$$

and $C^m(\Omega)$ is a Fréchet space (complete and with a topology that can be obtained from a metric). U is a neighborhood of 0 in $C^m(\Omega)$ if there exists $j_0 \in \mathbb{N}$ and there exists $r > 0$ such that

$$\{f \in C^m(\Omega) \mid p_{j_0}(f) < r\} \subseteq U.$$

Similarly, considering $C^\infty(\Omega) = \cap_m C^m(\Omega)$, we take the topology of uniform convergence on compact subsets of Ω, for the functions and for all the derivatives. This topology is generated by the countable set of seminorms

$$\tilde{p}_j(f) = \sum_{|\alpha| \leq j} \sup_{\Omega_j} |D^\alpha f|.$$

and $C^\infty(\Omega)$ is a Fréchet space. It will be denoted by $\mathcal{E}(\Omega)$. U is a neighborhood of 0 in $\mathcal{E}(\Omega)$ if there exists $j_0 \in \mathbb{N}$ and there exists $r > 0$ such that

$$\{f \in C^\infty(\Omega) \mid \tilde{p}_{j_0}(f) < r\} \subseteq U.$$

Consider now a compact set in Ω. We denote by $C_0^\infty(K)$ the set of $C^\infty(\Omega)$ functions having support contained in K. The topology of $C^\infty(\Omega)$ induces on $C_0^\infty(K)$ the topology of uniform convergence of all derivatives, i. e. the topology generated by the family of norms

$$q_j(f) = \sum_{|\alpha| \leq j} \sup_{K} |D^\alpha f|.$$

Remark that, on $C_0^\infty(K)$, q_j is a norm and no more a seminorm, as $\tilde{p}_j$ on $C^\infty(\Omega)$. $C_0^\infty(K)$ is a Fréchet space. Consider finally

$$C_0^\infty(\Omega) = \bigcup_{j=1}^{+\infty} C_0^\infty(\overline{\Omega}_j).$$

The correct topology to consider on $C_0^\infty(\Omega)$ is the so called inductive limit topology from the topologies of the spaces $C_0^\infty(\overline{\Omega}_j)$, i. e. the maximal locally convex topology such that, for all j, the immersion

$$C_0^\infty(\overline{\Omega}_j) \to C_0^\infty(\Omega), \qquad \varphi \mapsto \varphi$$

is continuous. This topology makes $C_0^\infty(\Omega)$ complete but not metrizable (this last thing can be seen using Baire's theorem). It can be proved that a linear functional T on $C_0^\infty(\Omega)$ is continuous with respect to this topology if and only if T is a distribution according to Definition 1.

9.1.4 Radon measures (see [2, Ch. 4])

Let Ω be an open set in $\mathbb{R}^n$. Consider $C_0(\Omega)$ the set of continuous functions having compact support on Ω. We call Radon measure a linear functional

$$\mu : C_0(\Omega) \to \mathbb{R} \text{ (or } \mathbb{C})$$

such that, for all K compact set in Ω, there exists $C_K > 0$ such that

$$|\mu(f)| \leq C_K \sup_{\Omega} |f| \qquad \text{for all } f \in C_0(\Omega) \text{ with } \operatorname{Supp} f \subseteq K.$$

It can be proved that μ is a Radon measure if an only if μ is linear and continuous with respect to the inductive limit topology from the topologies of the spaces $C_0(\overline{\Omega}_j)$, i. e. the maximal locally convex topology such that, for all j, the immersion

$$C_0(\overline{\Omega}_j) \to C_0(\Omega), \qquad f \mapsto f$$

is continuous. It can also be proved that if μ is a Radon measure then $\mu\big|_{\mathcal{D}}$ is a distribution of order 0 and, conversely, T is a distribution of order 0 if there exists a Radon measure μ such that $\mu\big|_{\mathcal{D}} = T$.

Definition 26. *Let ν be a complex measure on $\mathcal{B}(\Omega)$, the Borelian sets of Ω. ν is said to be a regular Borelian measure if, denoting by $|\nu|$ its total variation,*

i) $|\nu|$ is finite on compact sets;

ii) for all $B \in \mathcal{B}(\Omega)$,

$$\sup\{|\nu|(C) \bigm| C \text{ compact}, C \subseteq B\} = |\nu|(B) = \inf\{|\nu|(A) \bigm| A \text{ open}, A \supseteq B\}.$$

Theorem 34 (Riesz's representation theorem, see Th. 6.19 in [14])**.** *μ is a Radon measure if and only if there exists ν regular Borelian measure such that, for all $f \in C_0(\Omega)$,*

$$\mu(f) = \int_{\Omega} f \, d\nu.$$

Exercise 3. *Let T be a linear functional from $\mathcal{D}(\Omega)$ to $\mathbb{R}$. Suppose that, for all $\varphi \in \mathcal{D}(\Omega)$, if $\varphi \geq 0$ (this means that, for all $x \in \Omega$, $\varphi(x) \geq 0$), then $T(\varphi) \geq 0$.*

Show that T is a distribution of order 0.

Hint. From the positivity of T we deduce the monotonicity, i. e. if φ, $\psi \in \mathcal{D}(\Omega)$ and for all $x \in \Omega$, $\varphi(x) \geq \psi(x)$, then $T(\varphi) \geq T(\psi)$. Let now $\phi \in \mathcal{D}(\Omega)$ and denote by K its support. Take $\chi \in \mathcal{D}(\Omega)$ such that, for all x, $0 \leq \chi(x) \leq 1$ and such that χ is identically equal to 1 in a neighborhood of K. Then, for all $x \in \Omega$,

$$-\max|\phi| \cdot \chi(x) \leq \phi(x) \leq \max|\phi| \cdot \chi(x),$$

so that

$$T(-\max|\phi| \cdot \chi) \leq T(\phi) \leq T(\max|\phi| \cdot \chi),$$

and then

$$|T(\phi)| \leq T(\chi) \max|\phi|.$$

Setting $C_K = T(\chi)$, the conclusion follows.

9.1.5 Local character and support of a distribution

The following results says that it is sufficient to know the behavior of a distribution in a neighborhood of each point, to know its behavior in general.

Theorem 35. *Let T_1 and T_2 be two distributions in $\mathcal{D}'(\Omega)$. Suppose that, for all $x_0 \in \Omega$, there exists a neighborhood U_0 of x_0 such that, for all $\varphi \in \mathcal{D}(\Omega)$, if $\operatorname{Supp} \varphi \subseteq U_0$, then $T_1(\varphi) = T_2(\varphi)$.*

Then $T_1 = T_2$.

Proof. Let $\psi \in \mathcal{D}(\Omega)$. Denote by K the support of ψ. We know that, for all $x \in K$, there exists an open neighborhood U_x of x such that, for all $\varphi \in \mathcal{D}(\Omega)$, if $\operatorname{Supp} \varphi \subseteq U_x$, then $T_1(\varphi) = T_2(\varphi)$. Form the open covering $\{U_x \mid x \in K\}$ we extract a finite subcovering of K,

$$U_1,\ U_2, \ldots, U_N.$$

We use now the theorem on partition of unity (Theorem 32). There exist $\varphi_1, \ldots, \varphi_N$ in $\mathcal{D}(\Omega)$, with, for all j, $\operatorname{Supp} \varphi_j \subseteq U_j$, such that, for all $x \in K$, $\sum_j \varphi_j(x) = 1$. Then

$$\begin{aligned}
T_1(\psi) &= T_1(\psi \textstyle\sum_j \varphi_j) \\
&= T_1(\textstyle\sum_j \psi\varphi_j) \\
&= \textstyle\sum_j T_1(\psi\varphi_j) \qquad \text{with} \quad \operatorname{Supp} \psi\varphi_j \subseteq U_j \\
&= \textstyle\sum_j T_2(\psi\varphi_j) \\
&= T_2(\textstyle\sum_j \psi\varphi_j) \\
&= T_2(\psi \textstyle\sum_j \varphi_j) = T_2(\psi).
\end{aligned}$$

□

Now we define the support of a distribution.

Definition 27. *Let $T \in \mathcal{D}'(\Omega)$. Let $x \in \Omega$. We say that $x \notin \operatorname{Supp} T$ if there exists a neighborhood U of x such that, for all $\varphi \in \mathcal{D}(\Omega)$, if $\operatorname{Supp} \varphi \subseteq U$, then $T(\varphi) = 0$. $\operatorname{Supp} T$ is the smallest relatively closed set in Ω outside of which T is identically equal to 0.*

Remark 20. *Let $f \in C(\Omega)$. Then $f \in L^1_{loc}(\Omega)$ and consequently we can consider the distribution T_f associated to f. The support of f as continuous function coincides with the support of f as L^1_{loc} function and with the support of T_f as distribution.*

10

10.1 Derivative of a distribution, multiplication of a distribution with smooth function

The content of this paragraph can be found in [8, Ch. 1.4] (see also [15]).

La plus belle nuit de ma vie

J'ai toujours appelé cette nuit de découverte ma nuit merveilleuse, ou la plus belle nuit de ma vie. Dans ma jeunesse, j'avais souvent des insomnies de plusieurs heures et ne prenais jamais de somnifères. Je restais dans mon lit, lumière éteinte, et faisais souvent, évidemment sans rien écrire, des mathématiques. Mon énergie inventive était décuplée, j'avançais avec rapidité sans ressentir de fatigue. J'étais alors totalement libre, sans aucun des freins qu'imposent la réalité du jour et l'écriture. Après quelques heures, la lassitude survenait quand même, surtout si une difficulté mathématique se présentait obstinément. Alors je m'arrêtais et dormais jusqu'au matin. J'étais fatigué tout le jour suivant, mais heureux ; il me fallait souvent plusieurs jours pour tout remettre en ordre. Cette fois-là, j'étais sûr de moi et plein d'exaltation. Dans ce genre de circonstance, je ne perdais pas de temps pour tout expliquer par le menu à Cartan qui, comme je l'ai dit, habitait à côté. Il était lui-même enthousiasmé : « Bon, voilà que tu viens de résoudre toutes les difficultés de la dérivation. Désormais, plus de fonctions sans dérivées », me dit-il. Si une fonction est sans dérivée (Weierstrass), c'est qu'elle a des dérivées qui sont des opérateurs mais ne sont pas des fonctions.

Il existe une propriété tout à fait essentielle des distributions, donc des opérateurs : sur tout ouvert relativement compact, tout opérateur est somme finie de dérivées (naturellement au sens des opérateurs) de fonctions continues. C'est un théorème de finitude comme il en existe un grand nombre dans cette théorie. J'en ai donné plusieurs démonstrations dans mon livre des distributions. Je ne parvins pas à trouver de tels théorèmes, que d'ailleurs je ne soupçonnais pas, avant plusieurs mois, à Grenoble.

Figure 16: Page 243 of Laurent Schwartz's autobiography [16]

10.1.1 Derivative of a distribution

Let $f \in C^1(\Omega)$. We notice that both f and $\partial_{x_j} f$ are in $L^1_{loc}(\Omega)$, so we can consider the associated distributions, i. e., for $\varphi \in \mathcal{D}(\Omega)$,

$$T_f(\varphi) = \int_\Omega f\varphi \qquad \text{and} \qquad T_{\partial_{x_j} f}(\varphi) = \int_\Omega (\partial_{x_j} f)\varphi.$$

But

$$T_{\partial_{x_j} f}(\varphi) = \underbrace{\int_\Omega (\partial_{x_j} f)\varphi = -\int_\Omega f(\partial_{x_j}\varphi)}_{\text{integration by parts}} = -T_f(\partial_{x_j}\varphi).$$

Consequently, if you want that a (to be defined) derivative, with respect to x_j, of the distribution T_f, associated to f, behaves like the distribution $T_{\partial_{x_j} f}$, associated to the classical derivative of f, you have to set

$$(\partial_{x_j} T_f)(\varphi) = -T_f(\partial_{x_j}\varphi).$$

Definition 28. *Let $T \in \mathcal{D}'(\Omega)$. For all $\varphi \in \mathcal{D}(\Omega)$, we define*

$$(\partial_{x_j} T)(\varphi) = -T(\partial_{x_j}\varphi).$$

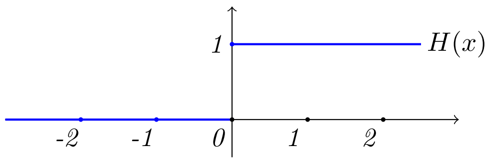
We have that $\partial_{x_j} T \in \mathcal{D}'(\Omega)$, in fact $\partial_{x_j} T$ is linear and

$$|(\partial_{x_j}T)(\varphi)| = |T(\partial_{x_j}\varphi)| \le C_K \sum_{|\alpha|\le m_K} \sup_{\Omega} |D^\alpha(\partial_{x_j}\varphi)| \le C_K \sum_{|\beta|\le m_K+1} \sup_{\Omega} |D^\beta \varphi|$$

for all $\varphi \in \mathcal{D}(\Omega)$ such that $\operatorname{Supp}\varphi \subseteq K$. Remark that, if T is a distribution of order m, then $\partial_{x_j} T$ is a distribution of order less or equal to $m+1$.

Example 8. *Let H be the Heavisde function, i. e.*

$$H : \mathbb{R} \to \mathbb{R}, \qquad H(x) = \begin{cases} 1 & \text{for} \quad x \ge 0, \\ 0 & \text{for} \quad x < 0. \end{cases}$$



H is a L^1_{loc} function. We denote by H also the associated distribution, i. e.

$$H : \mathcal{D}(\mathbb{R}) \to \mathbb{R}, \qquad H(\varphi) = \int_{\mathbb{R}} H\varphi = \int_0^{+\infty} \varphi(t)\, dt.$$

Let's compute the derivative of H as a distribution.

$$H'(\varphi) = -H(\varphi') = -\int_0^{+\infty} \varphi'(t)\, dt = -\varphi(t)\big|_0^{+\infty} = \varphi(0) = \delta_0(\varphi),$$

i. e. $H' = \delta_0$, the derivative of the Heaviside distribution is Dirac's delta at 0. Remark that the Heaviside function possess finite classical derivative equal to 0 for all $x \in \mathbb{R} \setminus \{0\}$. The derivative in the sense of distribution is more precise: Dirac's delta at 0 coincide, as distribution, to 0 in a neighborhood of each point of $\mathbb{R} \setminus \{0\}$, but gives a precise information also at 0.

Exercise 4. *Consider, for all $\varphi \in \mathcal{D}(\mathbb{R})$,*

$$PV_{\frac{1}{x}}(\varphi) = \lim_{\varepsilon \to 0^+} \int_{|x|\ge\varepsilon} \frac{\varphi(x)}{x}\, dx.$$

Prove that $PV_{\frac{1}{x}}$ is a distribution of order ≤ 1 (we call it principal value of $\frac{1}{x}$).

Denote by $T_{\log}$ the distribution associated to the $L^1_{loc}(\mathbb{R})$ function $x \mapsto \log|x|$. Prove that $T'_{\log} = PV_{\frac{1}{x}}$.

Hint. Let $\varphi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{Supp}\varphi \subseteq [-M, M]$. Remark that in this case

$$PV_{\frac{1}{x}}(\varphi) = \lim_{\varepsilon \to 0^+} \int_{\varepsilon \le |x| \le M} \frac{\varphi(x)}{x}\, dx.$$

Remarking that

$$\int_{\varepsilon \le |x| \le M} \frac{\varphi(0)}{x}\, dx = \varphi(0) \int_{\varepsilon \le |x| \le M} \frac{1}{x}\, dx = 0,$$

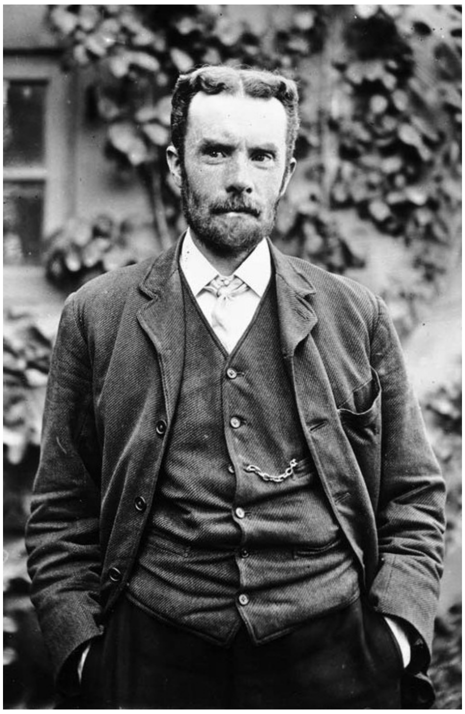


Figure 17: Olivier Heaviside (1850-1925)

we have that

$$\int_{\varepsilon \le |x| \le M} \frac{\varphi(x)}{x}\, dx = \int_{\varepsilon \le |x| \le M} \frac{\varphi(x) - \varphi(0)}{x}\, dx.$$

Consider now the function

$$\psi(x) = \begin{cases} \dfrac{\varphi(x) - \varphi(0)}{x} & \text{if } x \neq 0, \\ \varphi'(0) & \text{if } x = 0. \end{cases}$$

We have that $\psi \in C([-M, M])$. Consequently

$$\lim_{\varepsilon \to 0^+} \int_{|x| \ge \varepsilon} \frac{\varphi(x)}{x}\, dx = \lim_{\varepsilon \to 0^+} \int_{\varepsilon \le |x| \le M} \psi(x)\, dx = \int_{-M}^{M} \psi(x)\, dx,$$

so that the limit exists and it is finite. Moreover

$$\left|\int_{-M}^{M} \psi(x)\, dx\right| \le 2M \sup_{[-M,M]} |\psi| \qquad \text{and} \qquad \sup_{[-M,M]} |\psi| \le \sup_{\mathbb{R}} |\varphi'|.$$

We obtain finally

$$|PV_{\frac{1}{x}}(\varphi)| \le 2M \sup_{\mathbb{R}} |\varphi'| \quad \text{for all } \ \varphi \in \mathcal{D}(\mathbb{R}) \ \text{ with } \ \mathrm{Supp}\, \varphi \subseteq [-M, M],$$

i. e. $PV_{\frac{1}{x}}$ is a distribution of order ≤ 1.

Consider, for $\varphi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{Supp}\varphi \subseteq [-M, M]$,

$$T_{\log}(\varphi) = \int_{-M}^{M} (\log|x|)\varphi(x)\, dx.$$

We have

$$T'_{\log}(\varphi) = -T_{\log}(\varphi') = -\int_{-M}^{M} (\log|x|)\varphi'(x)\, dx.$$

Since the function $x \mapsto (\log|x|)\varphi'(x)$ is a L^1 function, we have that

$$\int_{-M}^{M} (\log|x|)\varphi'(x)\, dx = \lim_{\varepsilon\to 0^+} (\int_{-M}^{-\varepsilon} (\log|x|)\varphi'(x)\, dx + \int_{\varepsilon}^{M} (\log|x|)\varphi'(x)\, dx).$$

Now

$$\int_{-M}^{-\varepsilon} (\log|x|)\varphi'(x)\, dx = (\log|\varepsilon|)\varphi(-\varepsilon) - \int_{-M}^{-\varepsilon} \frac{\varphi(x)}{x}\, dx$$

and

$$\int_{\varepsilon}^{M} (\log|x|)\varphi'(x)\, dx = -(\log|\varepsilon|)\varphi(\varepsilon) - \int_{\varepsilon}^{M} \frac{\varphi(x)}{x}\, dx.$$

We finally obtain

$$\int_{-M}^{M} (\log|x|)\varphi'(x)\, dx = \lim_{\varepsilon\to 0^+} [(\log|\varepsilon|)(\varphi(-\varepsilon) - \varphi(\varepsilon)) - \int_{\varepsilon\le|x|\le M} \frac{\varphi(x)}{x}\, dx]$$

and the conclusion follows. Remark that the second part of the exercise already contains the first part, i. e. if one proves that $PV_{\frac{1}{x}}$ is the derivative of a distribution of order 0, then $PV_{\frac{1}{x}}$ is immediately a distribution of order ≤ 1.

Remark 21. *The function $x \mapsto \frac{1}{x}$ is not a $L^1_{loc}(\mathbb{R})$ function, so that it is not possible to define a distribution associated to this function. The distribution $PV_{\frac{1}{x}}$ is the correct substitute.*

Exercise 5. *Consider, for all $\varphi \in \mathcal{D}(\mathbb{R})$,*

$$FP_{\frac{1}{x^2}}(\varphi) = \lim_{\varepsilon\to 0^+} (\int_{|x|\ge\varepsilon} \frac{\varphi(x)}{x^2}\, dx - 2\frac{\varphi(0)}{\varepsilon}).$$

Prove that $FP_{\frac{1}{x^2}}$ is a distribution of order ≤ 2 (we call it finite part of $\frac{1}{x^2}$).

Prove that $PV'_{\frac{1}{x}} = -FP_{\frac{1}{x^2}}$.

The following result shows that, at least locally, a distribution is always a derivative (of order mn, in the sense of distributions) of a distribution associate to a bounded function.

Theorem 36 ("Structure locale d'une distribution" Th. XXI of [15])**.** *Let $T \in \mathcal{D}'(\Omega)$. Let ω be an open set in Ω such that $\overline{\omega}$ is a compact in Ω (i. e. ω is a relatively compact open subset of Ω).*

Then there exist $m \in \mathbb{N}$ and $f \in L^\infty(\omega)$ such that

$$T = D_1^m D_2^m \ldots D_n^m T_f \quad in \;\; \omega,$$

i. e., for all $\varphi \in \mathcal{D}(\omega)$,

$$T(\varphi) = (-1)^{mn} T_f(D_1^m D_2^m \ldots D_n^m \varphi) = (-1)^{mn} \int_\omega f(x) D_1^m D_2^m \ldots D_n^m \varphi(x)\, dx.$$

Proof. Suppose that

$$T(\varphi) = (-1)^{mn} \int_\omega f(x) D_1^m D_2^m \dots D_n^m \varphi(x)\, dx \quad \text{for all } \ \varphi \in \mathcal{D}(\omega). \tag{16}$$

Consequently

$$|T(\varphi)| \leq \|f\|_{L^\infty(\omega)} \int_\omega |D_1^m D_2^m \dots D_n^m \varphi(x)|\, dx$$

i. e. there exists $C > 0$ such that

$$|T(\varphi)| \leq C \int_\omega |D_1^m D_2^m \dots D_n^m \varphi(x)|\, dx \quad \text{for all } \ \varphi \in \mathcal{D}(\omega). \tag{17}$$

We prove now that (17) implies (16). In fact, suppose (17) holds. Let's define

$$V = \{D_1^m \dots D_n^m \varphi \;\big|\; \varphi \in \mathcal{D}(\omega)\}$$

and consider the functional

$$V \to \mathbb{C}, \qquad D_1^m \dots D_n^m \varphi \mapsto T(\varphi).$$

Thinking at V as a subspace of $L^1(\omega)$, we have that the above functional is linear and moreover condition (17) implies that it is continuous with respect to the norm of $L^1(\omega)$. We use now Hahn-Banach theorem. There exists $\Phi \in (L^1(\omega))'$ such that

$$\Phi(D_1^m \dots D_n^m \varphi) = T(\varphi) \qquad \text{for all } \ \varphi \in \mathcal{D}(\omega).$$

From Riesz's theorem we have that there exists $g \in L^\infty(\omega)$ such that

$$\Phi(v) = \int_\omega gv \qquad \text{for all } \ v \in L^1(\omega).$$

Consequently

$$T(\varphi) = \Phi(D_1^m \dots D_n^m \varphi) = \int_\omega g(x) D_1^m \dots D_n^m \varphi(x)\, dx \quad \text{for all } v \in L^1(\omega).$$

Taking $f = (-1)^{mn} g$, we have (16).

To conclude the proof it is sufficient to show (17). T is s distribution, then, in particular, there exist $C_{\overline{\omega}} > 0$ and $m_{\overline{\omega}} \in \mathbb{N}$ such that

$$|T(\varphi)| \leq C_{\overline{\omega}} \sum_{|\alpha| \leq m_{\overline{\omega}}} \sup_\Omega |D^\alpha \varphi| \quad \text{for all } \varphi \in \mathcal{D}(\Omega) \ \text{ with } \ \operatorname{Supp} \varphi \subseteq \overline{\omega}.$$

and consequently there exist $C > 0$ and $m \in \mathbb{N}$, such that

$$|T(\varphi)| \leq C \sum_{|\alpha| \leq m} \sup_\omega |D^\alpha \varphi| \quad \text{for all } \varphi \in \mathcal{D}(\omega).$$

Since ω is relatively compact, there exists $a > 0$ such that the diameter of ω is less or equal than a. Consequently, if $\psi \in \mathcal{D}(\omega)$, then

$$D^\alpha \psi(x) = \int_{-\infty}^{x_1} \partial_{x_1}(D^\alpha \psi(t, x')\, dt$$

and hence

$$\sup_\omega |D^\alpha \psi| \leq a \sup_\omega |D_1^{\alpha_1+1} \dots D_n^{\alpha_n} \psi|.$$

Using several times this argument we obtain that, for some $\tilde{C} > 0$ and k,

$$\sum_{|\alpha| \leq m} \sup_\omega |D^\alpha \varphi| \leq \tilde{C} \sup_\omega |D_1^k \dots D_n^k \psi|.$$

Consequently

$$|T(\varphi)| \leq C \sup_\omega |D_1^k \dots D_n^k \varphi| \quad \text{for all } \varphi \in \mathcal{D}(\omega). \tag{18}$$

Finally, remarking that, for $\psi \in \mathcal{D}(\mathbb{R}^n)$,

$$\psi(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \partial_{x_1} \dots \partial_{x_n} \psi(y_1, \dots, y_n)\, dy_1 \dots dy_n,$$

we have that, for $\varphi \in \mathcal{D}(\omega)$,

$$\sup_\omega |D_1^k \dots D_n^k \varphi| \leq \int_\omega |D_1^{k+1} \dots D_n^{k+1} \varphi|. \tag{19}$$

Putting together (18) and (19) we deduce (17) with $m = k + 1$. □

10.1.2 Multiplication of a distribution with a smooth function

Definition 29. *Let $T \in \mathcal{D}'(\Omega)$ and $a \in C^\infty(\Omega)$. We define, for all $\varphi \in \mathcal{D}(\Omega)$,*

$$aT(\varphi) = T(a\varphi).$$

We have that $aT \in \mathcal{D}'(\Omega)$, in fact aT is linear and

$$|aT(\varphi)| = |T(a\varphi)| \leq C_K \sum_{|\alpha| \leq m_K} \sup_\Omega |D^\alpha (a\varphi)|$$

for all $\varphi \in \mathcal{D}(\Omega)$ such that $\operatorname{Supp} \varphi \subseteq K$. *Remarking now that*

$$D^\alpha(a\varphi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^{\alpha-\beta} a D^\beta \varphi \qquad \textit{(Leibniz formula)},$$

then

$$\sup_\Omega |D^\alpha(a\varphi)| \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sup_K |D^{\alpha-\beta} a| \sup_\Omega |D^\beta \varphi|,$$

and consequently

$$|aT(\varphi)| \leq \tilde{C}_K \sum_{|\alpha| \leq m_K} \sup_\Omega |D^\alpha \varphi|,$$

where $\tilde{C}_K$ depends also on $\sup_K |D^\alpha a|$ *for all α such that $|\alpha| \leq m_K$.*

Example 9. *Let $T \in \mathcal{D}'(\Omega)$ and $a \in C^\infty(\Omega)$, with $\Omega \subseteq \mathbb{R}$. We want to compute $(aT)'$. Only for this time we will denote by $T'^{(d)}$ the derivative of T in the sense of distribution.*

$$\begin{aligned}
(aT)'^{(d)}(\varphi) &= -(aT)(\varphi') \\
&= -T(a\varphi') \\
&= -T((a\varphi)' - a'\varphi) \\
&= -T((a\varphi)') + T(a'\varphi) \\
&= T'^{(d)}(a\varphi) + T(a'\varphi) \\
&= (aT'^{(d)})(\varphi) + (a'T)(\varphi) \\
&= (aT'^{(d)} + a'T)(\varphi)
\end{aligned}$$

so that $(aT)'^{(d)} = aT'^{(d)} + a'T$, i. e. Leibniz formula remains valid also in the case of the multiplication of a distribution with a smooth function.

The following result shows that if a continuous function has a derivative in the sense of distributions which is another continuous function, then the function is classically differentiable and the classical derivative coincides with that one in distributional sense.

Theorem 37 (du Bois-Reymond). *Let $f, g \in C(\Omega)$. Suppose that $\partial_{x_j} T_f = T_g$. Then f is differentiable, in the direction of x_j, and $\partial_{x_j} f = g$.*

Proof. Suppose first that $f, g \in C_0(\Omega)$. Let $(\rho_n)_n$ be a mollifier. We have

$$f * \rho_n \xrightarrow{n} f \quad \text{and} \quad g * \rho_n \xrightarrow{n} g \quad \text{uniformly}$$

and

$$f * \rho_n(x) = \int_{\mathbb{R}^n} f(y)\rho_n(x-y)\,dy = T_f(\varphi_{n,x}),$$

$$g * \rho_n(x) = \int_{\mathbb{R}^n} g(y)\rho_n(x-y)\,dy = T_g(\varphi_{n,x}),$$

where $\varphi_{n,x}(y) = \rho_n(x-y)$ (here we have to think at x as a parameter). By hypothesis,

$$(\partial_{x_j} T_f)(\varphi_{n,x}) = T_g(\varphi_{n,x}). \tag{20}$$

Remarking now that

$$\partial_{y_j}\varphi_{n,x}(y) = -(\partial_{x_j}\rho_n)(x-y),$$

we have

$$\begin{aligned}
(\partial_{x_j} T_f)(\varphi_{n,x}) &= T_f(-\partial_{y_j}\varphi_{n,x}) \\
&= T_f(\partial_{x_j}\rho_n(x-\cdot)) \\
&= \int_{\mathbb{R}^n} f(y)\partial_{x_j}\rho_n(x-y)\,dy \\
&= \partial_{x_j}\Big(\int_{\mathbb{R}^n} f(y)\rho_n(x-y)\,dy\Big) \\
&= \partial_{x_j}(f * \rho_n)(x).
\end{aligned}$$

Condition (20) implies that

$$\partial_{x_j}(f * \rho_n)(x) = g * \rho_n(x).$$

Consequently

$$f * \rho_n \stackrel{n}{\longrightarrow} f \quad \text{and} \quad \partial_{x_j}(f * \rho_n) \stackrel{n}{\longrightarrow} g \quad \text{uniformly},$$

the conclusion follows from a classical result (see e. g. [6, Teor. 13.3]).

Suppose now that f, $g \in C(\Omega)$. Let $x_0 \in \Omega$ and let $\chi \in \mathcal{D}(\Omega)$ such that $\chi(x) = 1$ in a neighborhood of x_0. We know that

$$\partial_{x_j} T_f = T_g.$$

We have

$$\begin{aligned}
\partial_{x_j}(T_{\chi f}) &= \partial_{x_j}(\chi T_f) \\
&= \chi(\partial_{x_j} T_f) + (\partial_{x_j}\chi) T_f \\
&= \chi T_g + (\partial_{x_j}\chi) T_f \\
&= T_{\chi g + (\partial_{x_j}\chi) f}
\end{aligned}$$

Remark now that the functions χf and $\chi g + (\partial_{x_j}\chi) f$ are continuous functions with compact support in Ω, so that from the first part of the proof, the function χf is differentiable with respect to the direction of x_j and its partial derivative is the function $\chi g + (\partial_{x_j}\chi) f$. Finally in the neighborhood of x_0 in which the function χ is identically equal to 1 we have

$$\chi f = f \qquad \text{and} \qquad \chi g + (\partial_{x_j}\chi) f = g.$$

The theorem is proved.

□

11

11.1 Distributions with compact support

The content of this paragraph can be found in [8, Ch. 1.5] (see also [15]).

Remark 22. *Consider Ω, open set in $\mathbb{R}^n$. Suppose*

$$\Omega = \bigcup_j \Omega_j \quad \textit{with} \quad \Omega_j \textit{ open}, \quad \overline{\Omega}_j \textit{ compact} \quad \textit{and} \quad \overline{\Omega}_j \subseteq \Omega_{j+1} \textit{ for all } j.$$

Consider $f \in C^\infty(\Omega)$ and, for $j \in \mathbb{N}$,

$$\tilde{p}_j(f) = \sum_{|\alpha| \le j} \sup_{x \in \Omega_j} |D^\alpha f(x)|. \tag{21}$$

$\tilde{p}_j$ is a seminorm, i. e.

$$\tilde{p}_j(\lambda f) = |\lambda| \tilde{p}_j(f) \qquad \textit{and} \qquad \tilde{p}_j(f + g) \le \tilde{p}_j(f) + \tilde{p}_j(g).$$

We denote by $\mathcal{E}(\Omega)$ the Frechét space $C^\infty(\Omega)$ with the topology generated by the countable family of seminorms $(\tilde{p}_j)_j$.

Let $S : \mathcal{E}(\Omega) \to \mathbb{R}$ (or $\mathbb{C}$) be a linear functional. S is continuous if there exists $j_0 \in \mathbb{N}$ and $C_0 > 0$ such that

$$|S(f)| \leq C_0\, \tilde{p}_{j_0}(f) \qquad \text{for all} \quad f \in C^\infty(\Omega). \tag{22}$$

We denote by $\mathcal{E}'(\Omega)$ the dual space of $\mathcal{E}(\Omega)$.

We remark that $S \in \mathcal{E}'(\Omega)$ if and only if there exists K compact set in Ω and there exist $C_K > 0$ and $m_K \in \mathbb{N}$ such that

$$|S(f)| \leq C_K \sum_{|\alpha| \leq m_K} \sup_{x \in K} |D^\alpha f(x)| \qquad \text{for all} \quad f \in \mathcal{E}(\Omega). \tag{23}$$

In fact (21) and (22) imply (23) with $K = \overline{\Omega}_{j_0}$, $C_K = C_0$ and $m_K = j_0$. Conversely, if (23) holds, there exists j_0 such that $m_K \leq j_0$ and $K \subseteq \overline{\Omega}_{j_0}$. Consequently

$$|S(f)| \leq C_K\, \tilde{p}_{j_0}(f) \qquad \text{for all} \quad f \in C^\infty(\Omega).$$

The next result shows that the subspace of distributions with compact support in Ω can be identified with $\mathcal{E}'(\Omega)$.

Theorem 38. *Let $T \in \mathcal{D}'(\Omega)$ and let $\operatorname{Supp} T$ be a compact set in Ω.*

Then there exists a unique $S \in \mathcal{E}'(\Omega)$ such that $S\big|_{\mathcal{D}(\Omega)} = T$.

Conversely let $S \in \mathcal{E}'(\Omega)$.

Then $S\big|_{\mathcal{D}(\Omega)}$ is in $\mathcal{D}'(\Omega)$ and it has compact support.

Proof. Let $T \in \mathcal{D}'(\Omega)$ and let $\operatorname{Supp} T$ be a compact set in Ω. Let $\chi \in \mathcal{D}(\Omega)$ with $\chi = 1$ in a neighborhood of $\operatorname{Supp} T$. We define, for all $f \in \mathcal{E}(\Omega)$,

$$S(f) = T(\chi f).$$

We show that S is in $\mathcal{E}'(\Omega)$. In fact S is linear and, considering K' a compact set in Ω containing the support of χ, we have

$$|S(f)| = |T(\chi f)| \leq C_{K'} \sum_{|\alpha| \leq m_{K'}} \sup_{x \in \Omega} |D^\alpha(\chi(x) f(x))|$$

and

$$\sup_{x \in \Omega} |D^\alpha(\chi(x) f(x))| \leq \tilde{C} \sum_{\beta \leq \alpha} \sup_{x \in K'} |D^\beta f(x)|,$$

where $\tilde{C}$ depends on χ but not on f. Consequently

$$|S(f)| \leq \tilde{C}_{K'} \sum_{|\alpha| \leq m_{K'}} \sup_{x \in K'} |D^\alpha f(x)| \qquad \text{for all} \quad f \in \mathcal{E}(\Omega).$$

We show now that $S(\varphi) = T(\varphi)$ for all $\varphi \in \mathcal{D}(\Omega)$. In fact

$$S(\varphi) = T(\chi\varphi) = T(\varphi) + T((\chi - 1)\varphi)$$

but, remarking that the function $x \mapsto \chi(x) - 1$ is identically equal to 0 in a neighborhood of $\operatorname{Supp} T$, we have $T((\chi - 1)\varphi) = 0$, for all $\varphi \in \mathcal{D}(\Omega)$ (for this last fact, see Exercise 5 below). Finally we prove that S is unique. Suppose that $S_1,\ S_2 \in \mathcal{E}'(\Omega)$ such that

$$S_1(\varphi) = S_2(\varphi) = T(\varphi) \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

We know that, for $j = 1,\ 2$, there exists K_j compact set in Ω and there exist $C_j > 0$ and $m_j \in \mathbb{N}$ such that

$$|S_j(f)| \leq C_j \sum_{|\alpha| \leq m_j} \sup_{x \in K_j} |D^\alpha f(x)| \qquad \text{for all} \quad f \in \mathcal{E}(\Omega).$$

Let now $\tilde{\chi} \in \mathcal{D}(\Omega)$ with $\tilde{\chi} = 1$ in a neighborhood of $K_1 \cup K_2$. We have

$$\begin{aligned} S_1(f) &= S_1(\tilde{\chi} f + (1 - \tilde{\chi}) f) \\ &= S_1(\tilde{\chi} f) + S_1((1 - \tilde{\chi}) f) \qquad \text{with } S_1((1 - \tilde{\chi}) f) = 0 \\ &= S_1(\tilde{\chi} f) \\ &= S_2(\tilde{\chi} f) \\ &= S_2(\tilde{\chi} f) + S_2((1 - \tilde{\chi}) f) \qquad \text{with } S_2((1 - \tilde{\chi}) f) = 0 \\ &= S_2(\tilde{\chi} f + (1 - \tilde{\chi}) f) \\ &= S_2(f). \end{aligned}$$

Conversely, let $\in \mathcal{E}'(\Omega)$. Then S is linear and there exists K compact set in Ω and there exist $C_K > 0$ and $m_K \in \mathbb{N}$ such that

$$|S(f)| \leq C_K \sum_{|\alpha| \leq m_K} \sup_{x \in K} |D^\alpha f(x)| \qquad \text{for all} \quad f \in \mathcal{E}(\Omega).$$

Consequently, for all $\tilde{K}$ compact set in Ω,

$$|S(\varphi)| \leq C_K \sum_{|\alpha| \leq m_K} \sup_{x \in \Omega} |D^\alpha \varphi(x)| \quad \text{for all } \varphi \in \mathcal{D}(\Omega) \text{ with } \operatorname{Supp} \varphi \subseteq \tilde{K},$$

i. e. $S\big|_{\mathcal{D}(\Omega)} \in \mathcal{D}'(\Omega)$ and if $K \cap \operatorname{Supp} \varphi = \emptyset$, then $S(\varphi) = 0$. i. e. the support of $S\big|_{\mathcal{D}(\Omega)}$ is contained in K. □

Remark 23. *Let Ω be an open set in $\mathbb{R}^n$. $\mathcal{D}(\Omega)$ is dense in $\mathcal{E}(\Omega)$. Remember that U is a neighborhood of 0 in $\mathcal{E}(\Omega)$ if there exists $j_0 \in \mathbb{N}$ and there exists $r > 0$ such that*

$$\{f \in C^\infty(\Omega) \mid \tilde{p}_{j_0}(f) < r\} \subseteq U.$$

Consider a sequence of functions $(\chi_n)_n$ in $\mathcal{D}(\Omega)$, such that for every n, $\chi_n \in \mathcal{D}(\Omega_{j+1})$ and χ_n is identically equal to 1 in a neighborhood of $\overline{\Omega}_n$. Given $f \in \mathcal{E}(\Omega)$ and given W, a neighborhood of f in the topology of $\mathcal{E}(\Omega)$, we see that there exists $\bar{n}$ such that, for all $n \geq \bar{n}$, $\chi_n \cdot f \in W$.

Remark 24. *Let $(f_n)_n$ be a sequence in $\mathcal{E}(\Omega)$. The sequence will converge to 0 in the sense of $\mathcal{E}(\Omega)$ if, for all $k \in \mathbb{N}$,*

$$\lim_n \tilde{p}_k(f_n) = 0,$$

i. e., for all $k \in \mathbb{N}$,

$$\sum_{|\alpha|\le k} \sup_{x\in\Omega_k} |D^\alpha f_n(x)| \overset{n}{\longrightarrow} 0.$$

If a sequence $(\varphi_n)_n$ *in* $\mathcal{D}(\Omega)$ *converges to* 0 *in the sense of* $\mathcal{D}(\Omega)$*, then it converges to* 0 *also in the sense of* $\mathcal{E}(\Omega)$*.*

11.2 Solutions of some exercises

1) Let h be a function in $C^1([0,1]\times\mathbb{R})$. Let $g:\mathbb{R}\to\mathbb{R}$ such that

$$g(x) = \int_0^1 h(s,x)\, ds.$$

Prove that g is in $C^1(\mathbb{R})$ and

$$g'(x) = \int_0^1 \frac{\partial h}{\partial x}(s,x)\, ds.$$

Hint. Let $(t_n)_n$ be a sequence in $[-1, 1]$, such that $t_n \overset{n}{\longrightarrow} 0$. Denote by

$$f_n(s) = \frac{h(s,x+t_n)-h(s,x)}{t_n}.$$

We have

$$\begin{cases} f_n(s) \overset{n}{\longrightarrow} \frac{\partial h}{\partial x}(s,x) & \text{pointwise,} \\ |f_n(s)| \le \underbrace{\max_{\substack{\sigma\in[0,1] \\ y\in[x-1,x+1]}} |\frac{\partial h}{\partial x}(\sigma,y)|}_{\le C}. & \end{cases}$$

Using the dominated convergence theorem we have

$$\begin{aligned} \lim_n \frac{g(x+t_n)-g(x)}{t_n} &= \lim_n \int_0^1 \frac{h(s,x+t_n)-h(s,x)}{t_n}\, ds \\ &= \lim_n \int_0^1 f_n(s)\, ds \\ &= \int_0^1 \frac{\partial h}{\partial x}(s,x)\, ds. \end{aligned}$$

Since this is valid for all the sequence $(t_n)_n$, the conclusion follows.

2) Let ψ be a function in $C^\infty(\mathbb{R})$. Let $g:\mathbb{R}\to\mathbb{R}$ such that

$$g(x) = \begin{cases} \dfrac{\psi(x)-\psi(0)}{x} & \text{if } x\ne 0, \\ \psi'(0) & \text{if } x = 0. \end{cases}$$

Prove that g is in $C^\infty(\mathbb{R})$.

Hint. We have

$$\psi(x) - \psi(0) = \int_0^x \psi'(t)\,dt = \int_0^1 x\psi'(sx)\,ds,$$

so that

$$g(x) = \int_0^1 \psi'(sx)\,ds.$$

Define now $h(s,x) = \psi'(sx)$. Using recursively Exercise 1, we obtain that g is C^∞ with

$$g^{(n)}(x) = \int_0^1 s^n \psi^{(n+1)}(sx)\,ds.$$

3) Find all the distributions in $T \in \mathcal{D}'(\mathbb{R})$ such that $x \cdot T = 0$.

Hint. Remark that if $\varphi \in \mathcal{D}(\mathbb{R})$ and $\operatorname{Supp}\varphi \subseteq \mathbb{R} \setminus \{0\}$, then

$$x \cdot T = 0 \qquad \text{implies} \qquad T(\varphi) = 0.$$

In fact, if $\operatorname{Supp}\varphi \subseteq \mathbb{R} \setminus \{0\}$, then the function $x \mapsto \frac{\varphi(x)}{x}$ is in $\mathcal{D}(\mathbb{R})$ so that

$$T(\varphi) = x \cdot T(\frac{\varphi(x)}{x}) = 0.$$

Consequently, if $x \cdot T = 0$, then $\operatorname{Supp} T \subseteq \{0\}$.

Consider now $\chi \in \mathcal{D}(\mathbb{R})$, with χ equal to 1 in a neighborhood of 0. Let $\varphi \in \mathcal{D}(\mathbb{R})$. We have $\varphi = \chi\varphi + (1-\chi)\varphi$. Then

$$\begin{aligned}
T(\varphi) &= T(\chi\varphi) + T((1-\chi)\varphi) \qquad\qquad (\text{but } \ T((1-\chi)\varphi) = 0)\\
&= T(\chi \cdot (\varphi(x) - \varphi(0)) + T(\chi \cdot (\varphi(0)))\\
&= T(\chi \cdot \underbrace{\frac{\varphi(x)-\varphi(0)}{x}}_{\in C^\infty(\mathbb{R}) \text{ (Ex. 2)}} \cdot x) + \varphi(0)T(\chi)\\
&= x \cdot T(\chi \cdot \frac{\varphi(x)-\varphi(0)}{x}) + \varphi(0)T(\chi)\\
&= \varphi(0)T(\chi)\\
&= T(\chi)\delta_0(\varphi).
\end{aligned}$$

It is easy to see that $T(\chi)$ does not depend on χ, in the sense that taking χ_1 and χ_2 in $\mathcal{D}(\mathbb{R})$, with χ_1 and χ_2 equal to 1 in a neighborhood of 0, then $T(\chi_1) = T(\chi_2)$. We can conclude that

$$\text{if} \quad x \cdot T = 0 \qquad \text{then} \qquad T = c\delta_0, \text{ for some } \ c \in \mathbb{R}.$$

4) Find all the distributions in $T \in \mathcal{D}'(\mathbb{R})$ such that $x \cdot T = T_1$.

Hint. The problem, at level of functions, should be "find all the functions $f(x)$ such that $xf(x) = 1$". The solution would be, roughly speaking, $f(x) = \frac{1}{x}$. This suggests to try with $PV_{\frac{1}{x}}$. We have

$$\begin{aligned} x \cdot PV_{\frac{1}{x}}(\varphi) = PV_{\frac{1}{x}}(x\varphi(x)) &= \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{x\varphi(x)}{x}\, dx \\ &= \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \varphi(x)\, dx = \int_{\mathbb{R}} 1 \cdot \varphi(x)\, dx = T_1(\varphi), \end{aligned}$$

i. e.

$$x \cdot PV_{\frac{1}{x}} = T_1.$$

The problem is now to find all the the possible distributions with such a property. Suppose that there exists another S such that

$$x \cdot S = T_1.$$

We have

$$x \cdot PV_{\frac{1}{x}} - x \cdot S = T_1 - T_1 = 0, \qquad \text{i. e.} \qquad x \cdot (PV_{\frac{1}{x}} - S) = 0.$$

Using Exercise 4 we can conclude that

$$\text{if} \quad x \cdot T = T_1 \qquad \text{then} \qquad T = PV_{\frac{1}{x}} + c\delta_0, \text{ for some } \ c \in \mathbb{R}.$$

5) Let $T \in \mathcal{D}'(\Omega)$ and let $\varphi \in \mathcal{D}(\Omega)$ such that

$$\operatorname{Supp} T \cap \operatorname{Supp} \varphi = \emptyset.$$

Prove that $T(\varphi) = 0$.

Hint. Let $x \in \operatorname{Supp} \varphi$. Then $x \notin \operatorname{Supp} T$. Then there exists $r_x > 0$ such that, for all $\psi \in \mathcal{D}(\Omega)$, if $\operatorname{Supp} \psi \subseteq B(x, r_x)$, then $T(\psi) = 0$. Remark that $\{B(x, r_x) \mid x \in \operatorname{Supp} \varphi\}$ is an open covering of the compact set $\operatorname{Supp} \varphi$. Let

$$B(x_1, r_1), \ldots, B(x_N, r_N)$$

be a finite subcovering and let

$$\psi_1, \ldots, \psi_N$$

a partition of unity of $\operatorname{Supp} \varphi$, i. e. for all $j = 1, \ldots, N$, $\operatorname{Supp} \psi_j \subseteq B(x_j, r_j)$ and, for all $x \in \operatorname{Supp} \varphi$, $\sum_j \psi_j(x) = 1$. Then

$$T(\varphi) = T(\varphi \sum_j \psi_j) = T(\sum_j \varphi\psi_j) = \sum_j T(\varphi\psi_j) = 0.$$

12

12.1 Convolution of distributions

The content of this paragraph can be found in [8, Ch. 1.6] (see also [15]). We begin defining the convolution of a distribution with a test function. Also in this case we will use the analogy with the behavior of the convolution of functions. If we take $f \in L^1_{loc}(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then

$$(f * \varphi)(x) = \int_{\mathbb{R}^n} f(y)\varphi(x-y)\,dy = T_f(\psi_x), \qquad \text{where } \psi_x \,:\, y \mapsto \varphi(x-y).$$

This suggests the following definition.

Definition 30. *Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. For all $x \in \mathbb{R}^n$, we define*

$$(T * \varphi)(x) = T(\psi_x), \qquad \textit{where } \psi_x \,:\, y \mapsto \varphi(x-y).$$

We have seen in the preliminary results that, if $f \in L^1_{loc}(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then $f * \varphi \in C^\infty(\mathbb{R}^n)$ and

$$D_{x_j}(f * \varphi) = f * (D_{x_j}\varphi).$$

A similar result holds for the distributions.

Theorem 39. *Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$.*
Then $T * \varphi \in C^\infty(\mathbb{R}^n)$ and

$$D_{x_j}(T * \varphi) = (D_{x_j}T) * \varphi = T * (D_{x_j}\varphi).$$

Proof. Let's show only that $T * \varphi$ is a continuous function, letting the other part of the proof as an exercise. Fix $\bar{x}$ and consider $(x_n)_n$, a sequence in $\mathbb{R}^n$, such that $\lim_n x_n = \bar{x}$. Consider, for all $n \in \mathbb{N}$,

$$\psi_n(y) = \varphi(x_n - y) \quad \text{and} \quad \bar{\psi}(y) = \varphi(\bar{x} - y).$$

Since the functions ψ_n are translations of the compactly supported function $y \mapsto \varphi(-y)$ and the sequence $(x_n)_n$ is bounded, then there exists a compact set K such that, for all n, $\operatorname{Supp}\psi_n \subseteq K$. On the other hand, using the uniform continuity of φ and its derivatives, we have that $\psi_n \xrightarrow{n} \bar{\psi}$ uniformly with all its derivatives. This means that $(\psi_n)_n$ converges to $\bar{\psi}$ in the sense of $\mathcal{D}(\mathbb{R}^n)$. Consequently

$$T * \varphi(x_n) = T(\psi_n) \xrightarrow{n} T(\bar{\psi}) = T * \varphi(\bar{x}).$$

□

Now we give a results on the convergence of convolution of distributions.

Theorem 40. *Let T be a distribution in $\mathcal{D}'(\mathbb{R}^n)$ and $(\varphi_n)_n$ be a sequence in $\mathcal{D}(\mathbb{R}^n)$ converging to $\bar{\varphi}$ in the sense of $\mathcal{D}(\mathbb{R}^n)$ (this means that $(\varphi_n - \bar{\varphi})_n$ is converging to 0 in the sense of $\mathcal{D}(\mathbb{R}^n)$).*

Then $(T * \varphi_n)_n$ converges to $T * \bar{\varphi}$ in the sense on $\mathcal{E}(\mathbb{R}^n)$ (this means that, for all seminorms $\tilde{p}_j$, we have $\lim_n \tilde{p}_j(\varphi_n - \bar{\varphi}) = 0$).

Proof. Remarking that $T * \varphi$ is linear in φ, i. e. $T * (\alpha_1\varphi_1 + \alpha_2\varphi_2) = \alpha_1(T * \varphi_1) + \alpha_2(T * \varphi_2)$, it is not restrictive to suppose that $\varphi_n \xrightarrow{n} 0$ in the sense of $\mathcal{D}(\mathbb{R}^n)$. Let K be a compact set such that, for all n, $\operatorname{Supp}\varphi_n \subseteq K$. Recalling that, for $f \in \mathcal{E}(\mathbb{R}^n)$,

$$\tilde{p}_j(f) = \sum_{|\alpha|\le j} \sup_{x\in\Omega_j} |D^\alpha f(x)|,$$

we have to show that, for every fixed j,

$$\lim_n \Big(\sum_{|\alpha|\le j} \sup_{x\in\Omega_j} |D^\alpha (T * \varphi_n)(x)|\Big) = 0.$$

Now

$$D^\alpha(T*\varphi_n)(x) = (T*D^\alpha\varphi_n)(x) = T(\psi_{n,x}), \qquad \text{where } \psi_{n,x} : y \mapsto (D^\alpha\varphi_n)(x-y).$$

If $x \in \overline{\Omega}_j$, then the support of $y \mapsto \psi_{n,x}(y)$ is contained in the compact set $\tilde{K} = \overline{\Omega}_j - K$. Consequently, for all $x \in \overline{\Omega}_j$,

$$|T(\psi_{n,x})| \le C_{\tilde{K}} \sum_{\beta\le m_{\tilde{K}}} \sup_{y\in\mathbb{R}^n} |D^\beta \psi_{n,x}(y)|$$

and then

$$\sum_{|\alpha|\le j} \sup_{x\in\Omega_j} |D^\alpha(T * \varphi_n)(x)| \le C_{\tilde{K}} \sum_{\beta\le m_{\tilde{K}}+j} \sup_{x\in\mathbb{R}^n} |D^\beta \varphi_n(x)|$$

and the conclusion follows. □

The following result says that the convolution of a distribution with two test functions is, in some sense, associative.

Theorem 41. *Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and φ, $\psi \in \mathcal{D}(\mathbb{R}^n)$.*
Then

$$(T * \varphi) * \psi = T * (\varphi * \psi).$$

Proof. Remark that, if $f \in C_0(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} f(x)\,dx = \lim_{\varepsilon\to 0^+} \varepsilon^n \sum_{\nu\in\mathbb{Z}^n} f(\varepsilon\nu).$$

Consequently, denoting by $\tau_a f$ the function $x \mapsto f(x-a)$ for $a \in \mathbb{R}^n$,

$$\begin{aligned}\varphi * \psi(x) = \int_{\mathbb{R}^n} \varphi(x-y)\psi(y)\,dy &= \lim_{\varepsilon\to 0^+} \varepsilon^n \sum_{\nu\in\mathbb{Z}^n} \varphi(x-\varepsilon\nu)\psi(\varepsilon\nu) \\ &= \lim_{\varepsilon\to 0^+} \varepsilon^n \sum_{\nu\in\mathbb{Z}^n} \psi(\varepsilon\nu)\tau_{\varepsilon\nu}\varphi(x).\end{aligned}$$

Moreover, defining

$$f_\varepsilon(x) = \varepsilon^n \sum_{\nu\in\mathbb{Z}^n} \varphi(x-\varepsilon\nu)\psi(\varepsilon\nu) = \varepsilon^n \sum_{\nu\in\mathbb{Z}^n} \psi(\varepsilon\nu)\tau_{\varepsilon\nu}\varphi(x),$$

f_ε is in $\mathcal{D}(\mathbb{R}^n)$ and it is possible to see that

$$f_\varepsilon \xrightarrow{\varepsilon} \varphi * \psi \qquad \text{in the sense of } \mathcal{D}(\mathbb{R}^n),$$

and, using Theorem 40,

$$T * f_\varepsilon \xrightarrow{\varepsilon} T * (\varphi * \psi) \qquad \text{in the sense of } \mathcal{E}(\mathbb{R}^n). \tag{24}$$

Now

$$\begin{aligned}
(T * f_\varepsilon)(x) &= T * (\varepsilon^n \sum_{\nu\in\mathbb{Z}^n} \psi(\varepsilon\nu)\tau_{\varepsilon\nu}\varphi)(x) \\
&= \varepsilon^n \sum_{\nu\in\mathbb{Z}^n} \psi(\varepsilon\nu)((T * \tau_{\varepsilon\nu}\varphi)(x)) \\
&= \varepsilon^n \sum_{\nu\in\mathbb{Z}^n} \psi(\varepsilon\nu)T(\theta_{\nu,x}) \qquad \text{where } \ \theta_{\nu,x} : y \mapsto \varphi(x - \varepsilon\nu - y), \\
&= \varepsilon^n \sum_{\nu\in\mathbb{Z}^n} \psi(\varepsilon\nu)((T * \varphi)(x - \varepsilon\nu)) \\
&= \varepsilon^n \sum_{\nu\in\mathbb{Z}^n} (T * \varphi)(x - \varepsilon\nu)\psi(\varepsilon\nu).
\end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0^+} (T * f_\varepsilon)(x) = \lim_{\varepsilon\to 0^+} \varepsilon^n \sum_{\nu\in\mathbb{Z}^n} (T * \varphi)(x - \varepsilon\nu)\psi(\varepsilon\nu) = (T * \varphi) * \psi(x). \tag{25}$$

Putting together (24) and (25) we have the conclusion.

□

Remark 25. *Let $T \in \mathcal{D}'(\mathbb{R}^n)$. There exists $(f_n)_n$, sequence in $\mathcal{E}(\mathbb{R}^n)$, such that*

$$T_{f_n} \xrightarrow{n} T \qquad \textit{in the sense of } \mathcal{D}'(\mathbb{R}^n),$$

i. e. for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $T(\varphi) = \lim_n T_{f_n}(\varphi)$.

In fact let $(\rho_n)_n$ be a mollifier. We have

$$\rho_n * \varphi \xrightarrow{n} \varphi \qquad \textit{in the sense of } \mathcal{D}(\mathbb{R}^n)$$

and consequently

$$T * (\rho_n * \varphi) \xrightarrow{n} T * \varphi \qquad \textit{in the sense of } \mathcal{E}(\mathbb{R}^n).$$

But, form the Theorem 41, $T * (\rho_n * \varphi) = (T * \rho_n) * \varphi$, so that

$$(T * \rho_n) * \varphi \xrightarrow{n} T * \varphi \qquad \textit{in the sense of } \mathcal{E}(\mathbb{R}^n),$$

and consequently

$$((T * \rho_n) * \varphi)(0) \xrightarrow{n} (T * \varphi)(0). \tag{26}$$

Remark that $T * \rho_n$ is in $\mathcal{E}(\mathbb{R}^n)$.

Denote now by $\check{\varphi}$ the function $x \mapsto \varphi(-x)$. We have, for $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$(T * \check{\varphi})(0) = T(\varphi).$$

Considering (26) with $\check{\varphi}$ at the place of φ, we deduce that

$$T_{T*\rho_n}(\varphi) = ((T * \rho_n) * \check{\varphi})(0) \xrightarrow{n} (T * \check{\varphi})(0) = T(\varphi).$$

The following result gives another property of the convolution of a distribution with a test function, in particular the convolution of a distribution with a test function commutes with the translation operator τ_h.

Theorem 42. *Let $T \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$.*
Then
$$\tau_h(T * \varphi) = T * \tau_h \varphi.$$

Proof. We have

$$\tau_h(T * \varphi)(x) = (T * \varphi)(x - h) = T(\psi_{x-h}), \quad \text{where } \psi_{x-h} : y \mapsto \varphi(x - h - y).$$

But
$$\varphi(x - h - y) = \tau_h \varphi(x - y),$$
so that
$$T(\psi_{x-h}) = T(\tilde{\psi}_x), \qquad \text{where } \tilde{\psi}_x : y \mapsto \tau_h \varphi(x - y).$$
Finally
$$T(\tilde{\psi}_x) = (T * \tau_h \varphi)(x).$$
□

The final results gives a characterization of the convolution of a distribution with a test function.

Theorem 43. *Let $\Phi : \mathcal{D}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n)$ be a functional such that*

i) Φ is linear.

ii) Φ is continuous (i. e. if $(\varphi_n)_n$ is converging to 0 in the sense of $\mathcal{D}(\mathbb{R}^n)$, then $(\Phi(\varphi_n))_n$ is converging to 0 in the sense of $\mathcal{E}(\mathbb{R}^n)$).

iii) Φ commutes with τ_h (i. e., for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$, $\tau_h \Phi(\varphi) = \Phi(\tau_h \varphi)$).

Then there exists a unique $T \in \mathcal{D}'(\mathbb{R}^n)$ such that, for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$,
$$\Phi(\varphi) = T * \varphi. \tag{27}$$

Proof. Define
$$T(\varphi) = \Phi(\check{\varphi})(0), \qquad \text{where } \check{\varphi}(x) = \varphi(-x).$$
We verify now that T is a distribution. From *i)* and *ii)* we deduce that T is linear and if $(\varphi_n)_n$ is a sequence in $\mathcal{D}(\mathbb{R}^n)$ which goes to 0 in the sense of $\mathcal{D}(\mathbb{R}^n)$, then $(\Phi(\check{\varphi}_n))_n$ goes to 0 in the sense of $\mathcal{E}(\mathbb{R}^n)$ and consequently $(\Phi(\check{\varphi}_n)(0))_n$ goes to 0 in $\mathbb{R}$ (or $\mathbb{C}$), so that T is a distribution.

We verify that T satisfies (27). We have
$$T * \varphi(x) = T(\psi_x) = \Phi(\check{\psi}_x)(0),$$
where $\psi_x : y \mapsto \varphi(x - y)$ and $\check{\psi}_x : y \mapsto \varphi(x + y) = \tau_{-x}\varphi(y)$. Consequently,
$$T * \varphi(x) = \Phi(\check{\psi}_x)(0) = \underbrace{\Phi(\tau_{-x}\varphi)(0) = \tau_{-x}\Phi(\varphi)(0)}_{\text{from } iii)} = \Phi(\varphi)(x).$$
□

After all this work, we are now ready to define the convolution of two distributions.

Definition 31. *Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and $S \in \mathcal{E}'(\mathbb{R}^n)$ (i. e. S is a distribution with compact support). Consider*

$$\Phi : \mathcal{D}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n), \qquad \Phi(\varphi) = T * (S * \varphi).$$

i) Φ is linear. In fact, e. g.,

$$T * (S * (\varphi_1 + \varphi_2)) = T * ((S * \varphi_1) + (S * \varphi_2)) = T * (S * \varphi_1) + T * (S * \varphi_2).$$

ii) Φ is continuous. In fact if $(\varphi_n)_n$ is converging to 0 in the sense of $\mathcal{D}(\mathbb{R}^n)$, then $(S * \varphi_n)_n$ is converging to 0 in the sense of $\mathcal{E}(\mathbb{R}^n)$, but, since S has compact support, $(S * \varphi_n)_n$ is converging to 0 in the sense of $\mathcal{D}(\mathbb{R}^n)$, so that $(T * (S * \varphi_n))_n$ is converging to 0 in the sense of $\mathcal{E}(\mathbb{R}^n)$.

iii) Φ commutes with τ_h. In fact, from Theorem 42,

$$T * (S * \tau_h \varphi) = T * (\tau_h(S * \varphi)) = \tau_h(T * (S * \varphi)).$$

From Theorem 43 we deduce that there exists $U \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$T * (S * \varphi) = U * \varphi.$$

We define

$$U = T * S.$$

Remark 26. *It is possible to define, in a similar way, $S * \psi$, for $S \in \mathcal{E}'(\mathbb{R}^n)$ and $\psi \in \mathcal{E}(\mathbb{R}^n)$ and so on. It is also possible to show that convolution of two distributions, one of them with compact support, is commutative. Similarly the convolution of three distributions one of them with compact support, is associative. All the details can be found in [8, Ch. 1.6].*

13

13.1 Fourier transform of functions

The content of this paragraph can be found in [8, Ch. 1.7] (see also [15]).

13.1.1 Fourier transform of L^1 functions

We introduce here the Fourier transform of a L^1 function.

Definition 32. *Let $f \in L^1(\mathbb{R}^n)$. Let $\xi \in \mathbb{R}^n$. We define*

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx,$$

where $x \cdot \xi = x_1\xi_1 + \ldots + x_n\xi_n$. $\hat{f}$ is called Fourier transform of f. The Fourier transform of f will be denoted also with $\mathcal{F}(f)$.

Theorem 44. *Let $f \in L^1(\mathbb{R}^n)$.*
Then

i) $\hat{f} \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

ii) $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$.

iii) $\lim_{|\xi| \to +\infty} \hat{f}(\xi) = 0$ *(Riemann-Lebesgue lemma).*

Proof. The boundedness of $\hat{f}$ and the point *ii)* are consequence of

$$|\hat{f}(\xi)| \leq |\int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx| \leq \int_{\mathbb{R}^n} |f(x)|\, dx = \|f\|_{L^1},$$

while the continuity of $\hat{f}$ can be obtained, e. g., using the dominated convergence theorem. In fact, if $(\xi_n)_n$ is a sequence in $\mathbb{R}^n$ converging to $\bar{\xi}$, then,

$$\begin{cases} e^{-ix\cdot\xi_n} f(x) \xrightarrow{n} e^{-ix\cdot\bar{\xi}} f(x) & \text{almost everywhere,} \\ |e^{-ix_n\cdot\xi} f(x)| \leq |f(x)|. \end{cases}$$

Consequently, for all the sequences $(\xi_n)_n$ converging to $\bar{\xi}$,

$$\lim_n \hat{f}(\xi_n) = \lim_n \int_{\mathbb{R}^n} e^{-ix\cdot\xi_n} f(x)\, dx = \int_{\mathbb{R}^n} e^{-ix\cdot\bar{\xi}} f(x)\, dx = \hat{f}(\bar{\xi})$$

and the continuity of $\hat{f}$ follows.

Let's prove the Riemann-Lebesgue lemma. Consider $\varphi \in C_0^\infty(\mathbb{R}^n)$. Let $j = 1, \dots, n$. We have

$$\begin{aligned} \xi_j \hat{\varphi}(\xi) &= \int_{\mathbb{R}^n} \xi_j e^{-ix\cdot\xi} \varphi(x)\, dx \\ &= \int_{\mathbb{R}^n} -D_{x_j}(e^{-ix\cdot\xi})\varphi(x)\, dx \\ &= \underbrace{\int_{\mathbb{R}^n} -D_{x_j}(e^{-ix\cdot\xi}\varphi(x))\, dx}_{=0} + \int_{\mathbb{R}^n} e^{-ix\cdot\xi} D_{x_j}\varphi(x)\, dx \\ &= \widehat{D_{x_j}\varphi}(\xi). \end{aligned}$$

Consequently

$$(1 + |\xi|^2)\hat{\varphi}(\xi) = (\widehat{(1-\Delta)}\varphi)(\xi),$$

where Δ is the Laplacian operator $\partial_1^2 + \ldots + \partial_n^2 = -(D_j^2 + \ldots + D_n^2)$. Remarking that $(1-\Delta)\varphi$ is in L^1, we have, from the point *i)*, that $\widehat{(1-\Delta)}\varphi$ is in L^∞ and then

$$|\hat{\varphi}(\xi)| \leq \frac{\|\widehat{(1-\Delta)}\varphi\|_{L^\infty}}{1 + |\xi|^2}, \qquad \text{for all } \xi \in \mathbb{R}^n.$$

Hence

$$\lim_{|\xi| \to +\infty} \hat{\varphi}(\xi) = 0. \tag{28}$$

Suppose now that $f \in L^1(\mathbb{R}^n)$ and fix $\varepsilon > 0$. We know that there exists $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\|f - \varphi\|_{L^1} \leq \frac{\varepsilon}{2}.$$

From (28) we have that there exists $R > 0$ such that, for all $\xi \in \mathbb{R}^n$, if $|\xi| > R$, then $|\hat{\varphi}(\xi)| < \frac{\varepsilon}{2}$. As a consequence, if $|\xi| > R$, then

$$|\hat{f}(\xi)| \leq |\hat{\varphi}(\xi)| + |\hat{f}(\xi) - \hat{\varphi}(\xi)| \leq |\hat{\varphi}(\xi)| + \|\hat{f} - \hat{\varphi}\|_{L^\infty} < \frac{\varepsilon}{2} + \|f - \varphi\|_{L^1} < \varepsilon.$$

□

Theorem 45. *Let f in $C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Suppose that $D_{x_j} f$ in $L^1(\mathbb{R}^n)$. Then*

$$\widehat{D_{x_j} f}(\xi) = \xi_j \hat{f}(\xi).$$

Proof. We have

$$\begin{aligned} \widehat{D_{x_j} f}(\xi) &= \int_{\mathbb{R}^n} e^{-ix\cdot\xi} D_{x_j} f(x)\, dx \\ &= \int_{\mathbb{R}^n} D_{x_j}(e^{-ix\cdot\xi} f(x))\, dx + \int_{\mathbb{R}^n} \xi_j e^{-ix\cdot\xi} f(x)\, dx \end{aligned}$$

The conclusion follows from observing that

$$\int_{\mathbb{R}^n} D_{x_j}(e^{-ix\cdot\xi} f(x))\, dx = 0.$$

In fact, we know that if g is in $L^1(\mathbb{R})$ and $\lim_{x\to+\infty} g(x)$ exists, then

$$\lim_{x\to+\infty} g(x) = 0.$$

Suppose now g is in $C^1(\mathbb{R})$ and both g and g' are in $L^1(\mathbb{R})$, then $\lim_{x\to+\infty} g(x)$ exists, since

$$\lim_{x\to+\infty} g(x) = g(0) + \lim_{x\to+\infty} \int_0^x g'(t)\, dt.$$

□

Theorem 46. *Let f and $x \mapsto x_j f(x)$ be $L^1(\mathbb{R}^n)$ functions. Then $\hat{f}$ is differentiable with respect to ξ_j and*

$$D_{\xi_j} \hat{f}(\xi) = -\widehat{x_j f(x)}(\xi).$$

Proof. Let's prove the result in the case of $n = 1$. Consider a sequence $(\xi_k)_k$ in $\mathbb{R} \setminus \{\bar{\xi}\}$, converging to $\bar{\xi}$. We have

$$\begin{aligned} \frac{\hat{f}(\xi_k) - \hat{f}(\bar{\xi})}{\xi_k - \bar{\xi}} &= \int_{\mathbb{R}} \frac{e^{-ix\xi_k} - e^{-ix\bar{\xi}}}{\xi_k - \bar{\xi}} f(x)\, dx \\ &= \int_{\mathbb{R}} \underbrace{-i\,(\frac{e^{-ix(\xi_k - \bar{\xi})} - 1}{-ix(\xi_k - \bar{\xi})}) e^{-ix\cdot\bar{\xi}} x f(x)}_{=\psi_k(x)}\, dx \end{aligned}$$

Now

$$\psi_k(x) \xrightarrow{k} -ie^{-ix\cdot\bar{\xi}} x f(x) \qquad \text{for almost every } x,$$

and

$$|\psi_k(x)| \leq C|xf(x)|, \qquad \text{where } C \text{ does not depend on } k.$$

The dominated convergence theorem gives

$$\hat{f}\,'(\bar{\xi}) = \lim_k \frac{\hat{f}(\xi_k) - \hat{f}(\bar{\xi})}{\xi_k - \bar{\xi}} = -i\widehat{xf(x)}(\bar{\xi}).$$

□

13.1.2 Explicit computation of some Fourier transforms

Example 10. *Let* $a,\ b > 0$*. Consider*

$$f(x) = \begin{cases} a & \text{if } \ -b < x < b, \\ 0 & \text{if } \ |x| \geq 0. \end{cases}$$

If $\xi = 0$*,*

$$\hat{f}(0) = \int_{-b}^{b} a\, dx = 2ab.$$

If $\xi \neq 0$*,*

$$\hat{f}(\xi) = \int_{-b}^{b} e^{-ix\xi} a\, dx = a\frac{1}{-i\xi} e^{-ix\xi}\Big|_{-b}^{b} = ab\frac{e^{-ib\xi} - e^{ib\xi}}{-ib\xi} = 2ab\frac{\sin(b\xi)}{b\xi}$$

Finally

$$\hat{f}(\xi) = \begin{cases} 2ab\dfrac{\sin(b\xi)}{b\xi} & \text{if } \xi \neq 0, \\ 2ab & \text{if } \xi = 0. \end{cases}$$

Example 11. *Let* $a,\ b > 0$*. Consider*

$$f(x) = \begin{cases} ae^{-bx} & \text{if } \ x \geq 0, \\ ae^{bx} & \text{if } \ x < 0. \end{cases}$$

Let's make the computation in the case $a = b = 1$*.*

$$\begin{aligned} \hat{f}(\xi) &= \int_{-\infty}^{0} e^{-ix\xi+x}\, dx + \int_{0}^{+\infty} e^{-ix\xi-x}\, dx \\ &= \frac{1}{-i\xi+1} e^{-ix\xi+x}\Big|_{-\infty}^{0} + \frac{1}{-i\xi-1} e^{-ix\xi-x}\Big|_{0}^{+\infty} = \frac{1}{1-i\xi} + \frac{1}{1+i\xi} \end{aligned}$$

Finally

$$\hat{f}(\xi) = \frac{2}{1+\xi^2}.$$

Example 12. *Let* $a > 0$*. Consider*

$$f(x) = e^{-ax^2}.$$

We have to compute

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi - ax^2}\, dx.$$

Since the function $x \mapsto xe^{-ax^2}$ *is in* L^1*, from Theorem 46 we have that*

$$\hat{f}\,'(\xi) = \int_{\mathbb{R}} -ixe^{-ix\xi - ax^2}\, dx.$$

On the other hand, from Theorem 45,

$$i\xi\hat{f}(\xi) = \widehat{f'}\,(\xi) = \int_{\mathbb{R}} e^{-ix\xi}(-2axe^{-ax^2})\, dx = -2a\int_{\mathbb{R}} xe^{-ix\xi - ax^2}\, dx.$$

Consequently

$$\xi\hat{f}(\xi) = -2a\hat{f}\,'(\xi)$$

and, from a standard computation,

$$\hat{f}(0) = \int_{\mathbb{R}} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}$$

i. e. the function $\hat{f}$ *is a solution of the Cauchy problem*

$$\begin{cases} 2au'(\xi) + \xi u(\xi) = 0 & \textit{in } \mathbb{R}, \\ u(0) = \sqrt{\frac{\pi}{a}}. \end{cases}$$

Finally

$$\hat{f}(\xi) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\xi^2}{4a}}.$$

In the case of $x \in \mathbb{R}^n$*, we have*

$$f(x) = e^{-a|x|^2} \qquad \textit{and} \qquad \hat{f}(\xi) = (\sqrt{\frac{\pi}{a}})^n\, e^{-\frac{|\xi|^2}{4a}}.$$

13.1.3 Rapidly decreasing functions and tempered distributions

We introduce the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and its dual $\mathcal{S}'(\mathbb{R}^n)$.

Definition 33. *Let*

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \;\big|\; \textit{for all } \alpha,\, \beta \in \mathbb{N}^n, \;\; \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty\}.$$

$\mathcal{S}(\mathbb{R}^n)$ *is a vector space which is called Schwartz space or space of rapidly decreasing (at infinity) functions.*

Remark 27. *For* $\alpha, \beta \in \mathbb{N}^n$*,*

$$r_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)|$$

is a seminorm on $\mathcal{S}(\mathbb{R}^n)$*. We put on* $\mathcal{S}(\mathbb{R}^n)$ *the Fréchet topology generated by this countable family of seminorms. Remark that*

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \;\big|\; \textit{for all } k \in \mathbb{N}, \;\; \sup_{x\in\mathbb{R}^n} ((1+|x|)^k \sum_{|\beta|\le k} |D^\beta f(x)|) < \infty\}.$$

and the topology of $\mathcal{S}(\mathbb{R}^n)$ can be obtained using the sequence of norms $(\tilde{r}_k)_k$ with

$$\tilde{r}_k(f) = \sup_{x\in\mathbb{R}^n} ((1+|x|)^k \sum_{|\beta|\le k} |D^\beta f(x)|).$$

Definition 34. *The dual space of $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ is called space of tempered (or temperate) distributions, i. e. the functional $S : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}$ (or $\mathbb{C}$) is a tempered distribution if*

i) S is linear;

ii) there exist $C > 0$ and $\alpha_1,\ \beta_1, \dots \alpha_k,\ \beta_k \in \mathbb{N}^n$ such that, for all $f \in \mathcal{S}(\mathbb{R}^n)$,

$$|S(f)| \le C \sum_{j=1}^{k} p_{\alpha_j,\beta_j}(f).$$

Given a sequence $(f_n)_n$ in $\mathcal{S}(\mathbb{R}^n)$, we will say that $(f_n)_n$ converges to 0 in the sense of $\mathcal{S}(\mathbb{R}^n)$, if, for all $\alpha,\ \beta \in \mathbb{N}^n$,

$$p_{\alpha,\beta}(f_n) = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f_n(x)| \xrightarrow{n} 0.$$

Remark 28. *We have*

$$\mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{E}(\mathbb{R}^n).$$

It is possible to show that

i) given $(\varphi_n)_n$ in $\mathcal{D}(\mathbb{R}^n)$ such that $\varphi_n \xrightarrow{n} 0$ in the sense of $\mathcal{D}(\mathbb{R}^n)$, then $\varphi_n \xrightarrow{n} 0$ in the sense of $\mathcal{S}(\mathbb{R}^n)$, i. e. the immersion $\mathcal{D}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous;

ii) given $(f_n)_n$ in $\mathcal{S}(\mathbb{R}^n)$ such that $f_n \xrightarrow{n} 0$ in the sense of $\mathcal{S}(\mathbb{R}^n)$, then $f_n \xrightarrow{n} 0$ in the sense of $\mathcal{E}(\mathbb{R}^n)$, i. e. the immersion $\mathcal{S}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n)$ is continuous;

iii) given $f \in \mathcal{S}(\mathbb{R}^n)$, there exists $(\varphi_n)_n$ in $\mathcal{D}(\mathbb{R}^n)$ such that $(\varphi_n - f) \xrightarrow{n} 0$ in the sense of $\mathcal{S}(\mathbb{R}^n)$, i. e. $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$;

iv) given $\psi \in \mathcal{E}(\mathbb{R}^n)$, there exists $(f_n)_n$ in $\mathcal{S}(\mathbb{R}^n)$ such that $(f_n - \psi) \xrightarrow{n} 0$ in the sense of $\mathcal{E}(\mathbb{R}^n)$, i. e. $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{E}(\mathbb{R}^n)$.

As a consequence

$$\mathcal{E}'(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n).$$

Exercise 6. *Let $f \in \mathcal{S}(\mathbb{R}^n)$. Prove that, for all $p \in [1, +\infty]$, $f \in L^p(\mathbb{R}^n)$.*

Exercise 7. *Let $f \in L^p(\mathbb{R}^n)$, for some $p \in [1, +\infty]$. Prove that $T_f \in \mathcal{S}'(\mathbb{R}^n)$.*

13.1.4 Fourier inversion formula

We study the behavior of Fourier transform on the Schwartz space.

Theorem 47. *Let $f \in \mathcal{S}(\mathbb{R}^n_x)$.*
Then

$$\hat{f} \in \mathcal{S}(\mathbb{R}^n_\xi)$$

and, for all $\alpha \in \mathbb{N}^n$ and $\xi \in \mathbb{R}^n$,

$$\widehat{D^\alpha f}(\xi) = \xi^\alpha \hat{f}(\xi), \qquad \widehat{x^\alpha f(x)}(\xi) = (-1)^{|\alpha|} D^\alpha \hat{f}(\xi). \tag{29}$$

Moreover the functional $\mathcal{F} : \mathcal{S}(\mathbb{R}^n_x) \to \mathcal{S}(\mathbb{R}^n_\xi)$, $\mathcal{F}(f) = \hat{f}$ is linear and continuous.

Proof. We give only a sketch of the proof. We remark that if $f \in \mathcal{S}(\mathbb{R}^n)$, then, for all $\alpha,\ \beta \in \mathbb{N}^n$, $x^\alpha D^\beta f \in L^1(\mathbb{R}^n)$, so that the inequalities in (29) are consequences of Theorem 45 and Theorem 46, applied recursively. We remark also that the function $\xi \mapsto \xi^\alpha D^\beta \hat{f}(\xi)$ is the Fourier transform of the function $x \mapsto (-1)^{|\beta|} D^\alpha (x^\beta f(x))$ which is a function of $\mathcal{S}(\mathbb{R}^n_x)$. Consequently

$$\sup_{\xi \in \mathbb{R}^n} |\xi^\alpha D^\beta \hat{f}(\xi)| = \|\xi^\alpha D^\beta \hat{f}(\xi)\|_{L^\infty} \leq \|D^\alpha (x^\beta f(x))\|_{L^1} < +\infty.$$

Finally the continuity of $\mathcal{F}$ can be deduced from the fact that, for all $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\|f(x)\|_{L^1} \leq C \|(1 + |x|)^{n+1} f(x)\|_{L^\infty}.$$

where C depends only on the dimension n. □

Theorem 48. *Let $f \in \mathcal{S}(\mathbb{R}^n_x)$.*
Then, for all $x \in \mathbb{R}^n$,

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi.$$

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n_x)$. We have

$$\int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi = \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} \Big(\int_{\mathbb{R}^n_y} e^{-iy\cdot\xi} f(y)\, dy \Big)\, d\xi. \tag{30}$$

We notice that the function

$$(\xi,\, y) \mapsto e^{i(x-y)\cdot\xi} f(y)$$

is not in $L^1(\mathbb{R}^n_\xi \times \mathbb{R}^n_y)$, so that it is not possible to exchange the order of integration in (30). Consider now $g \in \mathcal{S}(\mathbb{R}^n_\xi)$. We have that, for all $x \in \mathbb{R}^n$, the function

$$(\xi,\, y) \mapsto e^{i(x-y)\cdot\xi} g(\xi) f(y)$$

is in $L^1(\mathbb{R}^n_\xi \times \mathbb{R}^n_y)$, hence

$$\begin{aligned}\int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} g(\xi)\hat{f}(\xi)\, d\xi &= \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} g(\xi)(\int_{\mathbb{R}^n_y} e^{-iy\cdot\xi} f(y)\, dy)\, d\xi. \\ &= \int_{\mathbb{R}^n_\xi}\int_{\mathbb{R}^n_y} e^{i(x-y)\cdot\xi} g(\xi) f(y)\, dy\, d\xi. \\ &= \int_{\mathbb{R}^n_\xi}\int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} g(\xi) f(y)\, d\xi\, dy. \\ &= \int_{\mathbb{R}^n_\xi} f(y)(\int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} g(\xi)\, d\xi)\, dy. \\ &= \int_{\mathbb{R}^n_\xi} f(y)\hat{g}(y-x)\, dy.\end{aligned}$$

i. e.

$$\int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} g(\xi)\hat{f}(\xi)\, d\xi = \int_{\mathbb{R}^n_z} f(z+x)\hat{g}(z)\, dz. \tag{31}$$

Let, for $k \in \mathbb{N} \setminus \{0\}$,

$$g_k(\xi) = e^{-\frac{|\xi|^2}{2k}} \qquad \text{and consequently} \qquad \widehat{g_k}(z) = (2\pi k)^{\frac{n}{2}} e^{-\frac{k|z|^2}{2}}.$$

Substitute g with g_k in (31). We have

$$\int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} e^{-\frac{|\xi|^2}{2k}} \hat{f}(\xi)\, d\xi = (2\pi k)^{\frac{n}{2}} \int_{\mathbb{R}^n_z} f(z+x) e^{-\frac{k|z|^2}{2}}\, dz. \tag{32}$$

Using the dominated convergence theorem we have

$$\int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} e^{-\frac{|\xi|^2}{2k}} \hat{f}(\xi)\, d\xi \xrightarrow{k} \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi}\hat{f}(\xi)\, d\xi$$

and

$$\begin{aligned}&(2\pi k)^{\frac{n}{2}} \int_{\mathbb{R}^n_z} f(z+x) e^{-\frac{k|z|^2}{2}}\, dz \\ &\quad = (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n_y} f(x + \frac{y}{\sqrt{k}}) e^{-\frac{|y|^2}{2}}\, dy \xrightarrow{k} (2\pi)^{\frac{n}{2}} f(x) \int_{\mathbb{R}^n_y} e^{-\frac{|y|^2}{2}}\, dy.\end{aligned}$$

Recalling that

$$\int_{\mathbb{R}^n_y} e^{-\frac{|y|^2}{2}}\, dy = (2\pi)^{\frac{n}{2}},$$

we obtain the conclusion. □

Corollary 11. *$\mathcal{F} : \mathcal{S}(\mathbb{R}^n_x) \to \mathcal{S}(\mathbb{R}^n_\xi)$, $\mathcal{F}(f) = \hat{f}$ is an isomorphism.*

14

14.1 Fourier transform of tempered distributions

The content of this paragraph can be found in [8, Ch. 1.7] (see also [15]).

14.1.1 Fourier transform of tempered distributions

We introduce the Fourier transform of tempered distributions.

Definition 35. *Let $S \in \mathcal{S}'(\mathbb{R}^n)$. We define, for all $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\hat{S}(f) = S(\hat{f}).$$

$\hat{S}$ is called Fourier transform of the tempered distribution S.

Remark 29. *$\hat{S}$ is a tempered distribution. In fact*

$$\hat{S} = S \circ \mathcal{F}.$$

Let $f \in L^1(\mathbb{R}^n)$ such that the distribution associated to f, T_f, is a tempered distribution. Then

$$\widehat{T_f} = T_{\hat{f}}.$$

In fact, for all $g \in \mathcal{S}(\mathbb{R}^n)$, the function $(x,\xi) \mapsto e^{-ix\cdot\xi} g(x) f(\xi)$ is in $L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, so that

$$\begin{aligned}
\widehat{T_f}(g) = T_f(\hat{g}) &= \int_{\mathbb{R}^n_\xi} f(\xi)\hat{g}(\xi)\, d\xi \\
&= \int_{\mathbb{R}^n_\xi} f(\xi) \int_{\mathbb{R}^n_x} e^{-ix\cdot\xi} g(x)\, dx\, d\xi \\
&= \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_x} e^{-ix\cdot\xi} f(\xi) g(x)\, dx\, d\xi \\
&= \int_{\mathbb{R}^n_x} g(x) \int_{\mathbb{R}^n_\xi} e^{-ix\cdot\xi} f(\xi)\, d\xi\, dx \\
&= \int_{\mathbb{R}^n_x} g(x)\hat{f}(x)\, dx = T_{\hat{f}}(g).
\end{aligned}$$

Example 13. *We denote by T_1 the distribution associated to the constant function equal to 1. T_1 is a tempered distribution. Let's compute $\widehat{T_1}$. We have*

$$\widehat{T_1}(f) = T_1(\hat{f}) = \int_{\mathbb{R}^n_\xi} \hat{f}(\xi)\, d\xi = (2\pi)^n \underbrace{\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} e^{i0\cdot\xi} \hat{f}(\xi)\, d\xi}_{=f(0)\ \text{(inv. Fourier trans.)}} = (2\pi)^n f(0)$$

i. e.

$$\widehat{T_1} = (2\pi)^n \delta_0.$$

Similarly, let's compute $\widehat{\delta_0}$. We have

$$\widehat{\delta_0}(f) = \delta_0(\hat{f}) = \hat{f}(0) = \int_{\mathbb{R}^n_x} e^{ix\cdot 0} f(x)\, dx = \int_{\mathbb{R}^n_x} f(x)\, dx$$

i. e.

$$\widehat{\delta_0} = T_1.$$

Example 14. *We denote by* H *the distribution associated to the Heaviside function. Let's compute* $\hat{H}$*. We have*

$$x\hat{H}(f) = \hat{H}(xf(x)) = H(\widehat{xf(x)}).$$

Recalling that

$$\widehat{xf(x)}(\xi) = i\hat{f}\,'(\xi)$$

we obtain

$$x\hat{H}(f) = H(i\hat{f}\,') = i\int_0^{+\infty} \hat{f}\,'(\xi)\,d\xi = -i\hat{f}(0) = -i\int_{\mathbb{R}} f(x)\,dx.$$

This, in particular, implies that $ix\hat{H} = T_1$*, i. e.* $i\hat{H}$ *is a solution of the equation*

$$xT = T_1.$$

Consequently (see Exercise 4) of Lesson 11) there exists $c \in \mathbb{C}$ *such that*

$$\hat{H} = -iPV_{\frac{1}{x}} + c\,\delta_0.$$

It remains to compute c*. Consider* $f(x) = e^{-x^2/2}$*. We know that* $\hat{f}(\xi) = \sqrt{2\pi}e^{-\xi^2/2}$*. Then*

$$\underbrace{\pi = \int_0^{+\infty} \sqrt{2\pi}e^{-\xi^2/2}\,d\xi}_{\textit{direct computation}} = H(\hat{f}) = \hat{H}(f) = \underbrace{-iPV_{\frac{1}{x}}(f)}_{=0\ (f\ \textit{even})} + c\,\delta_0(f) = cf(0) = c.$$

Finally

$$\hat{H} = -iPV_{\frac{1}{x}} + \pi\delta_0.$$

14.1.2 Fourier transform of L^2 functions

In this subparagraph we show what happens to the Fourier transform of a distribution associated to a L^2 function.

Theorem 49. *Let* $f,\ g \in \mathcal{S}(\mathbb{R}^n)$*.*
Then

i) $\int_{\mathbb{R}^n_x} f(x)\hat{g}(x)\,dx = \int_{\mathbb{R}^n_\xi} \hat{f}(\xi)g(\xi)\,d\xi$;

ii) $\int_{\mathbb{R}^n_x} f(x)\overline{g(x)}\,dx = (2\pi)^{-n}\int_{\mathbb{R}^n_\xi} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi$ *(Parseval's identity);*

iii) $\widehat{fg}(\xi) = (2\pi)^{-n}\hat{f} * \hat{g}(\xi)$;

iv) $\widehat{f * g}(x) = \hat{f}(x)\hat{g}(x)$.

Proof. Identity *i)* is a consequence of the fact that in the integral

$$\int_{\mathbb{R}^n_x} f(x)\int_{\mathbb{R}^n_\xi} e^{-ix\cdot\xi}g(\xi)\,d\xi\,dx$$

we can exchange the order of integration.

The Fourier inversion formula gives

$$\overline{g(x)} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} e^{-ix\cdot\xi} \overline{\hat{g}(\xi)}\, d\xi,$$

and *ii)* follows in a similar way.

For proving *iii)*, we have

$$\begin{aligned}
\widehat{fg}(\xi) &= \int_{\mathbb{R}^n_x} e^{-ix\cdot\xi} f(x) g(x)\, dx \\
&= \int_{\mathbb{R}^n_x} e^{-ix\cdot\xi} f(x) \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\eta} e^{ix\cdot\eta} \hat{g}(\eta)\, d\eta\, dx \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\eta} \hat{g}(\eta) \int_{\mathbb{R}^n_x} e^{-ix\cdot(\xi-\eta)} f(x)\, dx\, d\eta \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\eta} \hat{g}(\eta) \hat{f}(\xi-\eta)\, d\eta = (2\pi)^{-n} \hat{f} * \hat{g}(\xi).
\end{aligned}$$

We let the proof of *iv)* as an exercise. □

Theorem 50 (Plancherel theorem)**.** *Let $f \in L^2(\mathbb{R}^n)$.*

Then there exists a unique $g \in L^2(\mathbb{R}^n)$ such that

$$\widehat{T_f} = T_g.$$

We write $g = \hat{f}$ and we call it Fourier transform of f. We have, moreover,

$$\|\hat{f}\|_{L^2} = (2\pi)^{\frac{n}{2}} \|f\|_{L^2}.$$

Proof. We know that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. Consequently, there exists a sequence $(\varphi_n)_n$ in $\mathcal{S}(\mathbb{R}^n)$ such that

$$\varphi_n \xrightarrow{n} f \qquad \text{in } L^2(\mathbb{R}^n).$$

Using Parseval's identity, we have

$$\|\varphi_n - \varphi_m\|_{L^2} = (2\pi)^{-\frac{n}{2}} \|\hat{\varphi}_n - \hat{\varphi}_m\|_{L^2}.$$

Since $(\varphi_n)_n$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$, the same is valid for $(\hat{\varphi}_n)_n$ and this implies that there exists $g \in L^2(\mathbb{R}^n)$ such that

$$\hat{\varphi}_n \xrightarrow{n} g \qquad \text{in } L^2(\mathbb{R}^n).$$

Since the strong convergence of $(\varphi_n)_n$ and $(\hat{\varphi}_n)_n$, to f and g respectively, implies the weak convergence, we deduce that, for all $h \in \mathcal{S}(\mathbb{R}^n)$,

$$T_g(h) = \int gh = \lim_n \int \hat{\varphi}_n h = \lim_n \int \varphi_n \hat{h} = \int f\hat{h} = T_f(\hat{h})$$

i. e. $T_g = \widehat{T_f}$. Similarly, the strong convergence of $(\varphi_n)_n$ and $(\hat{\varphi}_n)_n$, to f and g respectively, implies that

$$\|g\|_{L^2} = \lim_n \|\hat{\varphi}_n\|_{L^2} = \lim_n (2\pi)^{\frac{n}{2}} \|\varphi_n\|_{L^2} = (2\pi)^{\frac{n}{2}} \|f\|_{L^2}.$$

Remark that g does not depend on the sequence $(\varphi_n)_n$ but only on f. In fact if one choose another sequence $(\psi_n)_n$ in $\mathcal{S}(\mathbb{R}^n)$ such that

$$\psi_n \stackrel{n}{\longrightarrow} f \qquad \text{in } \; L^2(\mathbb{R}^n),$$

then $\psi_n - \varphi_n \stackrel{n}{\longrightarrow} 0$ in $L^2(\mathbb{R}^n)$ and consequently, again from Parseval's identity, $\hat{\psi}_n - \hat{\varphi}_n \stackrel{n}{\longrightarrow} 0$ in $L^2(\mathbb{R}^n)$, i. e.

$$\lim_n \hat{\psi}_n = \lim_n \hat{\varphi}_n = g.$$

To end, let's prove that g is unique. Suppose that there exist g_1 and $g_2 \in L^2(\mathbb{R}^n)$ such that, for all $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$T_{g_1}(\hat{\psi}) = \widehat{T_f}(\psi) = T_{g_2}(\hat{\psi}).$$

It is sufficient to remark that each function in $\mathcal{S}(\mathbb{R}^n)$ can be represented as Fourier transform of a function in $\mathcal{S}(\mathbb{R}^n)$. Consequently, for all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$T_{g_1}(\phi) = \; T_{g_2}(\phi),$$

so that $g_1 = g_2$ a. e. and consequently as L^2 functions. □

14.1.3 Fourier-Laplace Transform of a distribution with compact support

Definition 36. *Let $T \in \mathcal{E}'(\mathbb{R}^n)$ be a distribution with compact support. For $\zeta \in \mathbb{C}^n$, consider the $C^\infty(\mathbb{R}^n_x)$ function*

$$x \mapsto \psi_\zeta(x) = e^{-ix\cdot\zeta} = e^{x\cdot\Im\zeta}(\cos(x\cdot\Re\zeta) - i\sin(x\cdot\Re\zeta)).$$

The function

$$\mathbb{C}^n \to \mathbb{C}, \qquad \zeta \mapsto T(\psi_\zeta)$$

is called the Fourier-Laplace transform of the distribution T and we denote it with $\widehat{T}^{FL}$.

Lemma 12. *Let $T \in \mathcal{E}'(\mathbb{R}^n)$ be a distribution with compact support.*
Then $\widehat{T}^{FL}$ is a C^∞ function.

Proof. We prove that $\widehat{T}^{FL}$ is a continuous function on $\mathbb{C}^n$, letting the rest of the proof as an exercise. Let $\bar{\zeta} \in \mathbb{C}^n$ and suppose that the sequence $(\zeta_k)_k$ is converging to $\bar{\zeta}$. Then the sequence of C^∞ functions $(\psi_{\zeta_k})_k$, where $\psi_{\zeta_k}(x) = e^{-ix\cdot\zeta_k}$, is converging, in the sense of $\mathcal{E}(\mathbb{R}^n)$, to the function $\psi_{\bar{\zeta}}(x) = e^{-ix\cdot\bar{\zeta}}$. Consequently

$$\lim_k \widehat{T}^{FL}(\zeta_k) = \lim_k T(\psi_{\zeta_k}) = T(\psi_{\bar{\zeta}}) = \widehat{T}^{FL}(\bar{\zeta}),$$

i. e. T^{FL} is a continuous function. □

Remark 30. *Let f be a continuous function, with compact support, defined on $\mathbb{R}^n$. Let T_f be the distribution with compact support associated to f. The*

Fourier-Laplace transform of T_f, evaluated on $\xi \in \mathbb{R}^n$, coincides with the Fourier transform of f as L^1 function, in fact

$$\widehat{T_f}^{FL}(\xi) = T_f(\psi_\xi) = T_f(e^{-ix\cdot\xi}) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx = \hat{f}(\xi).$$

On the other hand, the tempered distribution associated to the Fourier transform of f as L^1 function, coincides with the Fourier transform of the tempered distribution associated to f, in fact, for all $\psi \in \mathcal{E}(\mathbb{R}^n)$,

$$T_{\hat{f}}(\psi) = \underbrace{\int_{\mathbb{R}^n_\xi} \hat{f}(\xi)\psi(\xi)\,d\xi = \int_{\mathbb{R}^n_x} f(x)\hat{\psi}(x)\,dx}_{\text{exchange the order of integration}} = T_f(\hat{\psi}) = \widehat{T_f}(\psi).$$

As a conclusion, if f is a continuous function with compact support,

$$\hat{f} = \widehat{T_f}^{FL} \text{ on } \mathbb{R}^n \qquad \text{and} \qquad T_{\hat{f}} = \widehat{T_f}.$$

Definition 37. *(see [4, Ch. IV.2]). Let $f \in C^\infty(\Omega)$, where Ω is an open set in $\mathbb{R}^n$. f is said to be analytic in Ω, if f is locally the sum of its Taylor series, i. e., for all $x_0 \in \Omega$, there exists $r > 0$ such that, for all $x \in B(x_0, r)$,*

$$f(x) = \sum_{\nu\in\mathbb{N}^n} \frac{1}{\nu!}\,\partial^\nu f(x_0)\,(x - x_0)^\nu.$$

f is said to be entire analytic if, for all $x \in \mathbb{R}^n$,

$$f(x) = \sum_{\nu\in\mathbb{N}^n} \frac{1}{\nu!}\,\partial^\nu f(0)\,x^\nu.$$

Remark that an entire analytic function can be extended to a function, defined in $\mathbb{C}^n$ (we continue to denote it with f), setting

$$f(z) = \sum_{\nu\in\mathbb{N}^n} \frac{1}{\nu!}\,\partial^\nu f(0)\,z^\nu.$$

Lemma 13. *Let $f \in C^\infty(\mathbb{R}^n)$. Suppose that, for all $R > 0$ there exist M_R, $C_R > 0$ such that, for all $\alpha \in \mathbb{N}^n$,*

$$\sup_{|x|<R} |D^\alpha f(x)| \le C_R M_R^{|\alpha|}.$$

Then f is entire analytic.

Proof. See [4, Ch. I.4.2]. □

Lemma 14 (Convergence of entire analytic functions)**.** *Let $(f_n)_n$ a sequence of entire analytic functions defined in $\mathbb{C}^n$. Suppose that there exists a function f on $\mathbb{C}^n$ such that*

$$f_n \xrightarrow{n} f \qquad \text{uniformly on compact set of } \mathbb{C}^n.$$

Then f is entire analytic.

Proof. See [4, Ch. V.1.1]. □

Lemma 15. *Let $T \in \mathcal{E}'(\mathbb{R}^n)$ be a distribution with compact support and let $(\rho_n)_n$ be a family of mollifiers.*

Then

$$T_{T*\rho_n} \xrightarrow{n} T \qquad \text{in the sense of } \mathcal{E}'(\mathbb{R}^n).$$

Proof. We recall that

$$(T * \rho_n)(x) = T(\psi_{n,x}), \qquad \text{where } \psi_{n,x} : y \mapsto \rho_n(x-y).$$

Consequently, for $\varphi \in \mathcal{E}(\mathbb{R}^n)$,

$$\begin{aligned} T_{T*\rho_n}(\varphi) &= \int_{\mathbb{R}^n} T(\psi_{n,x})\varphi(x)\,dx \\ &= \lim_{\varepsilon\to 0} \varepsilon^n \sum_{\nu\in\mathbb{N}^n} T(\psi_{n,\varepsilon\nu})\,\varphi(\varepsilon\nu) \\ &= \lim_{\varepsilon\to 0} T(\varepsilon^n \sum_{\nu\in\mathbb{N}^n} \psi_{n,\varepsilon\nu}\,\varphi(\varepsilon\nu)) \\ &= T(\lim_{\varepsilon\to 0} \varepsilon^n \sum_{\nu\in\mathbb{N}^n} \psi_{n,\varepsilon\nu}\,\varphi(\varepsilon\nu)) \\ &= T(\check{\rho}_n * \varphi). \end{aligned}$$

The conclusion follows remarking that $\check{\rho}_n * \varphi \xrightarrow{n} \varphi$ in the sense of $\mathcal{E}(\mathbb{R}^n)$ (actually $\check{\rho}_n * \varphi \xrightarrow{n} \varphi$ in the sense of $\mathcal{D}(\mathbb{R}^n)$, if $\varphi \in \mathcal{D}(\mathbb{R}^n)$). □

Remark 31. *In the previous situation*

$$T_{T*\rho_n}(\varphi) = ((T * \rho_n) * \check{\varphi})(0) = (T * (\rho_n * \check{\varphi}))(0) = T((\rho_n * \check{\varphi})\check{})$$

and

$$(\rho_n * \check{\varphi})\check{} \longrightarrow \varphi \qquad \text{in the sense of } \mathcal{E}(\mathbb{R}^n).$$

Theorem 51. *Let $T \in \mathcal{E}'(\mathbb{R}^n)$ be a distribution with compact support.*

Then the the Fourier-Laplace transform of T is an entire analytic function. Moreover the distribution associated to the Fourier-Laplace transform of T coincides with the Fourier transform of T in the sense of tempered distributions.

Proof. Recall that we denote by $\widehat{T}$ and $\widehat{T}^{FL}$ the Fourier transform and the Fourier-Laplace transform of T respectively.

Let f be a continuous function with compact support. We have already remarked that

$$\hat{f} = \widehat{T_f}^{FL} \qquad \text{and} \qquad T_{\hat{f}} = \widehat{T_f}.$$

We show that $\hat{f}$ is an entire analytic function. In fact, since, for all $\alpha \in \mathbb{N}^n$,

$$x \mapsto x^\alpha f(x)$$

is a $L^1(\mathbb{R}^n)$ function, we have, from Theorem 46, that $\hat{f}$ ia a $C^\infty(\mathbb{R}^n)$ function and

$$|D^\alpha \hat{f}(\xi)| \leq \int_{|x|<R} |x^\alpha f(x)|\,dx \leq CR^{|\alpha|},$$

where we have supposed that $\operatorname{Supp} f \subseteq B(0, R)$ and $C = \|f\|_{L^1}$. Consequently, from Lemma 13, $\hat{f}$ is an entire analytic function.

Let $T \in \mathcal{E}'(\mathbb{R}^n)$ and let $(\rho_k)_k$ be a family of mollifiers. From Lemma 15 we have that $T_{T*\rho_k} \xrightarrow{k} T$ in the sense of $\mathcal{E}'(\mathbb{R}^n)$, consequently

$$\widehat{T}_{T*\rho_k} = T_{\widehat{T*\rho_k}} \xrightarrow{k} \widehat{T} \qquad \text{in the sense of } \mathcal{D}'(\mathbb{R}^n). \tag{33}$$

We have

$$\begin{aligned}
\widehat{T * \rho_k}^{FL}(\zeta) &= T * \rho_k(\psi_\zeta) \qquad\qquad (\text{where } \psi_\zeta(x) = e^{-ix\cdot\zeta}) \\
&= ((T * \rho_k) * \check{\psi}_\zeta)(0) \\
&= (T * (\rho_k * \check{\psi}_\zeta))(0) \\
&= T((\rho_k * \check{\psi}_\zeta)\check{\ }) \\
&= T(\check{\rho}_k * \psi_\zeta).
\end{aligned}$$

Considering that

$$\begin{aligned}
\check{\rho}_k * \psi_\zeta(x) &= \int_{\mathbb{R}^n} \check{\rho}_k(y) e^{-i(x-y)\cdot\zeta}\, dy \\
&= e^{-ix\cdot\zeta} \int_{\mathbb{R}^n} \check{\rho}_k(y) e^{iy\cdot\zeta}\, dy = e^{-ix\cdot\zeta} \widehat{\rho}_k^{FL}(\zeta) = \widehat{\rho}_k^{FL}(\zeta)\psi_\zeta(x),
\end{aligned}$$

we deduce

$$\widehat{T * \rho_k}^{FL}(\zeta) = T(\check{\rho}_k * \psi_\zeta) = T(\widehat{\rho}_k^{FL}(\zeta)\psi_\zeta) = \widehat{\rho}_k^{FL}(\zeta) T(\psi_\zeta) = \widehat{\rho}_k^{FL}(\zeta)\widehat{T}^{FL}(\zeta).$$

Remark now that

$$\widehat{\rho}_k^{FL}(\zeta) = \widehat{\rho}^{FL}(\frac{\zeta}{k}) \xrightarrow{k} \widehat{\rho}^{FL}(0) = 1 \qquad \text{uniformly on compact sets of } \mathbb{C}^n.$$

Hence

$$\widehat{T * \rho_k}^{FL}(\zeta) = \widehat{\rho}_k^{FL}(\zeta)\widehat{T}^{FL}(\zeta) \xrightarrow{k} \widehat{T}^{FL}(\zeta) \tag{34}$$

uniformly on compact sets of $\mathbb{C}^n$, but the functions $\widehat{T * \rho_k}^{FL}$ are entire analytic functions (recall that the $T * \rho_k$'s are continuous functions with compact support), and (34) implies, from Lemma 14, that $\widehat{T}^{FL}$ is an entire analytic function. To conclude the proof we have to show that

$$T_{\widehat{T}^{FL}} = \widehat{T}.$$

We notice that (34) implies that

$$T_{\widehat{T*\rho_k}} \xrightarrow{k} T_{\widehat{T}^{FL}} \qquad \text{in the sense of } \mathcal{D}'(\mathbb{R}^n),$$

and the conclusion follows from (33). □

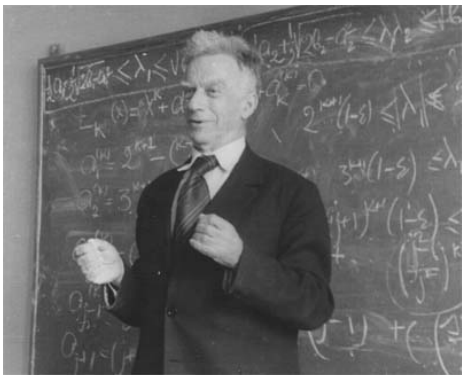


Figure 18: Sergej L'vovič Sobolev (1908-1989)

14.1.4 The Paley-Wiener theorem

Theorem 52. *Let U be an entire analytic function defined on $\mathbb{C}^n$.*

U is the Fourier-Laplace transform of a distribution with compact support contained in $\overline{B}(0,A)$ if and only if there exist $C>0$ and $N\in\mathbb{N}$ such that

$$|U(\zeta)| \le C(1+|\zeta|)^N e^{A|\Im\zeta|} \qquad \text{for all } \zeta\in\mathbb{C}^n. \tag{35}$$

U is the Fourier-Laplace transform of a C^∞ function with compact support contained in $\overline{B}(0,A)$ if and only if, for all $N\in\mathbb{N}$, there exist $C_N>0$ such that

$$|U(\zeta)| \le C_N(1+|\zeta|)^{-N} e^{A|\Im\zeta|} \qquad \text{for all } \zeta\in\mathbb{C}^n. \tag{36}$$

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15.1 Sobolev spaces in one space dimension - 1

The content of this paragraph can be found in [3, Ch. VIII].

15.1.1 First definitions

Definition 38. *Let I be an open interval in $\mathbb{R}$. Let $p\in[1,+\infty]$. We define*

$$W^{1,p}(I) = \{u\in L^p(I) \mid \exists g\in L^p(I):\ \forall\phi\in C^1_0(I),\ \int_I u\,\phi' = -\int_I g\,\phi\}.$$

$W^{1,p}(I)$ is called Sobolev space (of indexes 1 and p).

Remark 32.

i) In the definition of $W^{1,p}(I)$ it is sufficient to ask $\int_I u\,\varphi' = -\int_I g\,\varphi$ only for all $\varphi\in\mathcal{D}(I)$. In fact let $\phi\in C^1_0(I)$ and let $(\rho_n)_n$ be a family of mollifiers. We have $\rho_n*\phi,\ \rho_n*\phi'\in\mathcal{D}(I)$ and

$$\rho_n*\phi \xrightarrow{n} \phi, \qquad (\rho_n*\phi)' = \rho_n*\phi' \xrightarrow{n} \phi' \quad \textit{uniformly}$$

(actually in $L^q(\mathbb{R})$ for all $q\in[1,+\infty]$). We can pass to the limit in

$$\int_I u\,(\rho_n*\phi)' = -\int_I g\,(\rho_n*\phi).$$

ii) The function g in the definition of $W^{1,p}(I)$ is unique and we denote it with u'. In fact, if there exist $g_1,\ g_2 \in L^p(I)$ such that, for all $\phi \in C^1_0(I)$,

$$\int_I u\phi' = -\int_I g_1\phi = -\int_I g_2\phi$$

then $\int_I (g_1 - g_2)\phi = 0$ for all $\phi \in C^1_0(I)$ and this implies that $g_1 - g_2 = 0$. u' is called weak derivative of u.

iii) The definition of $W^{1,p}(I)$ can be given in the framework of distribution theory.

$$W^{1,p}(I) = \{u \in L^p(I) \mid \exists g \in L^p(I) :\ T'_u = T_g\},$$

i. e. $W^{1,p}(I)$ is the subset of $L^p(I)$ functions the derivative of which is a distribution associated to another $L^p(I)$ function. In this sense we can say that the weak derivative of u is the derivative in the sense of distributions.

Definition 39. *We set*

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|u'\|_{L^p} \quad \text{(or, equivalently, } (\|u\|^p_{L^p} + \|u'\|^p_{L^p})^{\frac{1}{p}} \text{ for } p \neq +\infty).$$

Theorem 53. *$W^{1,p}(I)$ is a Banach space. If $p \in [1, +\infty[$, then $W^{1,p}(I)$ is a separable Banach space. If $p \in]1, +\infty[$, then $W^{1,p}(I)$ is a reflexive Banach space. If $p = 2$ and $\|u\|_{W^{1,2}} = (\|u\|^2_{L^2} + \|u'\|^2_{L^2})^{\frac{1}{2}}$, $W^{1,2}(I) = H^1(I)$ is an Hilbert space.*

Proof. The proof is based on the following remark. Consider

$$\Phi : W^{1,p}(I) \to L^p(I) \times L^p(I), \qquad \Phi :\ u \mapsto (u,\ u').$$

Φ is an isometry between $W^{1,p}(I)$ and a subspace of $L^p(I) \times L^p(I)$. If this subspace is closed, then we obtain all the wanted properties, since $L^p(I) \times L^p(I)$ is a Banach space, a separable Banach space and a reflexive Banach space for $p \in [1, +\infty]$, $p \in [1, +\infty[$ and $p \in]1, +\infty[$ respectively.

Let $(u_n)_n$ be a sequence in $W^{1,p}(I)$ such that $(\Phi(u_n))_n$ converges to (u, v) in $L^p(I) \times L^p(I)$. This implies that

$$u_n \xrightarrow{n} u, \qquad u'_n \xrightarrow{n} v \quad \text{in} \quad L^p(I).$$

We know that, for all $\varphi \in \mathcal{D}(I)$,

$$\int_I u_n\varphi' = -\int_I u'_n\varphi$$

and, for $\frac{1}{p} + \frac{1}{p'} = 1$,

$$|\int_I (u_n - u)\varphi'| \leq \underbrace{\|u_n - u\|_{L^p}}_{\xrightarrow{n} 0} \|\varphi'\|_{L^{p'}}, \qquad |\int_I (u'_n - v)\varphi| \leq \underbrace{\|u'_n - v\|_{L^p}}_{\xrightarrow{n} 0} \|\varphi\|_{L^{p'}}.$$

Consequently

$$\begin{array}{ccc} \displaystyle\int_I u_n\varphi' & \xrightarrow{n} & \displaystyle\int_I u\varphi' \\ \| & & \\ \displaystyle -\int_I u'_n\varphi & \xrightarrow{n} & \displaystyle -\int_I v\varphi, \end{array}$$

i. e. $u \in W^{1,p}(I)$ and $v = u'$. As a consequence $\Phi(W^{1,p}(I))$ is closed in $L^p(I) \times L^p(I)$. $\square$

Remark 33. *It is interesting to remark that in the proof of the previous theorem we have seen also that if* $u_n \xrightarrow{n} u$ *and* $u'_n \xrightarrow{n} v$ *in* $L^p(I)$ *then* $u \in W^{1,p}(I)$ *and* $v = u'$.

Exercise 8. *Let* $p \in]1, +\infty]$. *Let* $(u_n)_n$ *in* $W^{1,p}(I)$. *Suppose that* $u_n \xrightarrow{n} u$ *in* $L^p(I)$ *and there exists* $C > 0$ *such that, for all* n, $\|u'_n\|_{L^p} \leq C$.

Prove that $u \in W^{1,p}(I)$.

Hint. Let $p' \in [1, +\infty[$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. From Banach-Alaoglu-Boubaki theorem there exist a subsequence $(u'_{n_k})_k$ and a function v in $L^p(I)$ such that $u'_{n_k} \stackrel{*}{\rightharpoonup} v$, i. e. for all $w \in L^{p'}(I)$,

$$\int_I u'_{n_k} w \xrightarrow{n} \int_I vw.$$

Consequently, for all $\varphi \in \mathcal{D}(I)$,

$$\begin{array}{ccc} \int_I u_{n_k} \varphi' & \xrightarrow{k} & \int_I u\varphi' \\ \| & & \\ -\int_I u'_{n_k} \varphi & \xrightarrow{k} & -\int_I v\varphi, \end{array}$$

i. e. $u \in W^{1,p}(I)$ with $u' = v$.

15.1.2 Continuous representative

Lemma 16. *Let* T *be a distribution on the open interval* $I \subseteq \mathbb{R}$. *Suppose that* $T' = 0$

Then there exists $c \in \mathbb{R}$ *such that* $T = T_c$.

Proof. Consider $\chi \in \mathcal{D}(I)$ such that $\int_I \chi = 1$ (obviously such a χ exists!). For all $\varphi \in \mathcal{D}(I)$, if one consider

$$\psi(x) = \int_{-\infty}^{x} \varphi(y)\,dy - \left(\int_{-\infty}^{+\infty} \varphi(y)\,dy\right) \int_{-\infty}^{x} \chi(y)\,dy,$$

then $\psi \in \mathcal{D}(I)$ and $\psi' = \varphi - c\chi$, where $c = \int_{-\infty}^{+\infty} \varphi(y)\,dy$. We have

$$0 = T'(\psi) = -T(\psi') = T(\varphi - c\chi) = T(\varphi) - cT(\chi),$$

i. e., for all $\varphi \in \mathcal{D}(I)$,

$$T(\varphi) = T(\chi) \int_{-\infty}^{+\infty} \varphi(y)\,dy = T(\chi)T_1(\varphi) = T_{\underbrace{T(\chi)}_{\text{const.}}}(\varphi).$$

$\square$

Theorem 54. *Let $u \in W^{1,p}(I)$.*
Then there exists $\tilde{u} \in C(\bar{I})$ such that $u = \tilde{u}$ a. e. in I and, for all $x,\ y \in \bar{I}$,

$$\tilde{u}(y) - \tilde{u}(x) = \int_x^y u'(t)\,dt.$$

Proof. We have $u' \in L^p(I) \subseteq L^1_{loc}(I)$. Fix $x_0 \in I$ and denote by

$$w(x) = \int_{x_0}^x u'(t)\,dt.$$

We deduce that $w \in AC(\tilde{I})$ for all bounded interval $\tilde{I}$ which contains x_0 and which is contained in I. We use the theorem on integration by parts in AC (Corollary 5 and subsequent topics, p. 27). We have, for all $\varphi \in \mathcal{D}(I)$,

$$\int_I w\varphi' = -\int_I u'\varphi.$$

But, since $u \in W^{1,p}(I)$, we have, by definition,

$$\int_I u\varphi' = -\int_I u'\varphi.$$

Consequently, for all $\varphi \in \mathcal{D}(I)$,

$$\int_I (u - w)\varphi' = 0,$$

i. e. $T'_{u-w} = 0$ and, from Lemma 16, there exists $c \in \mathbb{R}$ such that $T_{u-w} = T_c$. We define

$$\tilde{u}(x) = w(x) + c.$$

$\tilde{u}$ is continuous on $\bar{I}$ and, since $T_{\tilde{u}} = T_{w+c} = T_u$, for all $\varphi \in \mathcal{D}(I)$,

$$\int_I (u - \tilde{u})\varphi = 0,$$

i. e. $u(x) = \tilde{u}(x)$, for almost all $x \in I$. □

Corollary 12. *$W^{1,1}(I) = AC(I)$.*

15.1.3 Characterization of $W^{1,p}(I)$, for $p \in]1,\ +\infty]$

Theorem 55. *Let $p \in]1,\ +\infty]$. Suppose $u \in L^p(I)$.*
Then the following conditions are equivalent.

i) *$u \in W^{1,p}(I)$.*

ii) *There exists $C > 0$ such that, for all $\varphi \in \mathcal{D}(I)$,*

$$|\int_I u\varphi'| \le C\|\varphi\|_{L^{p'}}, \qquad \text{where } \frac{1}{p} + \frac{1}{p'} = 1.$$

iii) *There exists $C > 0$ such that, for all ω relatively compact open set in I and for all $h \in \mathbb{R}$ such that $|h| \le \operatorname{dist}(\omega,\ \partial I)$,*

$$\|\tau_h u - u\|_{L^p(\omega)} \le C|h|, \qquad \text{where } \tau_h u(x) = u(x - \tau).$$

Proof. Let $u \in W^{1,p}(I)$. The definition of $W^{1,p}(I)$ and Hölder inequality give that, for all $\varphi \in \mathcal{D}(I)$,

$$|\int_I u\varphi'| = |\int_I u'\varphi| \leq \|u'\|_{L^p} \|\varphi\|_{L^{p'}},$$

and *ii)* follows with $C = \|u'\|_{L^p}$.

Suppose *ii)*. Thinking at $\mathcal{D}(I)$ as a subspace of $L^{p'}(I)$ and considering the linear functional

$$\Phi : \mathcal{D}(I) \to \mathbb{R} \ \ (\text{or } \mathbb{C}), \qquad \Phi(\varphi) = \int_I u\varphi'.$$

condition *ii)* gives the continuity of Φ. Using Hahn-Banach theorem it is possible to extend Φ to a (unique) element $\tilde{\Phi}$ of the dual space of $L^{p'}(I)$. Recalling that $p' \in [1, +\infty[$, from Riezs's theorem we obtain that there exists $g \in L^p(I)$ such that, for all $v \in L^{p'}(I)$,

$$\tilde{\Phi}(v) = \int_I gv.$$

Consequently $u \in W^{1,p}(I)$ with $u' = -g$. We notice that this implication is not valid if $p = 1$.

Suppose that the condition *i)* is valid, i. e. $u \in W^{1,p}(I)$. Let ω be a relatively compact open set in I and suppose $h \in \mathbb{R}$, such that $|h| \leq \text{dist}(\omega, \partial I)$. Let $x \in \omega$. We have, using the same symbol u to denote the continuous representative of the function u,

$$u(x-h) - u(x) = \int_{x-h}^{x} u'(t)\, dt = h \int_0^1 u'(x - sh)\, ds.$$

Consequently

$$|u(x-h) - u(x)| \leq |h| |\int_0^1 u'(x-sh)\, ds|$$

and, from Hölder inequality,

$$|u(x-h) - u(x)|^p \leq |h|^p \int_0^1 |u'(x-sh)|^p\, ds.$$

Hence

$$\begin{aligned}\int_\omega |u(x-h) - u(x)|^p\, dx \ &\leq \ |h|^p \int_\omega \int_0^1 |u'(x-sh)|^p\, ds\, dx \\ &\leq \ |h|^p \int_0^1 \underbrace{\int_\omega |u'(x-sh)|^p\, dx}_{\leq \|u'\|^p_{L^p(I)}}\, ds,\end{aligned}$$

so that

$$\int_\omega |u(x-h) - u(x)|^p\, dx \leq |h|^p \|u'\|^p_{L^p(I)},$$

and *iii)* follows with $C = \|u'\|_{L^p(I)}$.

Suppose now that the condition *iii)* holds. We will deduce *ii)*. Let $\varphi \in \mathcal{D}(I)$ and take ω relatively compact open set in I such that $\operatorname{Supp}\varphi \subseteq \omega$. Let $h > 0$ such that $|h| \leq \operatorname{dist}(\omega, \partial I)$. Then

$$\begin{aligned}\int_\omega (u(x-h) - u(x))\varphi(x)\,dx &= \int_I u(x-h)\varphi(x)\,dx - \int_I u(x)\varphi(x)\,dx \\ &= \int_I u(x)\varphi(x+h)\,dx - \int_I u(x)\varphi(x)\,dx \\ &= \int_I u(x)(\varphi(x+h) - \varphi(x))\,dx.\end{aligned}$$

The previous identity and condition *iii)* imply

$$\begin{aligned}|\int_I u(x)(\varphi(x+h) - \varphi(x))\,dx| &\leq \int_\omega |u(x-h) - u(x)|\,|\varphi(x)|\,dx \\ &\leq \|\tau_h u - u\|_{L^p(\omega)}\|\varphi\|_{L^{p'}(I)} \\ &\leq C|h|\|\varphi\|_{L^{p'}(I)},\end{aligned}$$

hence

$$|\int_I u(x)(\frac{\varphi(x+h) - \varphi(x)}{h})\,dx| \leq C\|\varphi\|_{L^{p'}(I)}. \tag{37}$$

Passing to the limit, for $h \to 0$, in (37) we obtain that

$$|\int_I u\varphi'| \leq C\|\varphi\|_{L^{p'}(I)}.$$

We remark that only in the proof of the fact that *ii)* implies *i)*, we have used the hypothesis that $p \neq 1$. All the other implications are valid also in this case. □

Remark 34. *It is possible to prove that, in the case $p = 1$, we have*

$$i) \Longrightarrow ii) \iff iii).$$

(what it would remain to prove, considering what it has been done in the proof of the previous theorem, is ii) $\Longrightarrow$ iii)). Moreover, it is possible to prove that

$$\|\tau_h u - u\|_{L^1(\omega)} \leq C|h| \quad \iff \quad u \in BV(I).$$

15.1.4 Characterization of $W^{1,2}(\mathbb{R})$

We want to characterize $W^{1,2}(\mathbb{R}) = H^1(\mathbb{R})$. We need the following lemma, the proof of which we let as an exercise.

Lemma 17. *Let $f,\ g \in L^2(\mathbb{R})$.*
Then

$$\int_\mathbb{R} f\,\hat{g} = \int_\mathbb{R} \hat{f}\,g \qquad \textit{and} \qquad \int_\mathbb{R} f\,\overline{g} = \frac{1}{2\pi}\int_\mathbb{R} \hat{f}\,\overline{\hat{g}}.$$

Remark 35. *In the proof of the previous lemma it is not possible to write*

$$\hat{f}(\xi) = \int_\mathbb{R} e^{-ix\xi} f(x)\,dx,$$

since $f \notin L^1(\mathbb{R})$.

Theorem 56. *Let $u \in L^2(\mathbb{R})$.*

$$u \in H^1(\mathbb{R}) \qquad \textit{if and only if} \qquad (1+|\xi|^2)^{\frac{1}{2}}\hat{u}(\xi) \in L^2(\mathbb{R}).$$

Moreover

$$\|u\|_{H^1} = (\int_{\mathbb{R}} (1+|\xi|^2)|\hat{u}(\xi)|^2\, d\xi)^{\frac{1}{2}}. \tag{38}$$

Proof. Let $u \in H^1(\mathbb{R})$. We have $u,\ u',\ \hat{u},\ \widehat{u'} \in L^2(\mathbb{R})$. Consider $\psi \in \mathcal{S}(\mathbb{R})$. We have

$$\begin{aligned}\int_{\mathbb{R}} \widehat{u'}(\xi)\psi(\xi)\, d\xi &= \int_{\mathbb{R}} u'(x)\hat{\psi}(x)\, dx = -\int_{\mathbb{R}} u(x)\hat{\psi}'(x)\, dx \\ &= -i\int_{\mathbb{R}} u(x)\widehat{\xi\psi(\xi)}(x)\, dx = -i\int_{\mathbb{R}} \hat{u}(\xi)\,\xi\,\psi(\xi)\, d\xi.\end{aligned}$$

Consequently $i\widehat{u'}(\xi) = \xi\,\hat{u}(\xi)$ a. e. in $\mathbb{R}$ and then $|\xi|\hat{u}(\xi) \in L^2(\mathbb{R})$.

Conversely suppose $\hat{u}(\xi),\ \xi\hat{u}(\xi) \in L^2(\mathbb{R})$. Let $\varphi \in \mathcal{D}(\mathbb{R})$. We have

$$\begin{aligned}|\int_{\mathbb{R}} u(x)\varphi'(x)\, dx| &= \frac{1}{2\pi}|\int_{\mathbb{R}} \hat{u}(\xi)\widehat{\varphi'}(\xi)\, d\xi| \\ &= \frac{1}{2\pi}|\int_{\mathbb{R}} \hat{u}(\xi)\, i\xi\,\hat{\varphi}(\xi)\, d\xi| \\ &\le \frac{1}{2\pi}\|\xi\hat{u}(\xi)\|_{L^2}\|\hat{\varphi}\|_{L^2} \le C\|\varphi\|_{L^2}\end{aligned}$$

and the fact that $u \in H^1(\mathbb{R})$ follows from condition *ii)* of Theorem 55. The identity (38) is a consequence of Plancherel's theorem. □

Remark 36. *Let $s \in \mathbb{R}$. We will define*

$$H^s(\mathbb{R}) = \{u \in \mathcal{S}'(\mathbb{R}) \bigm| (1+|\xi|^2)^{\frac{s}{2}}\hat{u}(\xi) \in L^2(\mathbb{R})\},$$

with

$$\|u\|_{H^s} = (\int_{\mathbb{R}} (1+|\xi|^2)^s|\hat{u}(\xi)|^2\, d\xi)^{\frac{1}{2}}.$$

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16.1 Sobolev spaces in one space dimension - 2

The content of this paragraph can be found in [3, Ch. VIII].

16.1.1 Extension operator

Consider $u \in W^{1,p}(I)$. The problem is now to find a function in $w \in W^{1,p}(\mathbb{R})$ such that $w_{|I} = u$ with a fixed relation between the norms of u and w. We need the following lemma.

Lemma 18. *Let $\theta \in C^\infty([0,+\infty[)$ such that, for all $x \in [0,+\infty[$, $0 \le \theta(x) \le 1$ and*

$$\theta(x) = \begin{cases} 1 & \textit{if } 0 \le x \le \frac{1}{4}, \\ 0 & \textit{if } x \ge \frac{3}{4}. \end{cases}$$

Let $u \in W^{1,p}(\,]0,1[\,)$. *Denote by*

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{if } 0 < x \le 1, \\ 0 & \text{if } x > 1, \end{cases} \qquad \widetilde{u'}(x) = \begin{cases} u'(x) & \text{if } 0 < x \le 1, \\ 0 & \text{if } x > 1, \end{cases}$$

and set $v = \theta\widetilde{u}$.

Then $v \in W^{1,p}(\,]0,+\infty[\,)$ *and* $v' = \theta'\widetilde{u} + \theta\widetilde{u'}$.

Proof. Since $\theta \in L^\infty(]0,+\infty[)$ and $\widetilde{u} \in L^p(]0,+\infty[)$, we have $\theta\widetilde{u} \in L^p(]0,+\infty[)$. Let $\varphi \in \mathcal{D}(]0,+\infty[)$. Remark that $\theta\varphi$ is a test function on $]0,+\infty[$ that can be considered also as a test function on $]0,1[$. Hence

$$\begin{aligned}
\int_{]0,+\infty[} \theta\widetilde{u}\,\varphi' &= \int_{]0,+\infty[} \widetilde{u}\,(\theta\varphi') \\
&= \int_{]0,1[} u\,(\theta\varphi') \\
&= \int_{]0,1[} u\,((\theta\varphi)' - \theta'\varphi) \\
&= \int_{]0,1[} -u'\,(\theta\varphi) - u\,(\theta'\varphi) \\
&= -\int_{]0,1[} (u'\,\theta + u\,\theta')\varphi \\
&= -\int_{]0,+\infty[} (\widetilde{u'}\,\theta + \widetilde{u}\,\theta')\varphi.
\end{aligned}$$

The lemma is proved. □

We state the extension theorem.

Theorem 57. *Let* I *be an open interval in* $\mathbb{R}$. *Let* $p \in [1,+\infty]$.

Then there exists an operator $P : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$ *such that:*

i) for all $u \in W^{1,p}(I)$,
$$Pu_{|I} = u;$$

ii) there exists $C_0 > 0$ *such that, for all* $u \in W^{1,p}(I)$,
$$\|Pu\|_{L^p(\mathbb{R})} \le C_0\|u\|_{L^p(I)};$$

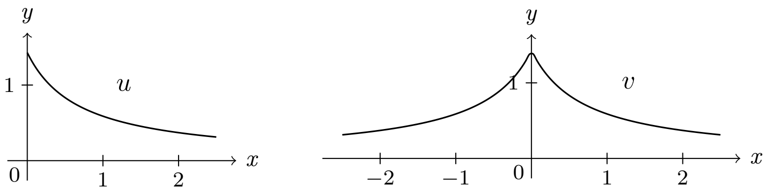
iii) there exists $C_1 > 0$ *such that, for all* $u \in W^{1,p}(I)$,
$$\|Pu\|_{W^{1,p}(\mathbb{R})} \le C_1\|u\|_{W^{1,p}(I)}.$$

Moreover C_0 *and* C_1 *depend only on* $|I|$.

Proof. Let $I =]0,+\infty[$ and $u \in W^{1,p}(I)$. Remark that we think at u as its continuous representative, so that $u \in C[0,+\infty[$. Consider now

$$v(x) = \begin{cases} u(x) & \text{if } x \ge 0, \\ u(-x) & \text{if } x < 0, \end{cases}$$

i. e. v is the extension of u to the whole $\mathbb{R}$, made by reflection with respect to the y axis.



We have that $v \in L^p(\mathbb{R})$ and

$$v(x) = \begin{cases} u(0) + \displaystyle\int_0^x u'(t)\,dt & \text{if } x \geq 0, \\ u(0) + \displaystyle\int_0^x -u'(-t)\,dt & \text{if } x < 0. \end{cases}$$

so that $v \in AC(\mathbb{R})$ and

$$v'(x) = \begin{cases} u'(x) & \text{for a. e. } x \geq 0, \\ -u'(-x) & \text{for a. e. } x < 0. \end{cases}$$

Consequently v' is an L^p function and moreover, from the result on integration by parts on AC functions, we have that v' is the weak derivative of v. As a conclusion we set

$$Pu = v.$$

We have

$$\|Pu\|_{L^p(\mathbb{R})} \leq 2\|u\|_{L^p(I)} \quad \text{and} \quad \|Pu\|_{W^{1,p}(\mathbb{R})} \leq 2\|u\|_{W^{1,p}(I)}.$$

Suppose now $I =]0,\, 1[$. Using Lemma 18 we construct $\theta\widetilde{u}$ and we extend it by reflection to the whole $\mathbb{R}$. Similarly, if we consider

$$\widetilde{\widetilde{u}}(x) = \begin{cases} u(x) & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x < 0, \end{cases} \qquad \widetilde{\widetilde{u'}}(x) = \begin{cases} u'(x) & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x < 0, \end{cases}$$

we have that $(1-\theta)\widetilde{\widetilde{u}} \in W^{1,p}(]-\infty,\, 1[)$ and $((1-\theta)\widetilde{\widetilde{u}})' = -\theta'\widetilde{\widetilde{u}} + (1-\theta)\widetilde{\widetilde{u'}}$. We extend $(1-\theta)\widetilde{\widetilde{u}}$ to $\mathbb{R}$ with a reflection in 1. We set

$$Pu = \theta\widetilde{u} + (1-\theta)\widetilde{\widetilde{u}}.$$

We let as an exercise to check the correctness of the points *i)*, *ii)* and *iii)*. □

16.1.2 An approximation result

We want to prove an approximation result. We need two lemmas.

Lemma 19. *Let $f \in L^1(\mathbb{R})$ and $u \in W^{1,p}(\mathbb{R})$, where $p \in [1, +\infty]$. Then $f * u \in W^{1,p}(\mathbb{R})$ and $(f * u)' = f * u'$.*

Proof. We know that if $f \in L^1(\mathbb{R})$ and $u \in L^p(\mathbb{R})$, then $f * u \in L^p(\mathbb{R})$. Similarly $f * u' \in L^p(\mathbb{R})$.

Suppose first that f has compact support. For all $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\begin{aligned}\int_{\mathbb{R}} (f * u)\, \varphi' &= \int_{\mathbb{R}} (\int_{\mathbb{R}} f(x-y)u(y)\, dy)\varphi'(x)\, dx \\ &= \int_{\mathbb{R}} u(y)(\int_{\mathbb{R}} f(x-y)\varphi'(x)\, dx)\, dy \\ &= \int_{\mathbb{R}} u\, (\check{f} * \varphi') \\ &= \int_{\mathbb{R}} u\, (\check{f} * \varphi)' \\ &= -\int_{\mathbb{R}} u'\, (\check{f} * \varphi) = -\int_{\mathbb{R}} (u' * f)\, \varphi.\end{aligned}$$

Consequently $f * u \in W^{1,p}(\mathbb{R})$ and $(f * u)' = f * u'$.

Suppose now that f has not compact support. Let $(f_n)_n$ be a sequence in $L^1(\mathbb{R})$ such that, for all n, f_n has compact support and $f_n \xrightarrow{n} f$ in $L^1(\mathbb{R})$ (it is sufficient to take $f_n = \chi_{B(0,n)} \cdot f$, see the next lemma). Then

$$f_n * u \xrightarrow{n} f * u \qquad \text{and} \qquad (f_n * u)' = f_n * u' \xrightarrow{n} f * u' \quad \text{in} \quad L^p(\mathbb{R}).$$

The conclusion follows from Remark 33. □

Lemma 20. *Let $\chi \in \mathcal{D}(\mathbb{R})$ such that*

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

Consider, for all $n \geq 1$, $\chi_n(x) = \chi(\frac{x}{n})$. Let $f \in L^p(\mathbb{R})$, with $p \in [1, +\infty[$.
Then

$$\chi_n \cdot f \xrightarrow{n} f \quad \text{in} \quad L^p(\mathbb{R}).$$

Proof. It is an application of the dominated convergence theorem. □

Theorem 58. *Let $u \in W^{1,p}(I)$, with $p \in [1, +\infty[$.*
Then there exists a sequence $(u_n)_n$ in $\mathcal{D}(\mathbb{R})$ such that

$$u_{n|I} \xrightarrow{n} u \quad \text{in} \quad W^{1,p}(I).$$

Proof. Suppose first that $I = \mathbb{R}$. Let $u \in W^{1,p}(\mathbb{R})$ with $p \in [1, +\infty[$. Let $(\rho_n)_n$ be a family of mollifier and let $(\chi_n)_n$ the sequence of truncation (or cut off) functions defined in Lemma 20. We set, for all n,

$$u_n = \chi_n \cdot (\rho_n * u).$$

We have $u_n \in \mathcal{D}(\mathbb{R})$ and, using also Lemma 20,

$$\begin{aligned}\|u_n - u\|_{L^p} &= \|\chi_n \cdot (\rho_n * u) - u\| \\ &\leq \|\chi_n \cdot (\rho_n * u) - \chi_n \cdot u\|_{L^p} + \|\chi_n \cdot u - u\|_{L^p} \\ &\leq \underbrace{\|\chi_n\|_{L^\infty}}_{\leq 1} \underbrace{\|(\rho_n * u) - u\|_{L^p}}_{\xrightarrow{n} 0} + \underbrace{\|\chi_n \cdot u - u\|_{L^p}}_{\xrightarrow{n} 0}.\end{aligned}$$

Finally, also from Lemma 19,

$$\begin{aligned}
&\|u_n' - u'\|_{L^p} \\
&\quad = \|\chi_n' \cdot (\rho_n * u) + \chi_n \cdot (\rho_n * u)' - u'\|_{L^p} \\
&\quad \le \underbrace{\|\chi_n'\|_{L^\infty}}_{\le \frac{C}{n}} \underbrace{\|\rho_n * u\|_{L^p}}_{\le \|u\|_{L^p}} + \underbrace{\|\chi_n \cdot ((\rho_n * u') - u')\|_{L^p}}_{\le \|(\rho_n * u') - u'\|_{L^p} \xrightarrow{n} 0} + \underbrace{\|\chi_n \cdot u' - u'\|_{L^p}}_{\xrightarrow{n} 0}.
\end{aligned}$$

Consequently

$$u_n \xrightarrow{n} u \qquad \text{and} \qquad u_n' \xrightarrow{n} u' \quad \text{in} \quad L^p(\mathbb{R}).$$

Suppose now $u \in W^{1,p}(I)$, with $I \subseteq \mathbb{R}$. Consider, from Theorem 57, Pu extension of u to $\mathbb{R}$. From the first part of the proof, we have that there exists a sequence $(w_n)_n$ in $C_0^\infty(\mathbb{R})$ such that

$$w_n \xrightarrow{n} Pu \quad \text{in} \quad W^{1,p}(\mathbb{R}).$$

Hence

$$w_{n|I} \xrightarrow{n} Pu_{|I} = u \quad \text{in} \quad W^{1,p}(I).$$

□

Corollary 13. *Let $p \in [1, +\infty[$.*
Then $\mathcal{D}(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$.

Remark 37. *As we will see, if $I \neq \mathbb{R}$, then $\mathcal{D}(I)$ is not dense in $W^{1,p}(I)$.*

17

17.1 Sobolev spaces in one space dimension - 3

The content of this paragraph can be found in [3, Ch. VIII].

17.1.1 Sobolev embeddings

Theorem 59 (Sobolev)**.** *Let I be an open interval in $\mathbb{R}$. Let $p \in [1, +\infty]$. Let $u \in W^{1,p}(I)$.*
Then $u \in L^\infty(I)$ and there exists $C > 0$, depending only on p and $|I|$, such that

$$\|u\|_{L^\infty(I)} \le C\|u\|_{W^{1,p}(I)},$$

i. e. $W^{1,p}(I)$ is contained in $L^\infty(I)$ with continuous immersion (embedding).

Proof. Suppose first that $I = \mathbb{R}$. If $p = +\infty$ there is nothing to prove. Let $p \in [1, +\infty[$. Consider the function

$$G : \mathbb{R} \to \mathbb{R}, \qquad G(t) = t\,|t|^{p-1}.$$

G is a $C^1(\mathbb{R})$ function and $G'(t) = p|t|^{p-1}$.
Let u be a $C_0^\infty(\mathbb{R})$ function and define

$$\psi(t) = G(u(t)).$$

We have

$$\psi \in C_0^1(\mathbb{R}), \qquad |\psi(t)| = |u(t)|^p \qquad \text{and} \qquad \psi'(t) = p|u(t)|^{p-1}u'(t).$$

Consequently

$$\|u\|_{L^\infty}^p = \sup_{t\in\mathbb{R}} |\psi(t)| \le \int_{\mathbb{R}} |\psi'(s)|\, ds = p\int_{\mathbb{R}} |u(s)|^{p-1}|u'(s)|\, ds, \tag{39}$$

but, thinking at $|u|^{p-1}$ as a function in $L^{p'}$, with $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\||u|^{p-1}\|_{L^{p'}} = (\int_{\mathbb{R}} |u(s)|^{(p-1)p'}\, ds)^{\frac{1}{p'}} = (\int_{\mathbb{R}} |u(s)|^p\, ds)^{\frac{p-1}{p}} = \|u\|_{L^p}^{p-1},$$

so that from (39) and Hölder inequality, we finally get

$$\|u\|_{L^\infty}^p \le p\|u\|_{L^p}^{p-1}\, \|u'\|_{L^p}.$$

Hence

$$\|u\|_{L^\infty} \le p^{\frac{1}{p}}\|u\|_{L^p}^{1-\frac{1}{p}}\, \|u'\|_{L^p}^{\frac{1}{p}}.$$

We apply the Young's inequality (see the Appendix) and we finally obtain

$$\|u\|_{L^\infty} \le p^{\frac{1}{p}}(\|u\|_{L^p} + \|u'\|_{L^p}) = p^{\frac{1}{p}}\|u\|_{W^{1,p}}. \tag{40}$$

Let now $u \in W^{1,p}(\mathbb{R})$. From Corollary 13, possibly passing to a subsequence, exists a sequence $(u_n)_n$ in $C_0^\infty(\mathbb{R})$ such that

$$u_n \xrightarrow{n} u \qquad \text{in } W^{1,p}(\mathbb{R}) \text{ and almost everywhere.}$$

Condition (40) implies that $(u_n)_n$ is a Cauchy sequence in $L^\infty(\mathbb{R})$ which converges almost everywhere to u. Consequently the convergence is in $L^\infty(\mathbb{R})$ and moreover $u \in L^\infty(\mathbb{R})$. Passing to the limit in

$$\|u_n\|_{L^\infty} \le p^{\frac{1}{p}}\|u_n\|_{W^{1,p}},$$

we obtain (40) for $u \in W^{1,p}(\mathbb{R})$.

Suppose finally that $u \in W^{1,p}(I)$ with $I \subseteq \mathbb{R}$. Consider the extension operator $P : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$. We apply to Pu the first part of the proof. Then

$$Pu \in L^\infty(\mathbb{R}) \quad \text{and} \quad \|Pu\|_{L^\infty(\mathbb{R})} \le p^{\frac{1}{p}}\, \|Pu\|_{W^{1,p}(\mathbb{R})}.$$

Consequently $u = Pu_{|I} \in L^\infty(I)$ and

$$\|u\|_{L^\infty(I)} \le \|Pu\|_{L^\infty(\mathbb{R})} \le p^{\frac{1}{p}}\, \|Pu\|_{W^{1,p}(\mathbb{R})} \le C\|u\|_{W^{1,p}(I)},$$

where C depends only on p and $|I|$. □

Theorem 60 (Rellich)**.** *Let I be an open interval in $\mathbb{R}$, with $|I| < +\infty$.*

i) If $p \in]1, +\infty]$, then

$$W^{1,p}(I) \subseteq C(\bar{I}) \qquad \textit{with compact embedding.}$$

ii) If $p = 1$ and $q \in [1, +\infty[$, then

$$W^{1,1}(I) \subseteq L^q(I) \qquad \textit{with compact embedding.}$$

Proof. Let's prove *i)*. We already know that $W^{1,p}(I) \subseteq C(\bar{I})$ (from the theorem on continuous representative) and that the immersion is continuous (from Sobolev theorem, when on $C(\bar{I})$ we put the sup-norm). It remains to prove that the embedding is a compact embedding, i. e. a bounded set in $W^{1,p}(I)$ is a relatively compact set in $C(\bar{I})$. To see this we use Ascoli-Arzelà's theorem, showing that the functions in a bounded set of $W^{1,p}(I)$ are equicontinuous. In fact, for all $u \in W^{1,p}(I)$ with $\|u\|_{W^{1,p}(I)} \leq C$, we have

$$|u(x) - u(y)| \leq \underbrace{|\int_x^y |u'(t)|\, dt \leq |x-y|^{\frac{1}{p'}}\, \|u'\|_{L^p(I)}}_{\text{Hölder}} \leq C|x-y|^{\frac{1}{p'}}.$$

Let's prove *ii)*. Also in this case the only thing to prove is that the embedding is a compact embedding, i. e. a bounded set in $W^{1,1}(I)$ is a relatively compact set in $L^q(I)$. We use, in this case, the Riesz-Fréchet-Kolmogorov theorem (see [3, Cor. IV.26]). We recall that a set B is relatively compact in $L^q(I)$ if the following two conditions hold.

a) For all $\varepsilon > 0$ and for all relatively compact set ω in I, there exists $0 < \delta < \operatorname{dist}(\omega, \partial I)$ such that,

$$\|\tau_h f - f\|_{L^q(\omega)} \leq \varepsilon, \quad \text{for all } h \in \mathbb{R} \text{ with } |h| < \delta, \text{ and for all } f \in B.$$

b) For all $\varepsilon > 0$ there exists a relatively compact set ω in I such that

$$\|f\|_{L^q(I\setminus\omega)} < \varepsilon, \quad \text{and for all } f \in B,$$

We set

$$B = \{u \in W^{1,1}(I) \,\big|\, \|u\|_{W^{1,1}} \leq C_0\}.$$

Let $u \in B$ and let ω be a relatively compact set in I and let $h \in \mathbb{R}$ with $|h| < \operatorname{dist}(\omega, \partial I)$. We know, from the Theorem 55 and Remark 34, that there exists $C > 0$ such that

$$\|\tau_h u - u\|_{L^1(\omega)} \leq C\,|h|.$$

Consequently

$$\int_\omega |u(x-h)-u(x)|^q\, dx \leq \int_\omega |u(x-h)-u(x)|(2\|u\|_{L^\infty})^{q-1}\, dx \leq C\,|h|(2\|u\|_{L^\infty})^{q-1},$$

hence

$$(\int_\omega |u(x-h) - u(x)|^q\, dx)^{\frac{1}{q}} \leq C'|h|^{\frac{1}{q}}\, \|u\|_{L^\infty}^{1-\frac{1}{q}},$$

where $C' = 2^{1-\frac{1}{q}} C^{\frac{1}{q}}$. From Sobolev theorem we have that there exists $C_1 > 0$ (depending only on q and $|I|$) such that

$$\|u\|_{L^\infty} \leq C_1 \|u\|_{W^{1,1}},$$

then

$$\|\tau_h u - u\|_{L^q(\omega)} \le C' \, (C_0 C_1)^{1-\frac{1}{q}} \, |h|^{\frac{1}{q}}$$

and a) follows taking $\delta = \varepsilon^q/((C')^q (C_0 C_1)^{q-1})$. Concerning the condition b), it is sufficient to remark that, for $u \in B$,

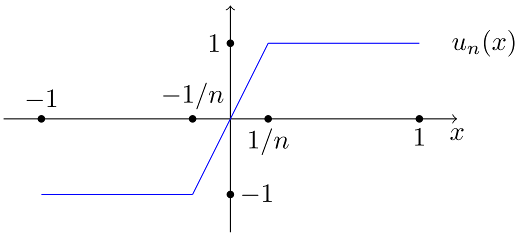
$$\|u\|_{L^q(I\setminus\omega)} \le \|u\|_{L^\infty} \, |I\setminus\omega|^{\frac{1}{q}} \le C_0 \, C_1 \, |I\setminus\omega|^{\frac{1}{q}}$$

and, since $|I| < +\infty$, it will be possible to find a relatively compact set ω such that $|I\setminus\omega| < \varepsilon^q/(C_0C_1)^{q-1}$. The proof is complete. □

Remark 38. *Let $u \in \mathcal{D}(\mathbb{R})$, with $u \ne 0$. Consider the sequence $(u_n)_n$, where $u_n(x) = u(x-n)$. The sequence $(u_n)_n$ is bounded in $W^{1,p}(\mathbb{R})$, for all $p \in [1, +\infty]$, but a subsequence which is converging in $C(\mathbb{R})$ or in $L^q(\mathbb{R})$ does not exists: the boundedness of I is a necessary condition in Rellich theorem.*

Remark 39. *For $n \ge 1$, consider the function*

$$u_n(x) = \begin{cases} 1 & \text{if } x \in [1/n, \, 1], \\ nx & \text{if } x \in [-1/n, \, 1/n], \\ -1 & \text{if } x \in [-1, \, -1/n]. \end{cases}$$



For all n, $u_n \in W^{1,1}(]-1,\,1[)$ and

$$\|u_n\|_{W^{1,1}} = \int_{]-1,\,1[} |u| + \int_{]-1,\,1[} |u'| \le 2 + 2 \le 4.$$

The sequence $(u_n)_n$ does not have a subsequence which converges in $C([-1,\,1])$. In fact the sequence $(u_n)_n$ converges pointwise to a non continuous function. The immersion of $W^{1,1}(]-1,\,1[)$ in $C([-1,\,1])$ is not a compact immersion.

17.1.2 Corollaries to Sobolev embedding theorem

Corollary 14. *Let $p \in [1, +\infty[$. Let $u \in W^{1,p}(\mathbb{R})$.*
Then

$$\lim_{x\to-\infty} u(x) = \lim_{x\to+\infty} u(x) = 0.$$

Proof. Let $\varepsilon > 0$ and consider $(u_n)_n$ in $\mathcal{D}(\mathbb{R})$ such that $u_n \xrightarrow{n} u$ in $W^{1,p}(\mathbb{R})$. Then, by Theorem 59, $u_n \xrightarrow{n} u$ in L^∞. As a consequence, there exists $\bar{n}$ such that, for all $n \ge \bar{n}$,

$$\|u_n - u\|_{L^\infty} < \varepsilon.$$

In particular $\|u_{\bar{n}} - u\|_{L^\infty} \le \varepsilon$ and, since $u_{\bar{n}}$ has compact support, there exists $R > 0$ such that, for all $x \in \mathbb{R}$, if $|x| > R$ then $u_{\bar{n}}(x) = 0$. In conclusion, for all $\varepsilon > 0$, there exists $R > 0$ such that, for all $x \in \mathbb{R}$, if $|x| > R$ then $|u(x)| \le |u_{\bar{n}}(x) - u(x)| < \varepsilon$. □

Corollary 15. *Let $p \in [1, +\infty]$. Let $u,\, v \in W^{1,p}(I)$.*
Then
$$uv \in W^{1,p}(I) \qquad \textit{and} \qquad (uv)' = u'v + uv'.$$

Proof. We show first that uv and $u'v + uv'$ are in $L^p(I)$. If $p = \infty$, there is nothing to prove. If, on the contrary, $p < \infty$, we use Sobolev theorem, deducing that u and v are in $L^p(I) \cap L^\infty(I)$ and the conclusion follows.
It remains to prove that
$$(uv)' = u'v + uv'.$$

Let $p \in [1, +\infty[$. Consider $(u_n)_n$ and $(v_n)_n$ in $\mathcal{D}(\mathbb{R})$ such that
$$u_{n|I} \xrightarrow{n} u \qquad \text{and} \qquad v_{n|I} \xrightarrow{n} v \quad \text{in} \;\; W^{1,p}(I).$$
Hence
$$u_{n|I} \xrightarrow{n} u \qquad \text{and} \qquad v_{n|I} \xrightarrow{n} v \quad \text{in} \;\; L^\infty(I).$$
We have
$$\begin{aligned}
&\|u_{n|I}v_{n|I} - uv\|_{L^p(I)} \\
&\quad \le \|u_{n|I}v_{n|I} - u\, v_{n|I}\|_{L^p(I)} + \|u\, v_{n|I} - uv\|_{L^p(I)} \\
&\quad \le \underbrace{\|v_{n|I}\|_{L^\infty(I)}}_{\text{bounded}} \underbrace{\|u_{n|I} - u\|_{L^p(I)}}_{\xrightarrow{n} 0} + \|u\|_{L^p(I)} \underbrace{\|v_{n|I} - v\|_{L^\infty(I)}}_{\xrightarrow{n} 0}
\end{aligned}$$
and
$$\begin{aligned}
&\|u'_{n|I}v_{n|I} - u'v\|_{L^p(I)} \\
&\quad \le \|u'_{n|I}v_{n|I} - u'\, v_{n|I}\|_{L^p(I)} + \|u'\, v_{n|I} - u'v\|_{L^p(I)} \\
&\quad \le \underbrace{\|v_{n|I}\|_{L^\infty(I)}}_{\text{bounded}} \underbrace{\|u'_{n|I} - u'\|_{L^p(I)}}_{\xrightarrow{n} 0} + \|u'\|_{L^p(I)} \underbrace{\|v_{n|I} - v\|_{L^\infty(I)}}_{\xrightarrow{n} 0}
\end{aligned}$$
Consequently
$$u_{n|I}v_{n|I} \xrightarrow{n} u\, v \qquad \text{and} \qquad u'_{n|I}v_{n|I} + u_{n|I}v'_{n|I} \xrightarrow{n} u'v + uv' \quad \text{in} \;\; L^p(I)$$
and the conclusion follows from Remark 33.
Let $p = \infty$. We have to prove that, for all $\varphi \in \mathcal{D}(I)$,
$$\int_I uv\, \varphi' = -\int_I (u'v + uv')\varphi. \tag{41}$$
Fix $\varphi \in \mathcal{D}(I)$ and consider an open relatively compact interval J such that $\operatorname{Supp} \varphi \subseteq J \subseteq I$. We have that $u_{|J},\; v_{|J} \in W^{1,p}(J)$ for all $p \in [1, +\infty[$. Then (41) is valid for the first part of the proof. ☐

Corollary 16. *Let $p \in [1, +\infty]$ and $u \in W^{1,p}(I)$. Let G be a $C^1(\mathbb{R})$ function, with $G(0) = 0$.*
Then
$$G \circ u \in W^{1,p}(I) \qquad \textit{and} \qquad (G \circ u)' = (G' \circ u)u'.$$

Proof. $u \in W^{1,p}(I)$, so that, from Sobolev theorem, $u \in L^\infty(I)$ and consequently there exists $M > 0$ such that $\|u\|_{L^\infty(I)} \leq M$. Then, considering the continuous representative,

$$\text{for all } \; x \in I, \quad u(x) \in [-M,\, M].$$

On the other hand, since $G \in C^1(\mathbb{R})$ and $G(0) = 0$, we have that there exists $C > 0$ such that,

$$\text{for all } \; s \in [-M,\, M], \quad |G(s)| \leq C|s|.$$

Hence

$$\text{for all } \; x \in I, \quad |G(u(x))| \leq C|u(x)|.$$

Consequently $G \circ u \in L^p(I) \cap L^\infty(I)$. Similarly there exists $C' > 0$ such that

$$\text{for all } \; x \in I, \quad |G'(u(x))| \leq C'$$

and then $(G' \circ u)\, u' \in L^p(I)$.

Suppose that $p \in [1,\, +\infty[$. We have to prove that, for all $\varphi \in \mathcal{D}(I)$,

$$\int_I (G \circ u)\, \varphi' = -\int_I ((G' \circ u)\, u')\, \varphi. \tag{42}$$

There exists $(u_n)_n$ in $\mathcal{D}(\mathbb{R})$ such that

$$u_{n|I} \xrightarrow{n} u \qquad \text{in } \; W^{1,p}(I), \text{ in } \; L^\infty(I) \; \text{ and a. e. in } I.$$

We remark that, since for all n, $G \circ u_n$ is in $C^1_0(\mathbb{R})$, then, for all $\varphi \in \mathcal{D}(I)$,

$$\int_I (G \circ u_n)\, \varphi' = -\int_I ((G' \circ u_n)\, u_n')\, \varphi.$$

We know that $\|u\|_{L^\infty(I)} \leq M$, so that we can suppose, without any restriction, that, for all n, $\|u_n\|_{L^\infty(I)} \leq M + 1$. Consequently

$$\|G \circ u_n - G \circ u\|_{L^\infty(I)} \leq (\sup_{|t| \leq M+1} |G'(t)|)\, \|u_n - u\|_{L^\infty(I)},$$

and we deduce

$$\int_I (G \circ u_n)\, \varphi' \xrightarrow{n} \int_I (G \circ u)\, \varphi'. \tag{43}$$

On the other hand

$$\begin{aligned} &\|(G' \circ u_n)\, u_n' - (G' \circ u)\, u'\|_{L^p(I)} \\ &\leq \|(G' \circ u_n)\, u_n' - (G' \circ u)\, u_n'\|_{L^p(I)} + \|(G' \circ u)\, u_n' - (G' \circ u)\, u'\|_{L^p(I)} \\ &\leq \|(G' \circ u_n) - (G' \circ u)\|_{L^\infty(I)} \|u_n'\|_{L^p(I)} + \|G' \circ u\|_{L^\infty(I)} \|u_n' - u'\|_{L^p(I)}. \end{aligned} \tag{44}$$

Remark now that, since G' on the interval $[-M-1, M+1]$ is uniformly continuous, we have that for all $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $t_1,\, t_2 \in [-M-1, M+1]$, if $|t_2 - t_1| \leq \delta$ then $|G'(t_2) - G'(t_1)| < \varepsilon$, consequently

$$\text{if } \; \|u_n - u\|_{L^\infty(I)} < \delta \qquad \text{then} \qquad \|(G' \circ u_n) - (G' \circ u)\|_{L^\infty(I)} < \varepsilon.$$

We deduce that

$$\lim_n \|(G' \circ u_n) - (G' \circ u)\|_{L^\infty(I)} = 0$$

and finally, from (44),

$$\lim_n \|(G' \circ u_n)\, u_n' - (G' \circ u)\, u'\|_{L^p(I)} = 0$$

and hence

$$\int_I ((G' \circ u_n)\, u_n')\, \varphi \overset{n}{\longrightarrow} \int_I ((G' \circ u)\, u')\, \varphi'. \tag{45}$$

The conclusion is reached from (43) and (45) in the usual way.

Let $p = \infty$. Also in this case we have to prove (42). Fix $\varphi \in \mathcal{D}(I)$ and consider an open relatively compact interval J such that $\operatorname{Supp} \varphi \subseteq J \subseteq I$. We have that $u_{|J},\ v_{|J} \in W^{1,p}(J)$ for all $p \in [1, +\infty[$. Then (42) is valid for the first part of the proof.

□

17.1.3 The space $W^{m,p}(I)$

Definition 40. *Let I be an open interval in $\mathbb{R}$. Let $p \in [1, +\infty]$ and $m \in \mathbb{N} \setminus \{0, 1\}$. We define*

$$W^{m,p}(I) = \{u \in W^{m-1,p}(I) \mid u' \in W^{m-1,p}(I)\}$$

i. e.

$$W^{m,p}(I) = \{u \in L^p(I) \mid u',\, u'',\, \ldots,\, u^{(m)} \in L^p(I)\},$$

where derivatives are to be considered in distributional sense. We set

$$\|u\|_{W^{m,p}} = \|u\|_{L^p} + \|u'\|_{L^p} + \ldots + \|u^{(m)}\|_{L^p}$$

or, equivalently, in the case of $p \in [1, +\infty[$,

$$\|u\|_{W^{m,p}} = (\|u\|_{L^p}^p + \|u'\|_{L^p}^p + \ldots + \|u^{(m)}\|_{L^p}^p)^{\frac{1}{p}}.$$

Remark 40. *We set $W^{m,2}(I) = H^m(I)$. For $m \geq 2$, it is possible to show that*

$$H^m(\mathbb{R}) = \{u \in L^2(\mathbb{R}) \mid (1 + |\xi|^2)^{\frac{m}{2}}\, \hat{u}(\xi) \in L^2(\mathbb{R})\}.$$

17.1.4 The space $W_0^{1,p}(I)$

Definition 41. *Let $p \in [1, +\infty[$. Let I be a bounded open interval in $\mathbb{R}$, i. e. an open interval such that $|I| < +\infty$. We denote by $W_0^{1,p}(I)$ the closure of $C_0^1(I)$ in $W^{1,p}(I)$.*

Remark 41. *$W_0^{1,p}(I)$ is the closure of $C_0^\infty(I)$ in $W^{1,p}(I)$. In fact, for every function f of $C_0^1(I)$ there exists a sequence $(u_n)_n$ in $C_0^\infty(I)$ such that $u_n \overset{n}{\longrightarrow} f$ and $u_n' \overset{n}{\longrightarrow} f'$ uniformly. This implies that $u_n \overset{n}{\longrightarrow} f$ in $W^{1,p}(I)$.*

Theorem 61. *Let $p \in [1, +\infty[$. Let I be a bounded open interval in $\mathbb{R}$. Let $u \in W^{1,p}(I)$ and consider the continuous representative of u (we indicate it with the same letter u).*

Then

$$u \in W_0^{1,p}(I) \qquad \text{if and only if} \qquad u_{|\partial I} = 0.$$

Proof. Let $u \in W_0^{1,p}(I)$. There exists a sequence $(u_n)_n$ in $C_0^\infty(I)$ such that $u_n \overset{n}{\longrightarrow} u$ in $W^{1,p}(I)$. From Sobolev theorem, $u_n \overset{n}{\longrightarrow} u$ uniformly in $C(\bar{I})$. In particular $0 = u_n(x) \overset{n}{\longrightarrow} u(x)$ for $x \in \partial I$. As a consequence $u_{|\partial I} = 0$.

Conversely suppose $u \in W^{1,p}(I)$ with $u_{|\partial I} = 0$. Consider a function $G \in C^1(\mathbb{R})$ such that,

$$G(s) = \begin{cases} s & \text{if } |s| \geq 2, \\ 0 & \text{if } |s| \leq 1, \end{cases}$$

and, for all $s \in \mathbb{R}$, $|G(s)| \leq |s|$. Consider

$$u_n(x) = \frac{1}{n} G(n\, u(x)).$$

From Corollary 16 we deduce that $u_n \in W^{1,p}(I)$. Moreover $u_n \in C(\bar{I})$ and $u_n(x) = 0$ if $|u(x)| < \frac{1}{n}$, hence $\operatorname{Supp} u_n$ is a compact set contained in I, i. e $u_n \in W_0^{1,p}(I)$. It remains to prove that $u_n \overset{n}{\longrightarrow} u$ in $W^{1,p}(I)$. We let it as an exercise. □

Corollary 17 (Poincaré inequality)**.** *Let $p \in [1, +\infty[$. Let I be a bounded open interval in $\mathbb{R}$.*

Then there exists $C > 0$ such that, for all $u \in W_0^{1,p}(I)$,

$$\|u\|_{W^{1,p}(I)} \leq C \|u'\|_{L^p(I)},$$

i. e. $\|u'\|_{L^p(I)}$ is an equivalent norm in $W_0^{1,p}(I)$.

Proof. Let $I =]a, b[$. We have, for $u \in W_0^{1,p}(]a, b[)$,

$$|u(x)| = |\int_a^x u'(t)\, dt| \leq \int_a^b |u'(x)|\, dx \leq (b-a)^{\frac{1}{p'}} \|u'\|_{L^p(I)}.$$

Consequently

$$\int_a^b |u(x)|^p\, dx \leq \int_a^b (b-a)^{\frac{p}{p'}} \|u'\|^p_{L^p(I)}\, dx = (b-a)^{1+\frac{p}{p'}} \|u'\|^p_{L^p(I)},$$

and hence

$$\|u\|_{L^p(I)} \leq (b-a) \|u'\|_{L^p(I)}.$$

We conclude

$$\|u\|_{W^{1,p}(I)} \leq ((b-a)+1) \|u'\|_{L^p(I)}.$$

□

Remark 42. *We denote by $H_0^1(I)$ the space $W_0^{1,2}(I)$.*

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18.1 Sobolev spaces in one space dimension - 4

The content of this paragraph can be found in [3, Ch. VIII].

18.1.1 Examples of boundary value problems

Example 15 (Homogeneous Dirichlet problem). *Let $f \in C([0, 1])$.*
Find $u \in C^2(]0, 1[) \cap C([0, 1])$ such that

$$\begin{cases} -u'' + u = f & \text{in }]0, 1[, \\ u(0) = u(1) = 0. \end{cases} \tag{46}$$

Problem (46) is known as the (classical) homogeneous Dirichlet problem. The strategy for solving it will be the following.

a) Introduce a modified (weak) problem. The correct setting of this modified problem will be crucial.

b) Solve the weak problem, using some suitable functional analysis results.

c) Check that the solution of the weak problem, with the conditions of the classical problem, is actually the solution of the classical problem.

a) Let $\tilde{f} \in L^2(]0, 1[)$.
Find $w \in H_0^1(]0, 1[)$ such that

$$\int_{]0,1[} w'\, v' + \int_{]0,1[} w\, v = \int_{]0,1[} \tilde{f}\, v, \qquad \text{for all } v \in H_0^1(]0, 1[). \tag{47}$$

Problem (47) is the weak homogeneous Dirichlet problem.

Remark 43. *If u is a solution of the classical problem then u is a solution of the weak one, if $\tilde{f} = f$. In fact, suppose that $u \in C^2(]0, 1[) \cap C([0, 1])$ is a solution to the classical problem, then*

$$\int_{]0,1[} (-u'' + u)\, v = \int_{]0,1[} f\, v, \qquad \text{for all } v \in H_0^1(]0, 1[).$$

and, integrating by parts,

$$\int_{]0,1[} u'\, v' + \int_{]0,1[} u\, v = \int_{]0,1[} f\, v, \qquad \text{for all } v \in H_0^1(]0, 1[).$$

Finally, since $u(0) = u(1) = 0$, $u \in H_0^1(]0, 1[)$.

b) Let's solve problem (47). We use Lax-Milgram theorem (see [3, Cor. V.8]). We choose as Hilbert space H the space $H_0^1(]0, 1[)$, as bilinear form a, the form

$$a(w, v) = \int_{]0,1[} w'\, v' + \int_{]0,1[} w\, v$$

and as ϕ, element of H', the functional

$$\phi : H_0^1(]0, 1[) \to \mathbb{R}, \qquad \phi(v) = \int_{]0,1[} \tilde{f}\, v.$$

The existence and uniqueness of the solution $w \in H_0^1(]0, 1[)$ follows. Remark that, in particular,

$$\int_{]0,1[} w'\, \varphi' = \int_{]0,1[} (\tilde{f} - w)\, \varphi, \qquad \text{for all } \varphi \in \mathcal{D}(]0, 1[),$$

with $\tilde{f} - w \in L^2$, so that $w' \in H^1(]0, 1[)$ and consequently $w \in H^2(]0, 1[) \cap H_0^1(]0, 1[)$.

c) We show that the solution of problem (47) with $\tilde{f} = f$ is the solution of problem (46). Suppose that $\tilde{f} = f \in C([0, 1])$. Take $u = w$. Since $u \in H_0^1(]0, 1[)$, we have $u \in C([0, 1])$ with $u(0) = u(1) = 0$. Moreover

$$\int_{]0,1[} u'\, v' = \int_{]0,1[} (f - u)\, v, \qquad \text{for all} \;\; v \in H_0^1(]0, 1[).$$

In particular

$$\int_{]0,1[} u'\, \varphi' = \int_{]0,1[} (f - u)\, \varphi, \qquad \text{for all} \;\; \varphi \in \mathcal{D}(]0, 1[),$$

i. e. $u' \in H^1(]0, 1[)$ (notice that, in particular, $f - u \in L^2$) and $(u')' = u - f$ in the sense of distributions. Consequently

$$\int_{]0,1[} (-(u')' + u - f)\, \varphi = 0, \qquad \text{for all} \;\; \varphi \in \mathcal{D}(]0, 1[).$$

It remains to prove that u is in C^2 and $(u')'$ is the classical second derivative. This is a consequence of the du Bois-Reymond thoerem. In fact u' is continuous (it is in $H^1(]0, 1[)$) and its derivative in the sense of distribution is $u - f$, which is continuous. Remark that we can obtain the same conclusion using the fundamental theorem of calculus. In fact the theorem on the continuous representative implies that

$$u'(x) - u'(y) = \int_y^x (u')'(t)\, dt = \int_y^x (u(t) - f(t))\, dt$$

but $u - f$ is continuous, so that u' is a C^1 function.

Example 16 (Non-homogeneous Dirichlet problem)**.** *Let* $f \in C([0, 1])$.
Find $u \in C^2(]0, 1[) \cap C([0, 1])$ *such that*

$$\begin{cases} -u'' + u = f & \textit{in} \;\;]0, 1[, \\ u(0) = a, \quad u(1) = b, & a, b \in \mathbb{R}. \end{cases} \tag{48}$$

Consider

$$v(x) = u(x) - (a + (b - a)x).$$

Then

$$v'' = u'' \qquad \text{and} \qquad -v''(x)+v(x) = -u''(x)+u(x)-(a+(b-a)x) = f(x)-(a+(b-a)x)$$

and

$$v(0) = a - a = 0, \qquad v(1) = b - (a + (b - a)) = 0,$$

i. e. v is the solution of

$$\begin{cases} -v'' + v = \tilde{f} & \text{in} \;\;]0, 1[, \\ v(0) = v(1) = 0, & \end{cases}$$

where $\tilde{f}(x) = f(x) - (a + (b - a)x)$.

Example 17 (Homogeneous Neumann problem). *Let $f \in C([0,\,1])$.*
Find $u \in C^2(]0,\,1[) \cap C^1([0,\,1])$ such that

$$\begin{cases} -u'' + u = f & \textit{in } \ ]0,\,1[, \\ u'(0) = u'(1) = 0. \end{cases} \tag{49}$$

a) We introduce the weak problem considering $\tilde{f} \in L^2(]0,\,1[)$ and looking for $w \in H^1(]0,\,1[)$ such that

$$\int_{]0,1[} w'\,v' + \int_{]0,1[} w\,v = \int_{]0,1[} \tilde{f}\,v, \qquad \text{for all } \ v \in H^1(]0,\,1[). \tag{50}$$

We remark also in this case that the solutions to (49) are solutions to (50) if $\tilde{f} = f$. In fact, if

$$\begin{cases} -u'' + u = f & \text{in } \ ]0,\,1[, \\ u'(0) = u'(1) = 0, \end{cases}$$

then

$$(-u'' + u - f)\,v = 0, \qquad \text{for all } \ v \in H^1(]0,\,1[),$$

so that

$$\int_{]0,\,1[} -u''\,v + \int_{]0,\,1[} u\,v = \int_{]0,\,1[} f\,v$$

and

$$-\int_{]0,\,1[} u''\,v = (-(u'v)\big|_0^1) + \int_{]0,\,1[} u'\,v' = (\underbrace{u'(0)}_{=0}\,v(0) - \underbrace{u'(1)}_{=0}\,v(1)) + \int_{]0,\,1[} u'\,v'.$$

We obtain

$$\int_{]0,\,1[} u'\,v' + \int_{]0,\,1[} u\,v = \int_{]0,\,1[} f\,v, \qquad \text{for all } \ v \in H^1(]0,\,1[).$$

b) We show now that, by Lax-Milgram theorem, problem (50) has a unique solution. We choose as Hilbert space H the space $H^1(]0,\,1[)$, as bilinear form a, the form

$$a(w, v) = \int_{]0,1[} w'\,v' + \int_{]0,1[} w\,v$$

and as ϕ, element of H', the functional

$$\phi : H^1(]0,\,1[) \to \mathbb{R}, \qquad \phi(v) = \int_{]0,1[} \tilde{f}\,v.$$

The existence and uniqueness of the solution $w \in H^1(]0,\,1[)$ follows.

c) Let w be the solution of (50) with $\tilde{f} = f \in C([0,\,1])$. We have, in particular,

$$\int_{]0,\,1[} w'\,\varphi' = \int_{]0,\,1[} (f - w)\,\varphi, \qquad \text{for all } \ \varphi \in \mathcal{D}(]0,\,1[),$$

so that $(w')' = -(f - w) \in C([0, 1])$ and, consequently, $w' \in H^1(]0, 1[)$, $w \in H^2(]0, 1[)$, and finally (remember du Bois-Reymond theorem or the fundamental theorem) $w \in C^2(]0, 1[) \cap C^1([0, 1])$ with

$$-w'' + w = f.$$

We have, for all $\psi \in H^1(]0, 1[)$,

$$\begin{aligned} 0 &= \int_{]0,\,1[} (-w'' + w - f)\,\psi \\ &= \underbrace{\int_{]0,\,1[} w'\,\psi' + \int_{]0,\,1[} w\,\psi - \int_{]0,\,1[} f\,\psi}_{=0 \quad \text{from (50)}} - (w'(0)\psi(0) - w'(1)\psi(1)) \\ &= w'(1)\psi(1) - w'(0)\psi(0). \end{aligned}$$

choosing ψ in such a way that $\psi(1) = 1$ and $\psi(0) = 0$ we obtain that $w(0) = 0$ and, similarly, choosing ψ in such a way that $\psi(0) = 0$ and $\psi(1) = 1$, we deduce that $w(1) = 0$.

18.1.2 Maximum principle for the Dirichlet problem

Theorem 62. *Let $f \in L^2(]0, 1[)$. Let $u \in H^2(]0, 1[)$ be the solution to*

$$\begin{cases} -u'' + u = f & \text{in }]0, 1[, \\ u(0) = \alpha, \quad u(1) = \beta, & \alpha, \beta \in \mathbb{R}. \end{cases} \tag{51}$$

Then, for all $x \in [0, 1]$,

$$\min\{\alpha, \beta, \operatorname{inf\,ess} f\} \le u(x) \le \max\{\alpha, \beta, \operatorname{sup\,ess} f\}.$$

Proof. We have

$$\int_{]0,1[} u'\,v' + \int_{]0,1[} u\,v = \int_{]0,1[} f\,v, \qquad \text{for all } \ v \in H_0^1(]0, 1[). \tag{52}$$

Let $G \in C^1(\mathbb{R})$,

$$G(s) = \begin{cases} 0 & \text{if } s \le 0, \\ \text{strictly increasing} & \text{if } s > 0. \end{cases}$$

Let $K = \max\{\alpha, \beta, \operatorname{sup\,ess} f\}$ and suppose that $K < +\infty$.

We show that $u(x) \le K$ for all $x \in [0, 1]$. Consider

$$v(x) = G(u(x) - K).$$

$v \in H^1(]0, 1[)$ and

$$\begin{aligned} v(0) &= G(u(0) - K) = G(\alpha - K) = 0, \\ v(1) &= G(u(1) - K) = G(\beta - K) = 0. \end{aligned}$$

Consequently $v \in H^1_0(]0,1[)$ and we use it inside (52), taking into account that

$$v'(x) = G'(u(x) - K)u'(x).$$

We have

$$\int_{]0,1[} u'\, G'(u-K)u' + \int_{]0,1[} u\, G(u-K) = \int_{]0,1[} f\, G(u-K)$$

i. e.

$$\underbrace{\int_{]0,1[} (u')^2\, G'(u-K)}_{\geq 0} + \int_{]0,1[} (u-K)\, G(u-K) = \underbrace{\int_{]0,1[} (f-K)\, G(u-K)}_{\leq 0}.$$

We obtain

$$\int_{]0,1[} (u-K)\, G(u-K) \leq 0.$$

Remarking finally that the function $x \mapsto x\, G(x)$ is nonnegative, we have that

$$(u(x) - K)\, G(u(x) - K) = 0 \qquad \text{for all} \;\; x \in [0,1],$$

and hence $u(x) - K \leq 0$ for all $x \in [0,1]$. The computation to show that $u(x) \geq \min\{\alpha, \beta, \inf\operatorname{ess} f\}$ is similar. □

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19.1 Sobolev spaces in N space dimensions - 1

The content of this paragraph can be found in [3, Ch. IX].

19.1.1 Generalities

Definition 42. *Let Ω be an open set in $\mathbb{R}^N$. Let $p \in [1, +\infty]$. We define*

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) \mid \exists g_1, \ldots, g_N \in L^p(\Omega) :$$
$$\forall \phi \in C^1_0(\Omega),\; \forall j = 1, \ldots, N, \;\; \int_\Omega u\, \partial_j \phi = -\int_\Omega g_j\, \phi\}.$$

$W^{1,2}(\Omega)$ will be denoted by $H^1(\Omega)$.

Remark 44. *i) In the definition of $W^{1,p}(\Omega)$ it is sufficient that*

$$\int_\Omega u\, \partial_j \varphi = -\int_\Omega g_j\, \varphi, \qquad \text{for all} \;\; \varphi \in \mathcal{D}(\Omega).$$

ii) The functions $g_1, \ldots, g_N$ are unique and, for each j, g_j is the (function associated to the) j^{th} partial derivative of u in the sense of distributions. We set $g_j = \partial_j u$.

iii)

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) \mid \nabla u \in (L^p(\Omega))^N\},$$

where ∇u is the gradient of u in in the sense of distributions.

Definition 43. *We define*

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{(L^p(\Omega))^N} = \|u\|_{L^p(\Omega)} + \sum_{j=1}^{N} \|\partial_j u\|_{L^p(\Omega)}$$

and

$$\|u\|_{H^1(\Omega)} = (\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{(L^2(\Omega))^N})^{\frac{1}{2}} = (\|u\|^2_{L^2(\Omega)} + \sum_{j=1}^{N} \|\partial_j u\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

Theorem 63. *$W^{1,p}(\Omega)$ is a Banach space. If $p \in [1, +\infty[$, then $W^{1,p}(\Omega)$ is a separable Banach space. If $p \in]1, +\infty[$, then $W^{1,p}(\Omega)$ is a reflexive Banach space. $H^1(\Omega)$ is an Hilbert space.*

Proof. The proof is the same in the case $N = 1$. □

Remark 45. *Also in the case of $W^{1,p}(\Omega)$, it will be useful to remember that if a sequence $(u_n)_n$ is such that $u_n \xrightarrow{n} u$ in $L^p(\Omega)$ and, for all j, $\partial_j u_n \xrightarrow{n} v_j$ in $L^p(\Omega)$, then $u \in W^{1,p}(\Omega)$ and, for all j, $\partial_j u = v_j$.*

19.1.2 Properties of $W^{1,p}(\Omega)$

We give a first result of density.

Theorem 64 (Friedrichs)**.** *Let $p \in [1, +\infty[$. Let $u \in W^{1,p}(\Omega)$.*
Then there exists $(u_n)_n \in \mathcal{D}(\mathbb{R}^N)$ such that

i)
$$u_{n|\Omega} \xrightarrow{n} u \qquad in \ \ L^p(\Omega).$$

ii) for all ω, relatively compact open set in Ω,

$$\nabla u_{n|\omega} \xrightarrow{n} \nabla u_{|\omega} \qquad in \ \ (L^p(\omega))^N.$$

In the case of $\Omega = \mathbb{R}^N$, there exists $(u_n)_n \in \mathcal{D}(\mathbb{R}^N)$ such that

$$u_n \xrightarrow{n} u \qquad in \ \ W^{1,p}(\mathbb{R}^N).$$

Proof (sketch). First of all, we consider the function

$$\overline{u}(x) = \begin{cases} u(x) & \text{if } \ x \in \Omega, \\ 0 & \text{if } \ x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

We have $\overline{u} \in L^p(\mathbb{R}^N)$. With $(\rho_n)_n$ a family of mollifiers, we define

$$v_n = \rho_n * \overline{u}.$$

Finally we take $\chi \in \mathcal{D}(\mathbb{R}^N)$ such that

$$\chi(x) = \begin{cases} 1 & \text{if } \ |x| \leq 1, \\ 0 & \text{if } \ |x| \geq 2, \end{cases}$$

and, for all $n \geq 1$, $\chi_n(x) = \chi(\frac{1}{n}x)$.

We define

$$u_n = \chi_n(\rho_n * \overline{u}).$$

While the proving of $u_{n|\Omega} \stackrel{n}{\longrightarrow} u$ in $L^p(\mathbb{R}^N)$ is standard, a little more complicated it will be to show that $\nabla u_{n|\omega} \stackrel{n}{\longrightarrow} \nabla u_{|\omega}$ in $(L^p(\omega))^N$, for all open relatively compact set ω of Ω. For this it will be useful to remember that if $f \in L^1(\mathbb{R}^N)$ and $v \in W^{1,p}(\mathbb{R}^N)$, with $p \in [1, +\infty]$, then $f * v \in W^{1,p}(\mathbb{R}^N)$ and $\partial_j(f * v) = f * \partial_j v$. $\square$

Remark 46. *In general, if $N > 1$, a sequence $(w_n)_n \in \mathcal{D}(\mathbb{R}^N)$ such that $w_{n|\Omega} \stackrel{n}{\longrightarrow} u$ in $W^{1,p}(\Omega)$ does not exist, differently from what happens for $N = 1$. We will se that this depends on the regularity of $\partial\Omega$.*

Remark 47. *Let Ω be an open set in $\mathbb{R}^N$. Consider $W^{1,p}(\Omega)$, with $p \in [1, +\infty[$. Let*

$$H = \{u \in C^\infty(\Omega) \mid u \in L^p(\Omega) \text{ and } \nabla u \in (L^p(\Omega))^N\} = C^\infty(\Omega) \cap W^{1,p}(\Omega).$$

It is possible to prove (see [12] and [1, Th.3.16]) that H is dense in $W^{1,p}(\Omega)$, i. e. for all $u \in W^{1,p}(\Omega)$ there exists $(v_n)_n \in C^\infty(\Omega)$ such that

i) $v_n \stackrel{n}{\longrightarrow} u$ *in* $L^p(\Omega)$,

ii) $\nabla v_n \stackrel{n}{\longrightarrow} \nabla u$ *in* $(L^p(\Omega))^N$.

Concerning this theorem, R. A. Adams says "it is surprising that this elementary result remained undiscovered for so long" [1, p. 45].

We give a result on $W^{1,p}(\Omega)$, for $p > 1$.

Theorem 65 (Characterization of $W^{1,p}(\Omega)$, for $p > 1$)**.** *Let $p \in]1, +\infty]$. Suppose $u \in L^p(\Omega)$.*

Then the following conditions are equivalent.

i) $u \in W^{1,p}(\Omega)$.

ii) There exists $C > 0$ such that, for all $\varphi \in \mathcal{D}(\Omega)$ and for all $j = 1, \ldots, N$,

$$|\int_\Omega u\, \partial_j \varphi| \leq C\|\varphi\|_{L^{p'}}, \qquad \text{where } \frac{1}{p} + \frac{1}{p'} = 1.$$

iii) There exists $C > 0$ such that, for all ω, relatively compact open set in Ω, and for all $h \in \mathbb{R}^N$, such that $|h| \leq \mathrm{dist}(\omega, \partial\Omega)$,

$$\|\tau_h u - u\|_{L^p(\omega)} \leq C|h|, \qquad \text{where } \tau_h u(x) = u(x - h).$$

Proof. The proof is the same as in the case $N = 1$, apart from the implication *i)* $\Rightarrow$ *iii)*. Let's see the details. Suppose first that $u \in \mathcal{D}(\mathbb{R}^N)$, $h \in \mathbb{R}^N$, and define

$$v(t) = u(x - th), \qquad t \in \mathbb{R}.$$

We have $v'(t) = -h \cdot \nabla u(x - th)$ and hence

$$u(x) - u(x-h) = v(0) - v(1) = -\int_0^1 v'(t)\,dt = \int_0^1 h \cdot \nabla u(x - th)\,dt.$$

As a consequence, using Hölder inequality,

$$|\tau_h u(x) - u(x)|^p \leq |h|^p \int_0^1 |\nabla u(x - th)|^p\,dt,$$

and

$$\int_\omega |\tau_h u(x) - u(x)|^p\,dx \leq |h|^p \int_\omega \int_0^1 |\nabla u(x - th)|^p\,dt\,dx$$
$$\leq |h|^p \int_0^1 \int_\omega |\nabla u(x - th)|^p\,dx\,dt \leq |h|^p \int_0^1 \int_{\omega + th} |\nabla u(x)|^p\,dx\,dt.$$

Considering now the fact that $|h| \leq \operatorname{dist}(\omega, \partial\Omega)$, it will exists a relatively compact open set ω' in Ω such that $\omega + th \subseteq \omega'$ for all $t \in [0, 1]$, so that

$$\|\tau_h u - u\|^p_{L^p(\omega)} \leq |h|^p \int_{\omega'} |\nabla u|^p. \tag{53}$$

Let now $u \in W^{1,p}(\Omega)$, with $p \in [1, +\infty[$. By Friedrichs' theorem there exist a sequence $(u_n)_n$ in $\mathcal{D}(\mathbb{R}^N)$ such that $u_{n|\Omega} \xrightarrow{n} u$ in $L^p(\Omega)$ and $\nabla u_{n|\omega'} \xrightarrow{n} \nabla u_{|\omega'}$ in $(L^p(\omega'))^N$. But from (53), for all n,

$$\|\tau_h u_n - u_n\|_{L^p(\omega)} \leq |h| (\int_{\omega'} |\nabla u_n|^p)^{\frac{1}{p}},$$

so that we can pass to the limit with respect to n. It remains to consider that case $p = +\infty$. Given $u \in W^{1,\infty}(\Omega)$ we have

$$\|\tau_h u - u\|_{L^p(\omega)} \leq |h| (\int_{\omega'} |\nabla u|^p)^{\frac{1}{p}},$$

for all $p \in [1, +\infty[$ and we can pass to the limit with respect to p, obtaining

$$\|\tau_h u - u\|_{L^\infty(\omega)} \leq |h|\, \|\nabla u\|_{L^\infty(\omega')}. \tag{54}$$

□

Remark 48. *The difference in the proof of Theorem 65 above, with respect to the Theorem 55, is due to the fact that for $u \in W^{1,p}(\Omega)$ with $N \geq 2$ there is no a continuous representative. An example is given by the function $u(x, y) = (x^2 + y^2)^{-\frac{1}{4}}$ in $\Omega = \{(x, y) \in \mathbb{R}^2 \,\big|\, x^2 + y^2 < 1\}$. $u \in W^{1,1}(\Omega)$ but there is no continuous functions defined on Ω which coincides with u almost everywhere.*

Remark 49. *From (54) it is possible to deduce the following fact. Let $u \in W^{1,\infty}(\Omega)$. Then for all $x_0 \in \Omega$ there exists $r_{x_0} > 0$ such that, for almost all x, $y \in B(x_0, r_{x_0})$,*

$$|u(x) - u(y)| \leq \|\nabla u\|_{L^\infty(\Omega)}\, |x - y|.$$

Remark 50. *Let $u \in L^1(\Omega)$ such that there exists $C > 0$ such that, for all $\varphi \in \mathcal{D}(\Omega)$ and for all $j = 1, \dots, N$,*

$$|\int_I u\,\partial_j\varphi| \le C\|\varphi\|_{L^\infty}.$$

This is equivalent to the fact that there exists $C > 0$ such that, for all ω relatively compact open set in Ω and for all $h \in \mathbb{R}^N$ such that $|h| \le \operatorname{dist}(\omega, \partial\Omega)$,

$$\|\tau_h u - u\|_{L^1(\omega)} \le C|h|.$$

The functions satisfying these properties are called $BV(\Omega)$ functions (functions with bounded variation on Ω). We have

$$W^{1,1}(\Omega) \subsetneqq BV(\Omega).$$

We list finally three various properties on $W^{1,p}(\Omega)$. The proofs of the theorems can be found in [3, Ch. IX].

Theorem 66 (Derivative of a product)**.** *Let $p \in [1, +\infty]$. Let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Then $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and, for all $j = 1, \dots, N$,

$$\partial_j(uv) = \partial_j u\, v + u\,\partial_j v.$$

Theorem 67 (Derivative of a composition)**.** *Let $G \in C^1(\mathbb{R})$ with $G(0) = 0$ and $|G'(s)| \le M$, for some $M > 0$ and for all $s \in \mathbb{R}$. Let $p \in [1, +\infty]$ and $u \in W^{1,p}(\Omega)$.*

Then $G \circ u \in W^{1,p}(\Omega)$ and, for all $j = 1, \dots, N$,

$$\partial_j(G \circ u) = (G' \circ u)\,\partial_j u.$$

Theorem 68 (Change of variables formula)**.** *Let Ω and Ω' open sets in $\mathbb{R}^N$ and suppose there exists a bijective function $\Phi : \Omega' \to \Omega$ such that*

$$\Phi \in C^1(\Omega'), \qquad \Phi^{-1} \in C^1(\Omega), \qquad \operatorname{Jac}\Phi \in L^\infty(\Omega'), \qquad \operatorname{Jac}\Phi^{-1} \in L^\infty(\Omega),$$

where $\operatorname{Jac}\Phi$ and $\operatorname{Jac}\Phi^{-1}$ are the Jacobian matrices of Φ and Φ^{-1} respectively. Let $u \in W^{1,p}(\Omega)$.

Then $u \circ \Phi \in W^{1,p}(\Omega')$ and, for all $j = 1, \dots, N$,

$$\partial_{y_j}(u \circ \Phi)(y) = \sum_{h=1}^{N} \partial_{x_h} u(\Phi(y))\,\partial_{y_j}\Phi_h(y).$$

19.1.3 The space $W^{m,p}(\Omega)$

Definition 44. *Let Ω be an open set in $\mathbb{R}^N$. Let $p \in [1, +\infty]$ and $m \in \mathbb{N}\backslash\{0, 1\}$. We define*

$$W^{m,p}(\Omega) = \{u \in W^{m-1,p}(\Omega) \bigm| \nabla u \in (W^{m-1,p}(\Omega))^N\}$$

i. e.

$$W^{m,p}(\Omega) = \{u \in L^p(I) \bigm| D^\alpha u \in L^p(\Omega),\ \text{for } |\alpha| \le m\},$$

where derivatives are to be considered in distributional sense. We set

$$\|u\|_{W^{m,p}} = \sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^p}$$

or, equivalently, in the case of $p \in [1, +\infty[$,

$$\|u\|_{W^{m,p}} = (\sum_{|\alpha|\leq m} \|D^\alpha u\|^p_{L^p})^{\frac{1}{p}}.$$

Remark 51. *We set* $W^{m,2}(\Omega) = H^m(\Omega)$. *It is possible to show that*

$$H^m(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) \bigm| (1 + |\xi|^2)^{\frac{m}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^N)\}.$$

19.1.4 Extension operator for $u \in W^{1,p}(\Omega)$

As we will see, the possibility of extending a function of $W^{1,p}(\Omega)$ to the whole $\mathbb{R}^N$ will depend on the regularity of the border of Ω.

Definition 45. *Let*

$$x = (x_1, \dots, x_N), \qquad x' = (x_1, \dots, x_{N-1}), \qquad \text{so that} \qquad x = (x', x_N).$$

Let R, $r > 0$ *and*

$$B_{R,\,r}(x'_0,\, x_{N0}) = \{(x', x_N) \in \mathbb{R}^N \bigm| |x' - x'_0| < R, \quad |x_N - x_{N0}| < r\}.$$

$B_{R,r}$ *will denote* $B_{R,\,r}(0, 0)$. *Let* Ω *be an open set in* $\mathbb{R}^N$. Ω *is said to be of class* C^1 *if, for all* $x_0 \in \partial\Omega$, *there exists* U, *open neighborhood of* x_0 *and there exists* $\Phi : U \to B_{R,r}$ *such that*

i) Φ *is invertible and* $\Phi \in C^1(\overline{U})$, $\Phi^{-1} \in C^1(\overline{B}_{R,r})$;

ii) $\Phi(U \cap \partial\Omega) = \{x \in B_{R,r} \bigm| x_N = 0\}$ *and* $\Phi(U \cap \Omega) = \{x \in B_{R,r} \bigm| x_N < 0\}$.

Theorem 69. *Let* $p \in [1, +\infty]$. *Let* Ω *be an open set of class* C^1 *in* $\mathbb{R}^N$. *Suppose that* $\partial\Omega$ *is bounded (or* Ω *is an half-space).*

Then there exists an operator $P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ *such that:*

i) for all $u \in W^{1,p}(\Omega)$,

$$Pu_{|\Omega} = u;$$

ii) there exists $C_0 > 0$ *such that, for all* $u \in W^{1,p}(\Omega)$,

$$\|Pu\|_{L^p(\mathbb{R}^N)} \leq C_0 \|u\|_{L^p(\Omega)};$$

iii) there exists $C_1 > 0$ *such that, for all* $u \in W^{1,p}(\Omega)$,

$$\|Pu\|_{W^{1,p}(\mathbb{R}^N)} \leq C_1 \|u\|_{W^{1,p}(\Omega)}.$$

Proof (sketch). The first result needed in the proof is the following lemma, the proof of which can be found in [3, Ch.IX].

Lemma 21. *Let $p \in [1, +\infty]$. Let $u \in W^{1,p}(\Omega)$, where*

$$\Omega = \{x \in B_{R,r} \bigm| x_N < 0\}.$$

Let

$$v(x', x_N) = \begin{cases} u(x', x_N) & \text{if } (x', x_N) \in \Omega, \\ u(x', -x_N) & \text{if } (x', -x_N) \in \Omega, \end{cases}$$

i. e. v is defined with a reflection with respect to the hyperplane $\{x_N = 0\}$.
Then $v \in W^{1,p}(B_{R,r})$ and

$$\|v\|_{L^p(B_{R,r})} \leq 2\|u\|_{L^p(\Omega)}, \qquad \|v\|_{W^{1,p}(B_{R,r})} \leq 2\|u\|_{W^{1,p}(\Omega)}.$$

Secondly, a modification of the usual partition of unity result will be essential. The proof can be obtained suitably fitting out that one of Theorem 32.

Lemma 22. *Let K be a compact set in $\mathbb{R}^N$. Let $U_1, \dots, U_k$ be open sets in $\mathbb{R}^N$, with $K \subseteq \bigcup_{j=1}^k U_j$.*
Then there exist $\varphi_0 \in C^\infty(\mathbb{R}^N)$ with $\operatorname{Supp}\varphi_0 \subseteq \mathbb{R}^N \setminus K$ and, for all $j = 1, \dots, k$, $\varphi_j \in C_0^\infty(U_j)$ such that,

$$\sum_{j=0}^{k} \varphi_j(x) = 1, \qquad \text{for all} \quad x \in \mathbb{R}^N.$$

If Ω is a bounded open set and $K = \partial\Omega$, then $\varphi_{0|\Omega} \in C_0^\infty(\Omega)$.

Let us come to the sketch of the proof of the extension theorem. If Ω is an half-plane, a reflection (Lemma 21) will be sufficient to obtain the conclusion. Suppose then that Ω is bounded and of class C^1. Every point of the border of Ω will have an open neighborhood U satisfying the requests of Definition 45. It is possible to consider a finite sub-covering $U_1, \dots, U_k$. We use Lemma 22 and we construct a partition of unity. Take now the function $u \in W^{1,p}(\Omega)$ and consider

$$u = \sum_{j=0}^{k} \varphi_j u = \sum_{j=0}^{k} u_j.$$

We extend each of the u_j to $\mathbb{R}^N$. In particular u_0 will be extended considering

$$\bar{u}_0(x) = \begin{cases} u_0(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

To conclude let's see how to do with u_1. We consider the function u on $U_1 \cap \Omega$. Using the function Φ from Definition 45 and the result of Theorem 68, we obtain a $W^{1,p}$ function on the set $\{x \in B_{R,r} \bigm| x_N < 0\}$. Lemma 21 extends this function to the whole $B_{R,r}$ and Φ^{-1} gives a function on $W^{1,p}(U_1)$ which coincides with u on $U_1 \cap \Omega$. We call it v_1. We set

$$\bar{u}_1(x) = \begin{cases} \varphi_1(x) v_1(x) & \text{if } x \in U_1, \\ 0 & \text{if } x \notin U_1. \end{cases}$$

To conclude the proof it will be sufficient to verify that

$$Pu = \sum_{j=0}^{k} \bar{u}_j$$

satisfies all the requested properties. $\square$

Corollary 18. *Let $p \in [1, +\infty[$. Let Ω be an open bounded set of class C^1 in $\mathbb{R}^N$. Let $u \in W^{1,p}(\Omega)$.*

Then there exists a sequence $(u_n)_n$ in $\mathcal{D}(\mathbb{R}^N)$ such that

$$u_{n|\Omega} \xrightarrow{n} u \qquad \text{in } \; W^{1,p}(\Omega),$$

i. e. the restrictions to Ω of the functions of $\mathcal{D}(\mathbb{R}^N)$ are dense in $W^{1,p}(\Omega)$.

Proof (sketch). We extend the function u to Pu defined on the whole $\mathbb{R}^N$ using Theorem 69. Then the sequence (see Theorem 64)

$$u_n = \chi_n(\rho_n * Pu)$$

will give the wanted conclusion. $\square$

Remark 52. *In Corollary 18, the hypothesis of boundedness for the open set Ω can be removed.*

20

20.1 Sobolev spaces in N space dimensions - 2

The content of this paragraph can be found in [3, Ch. IX].

20.1.1 Sobolev embeddings

Remark 53. *From now on we will use the notation*

$$\|\nabla u\|_{L^p(\mathbb{R}^N)} \quad \text{instead of} \quad \|\nabla u\|_{(L^p(\mathbb{R}^N))^N}.$$

Theorem 70 (Sobolev-Gagliardo-Nirenberg)**.** *Let $p \in [1, N[$ and let p^* such that*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N} \qquad \text{i. e.} \qquad p^* = \frac{pN}{N-p} \qquad \text{(note that } p^* > p\text{).}$$

Then

$$W^{1,p}(\mathbb{R}^N) \subseteq L^{p^*}(\mathbb{R}^N)$$

and there exists $C > 0$ such that, for all $u \in W^{1,p}(\mathbb{R}^N)$,

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^N)}. \tag{55}$$

Remark 54 (important)**.** *Let $p \in [1, N[$. Suppose that there exist $q \in [1, +\infty[$ and $C > 0$ such that*

$$W^{1,p}(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^N) \tag{56}$$

and

$$\|u\|_{L^q(\mathbb{R}^N)} \le C\|\nabla u\|_{L^p(\mathbb{R}^N)}, \qquad \textit{for all } \ u \in W^{1,p}(\mathbb{R}^N). \tag{57}$$

Then necessarily

$$q = \frac{pN}{N-p}.$$

This result is due to the so called property of "scaling". In fact, suppose (56) and (57) are valid and consider $u \in W^{1,p}(\mathbb{R}^N)$. Then, taking $\lambda > 0$, the inequality (57) should be true also for the function $v(x) = u(\lambda x)$ with the same value of C, i. e.

$$\|u(\lambda\cdot)\|_{L^q(\mathbb{R}^N)} = \|v\|_{L^q(\mathbb{R}^N)} \le C\|\nabla v\|_{L^p(\mathbb{R}^N)} = C\|\nabla(u(\lambda\cdot))\|_{L^p(\mathbb{R}^N)}.$$

We have

$$\|u(\lambda\cdot)\|_{L^q(\mathbb{R}^N)} = \underbrace{(\int_{\mathbb{R}^N} |u(\lambda x)|^q \, dx)^{\frac{1}{q}} = (\int_{\mathbb{R}^N} |u(y)|^q \lambda^{-N} \, dy)^{\frac{1}{q}}}_{change\ of\ variables\ \lambda x = y} = \lambda^{-\frac{N}{q}} \|u\|_{L^q(\mathbb{R}^N)}.$$

On the other hand

$$\partial_j v(x) = \partial_j(u(\lambda x)) = \lambda \partial_j u(x),$$

so that

$$\nabla(u(\lambda\cdot)) = \lambda(\nabla u)(\lambda\cdot)$$

and

$$\begin{aligned} \|\nabla(u(\lambda\cdot))\|_{L^p(\mathbb{R}^N)} &= (\int_{\mathbb{R}^N} |\lambda(\nabla u)(\lambda x)|^p \, dx)^{\frac{1}{p}} \\ &= (\int_{\mathbb{R}^N} |\lambda(\nabla u)(y)|^p \, \lambda^{-N} \, dy)^{\frac{1}{p}} = \lambda^{1-\frac{N}{p}} \|\nabla u\|_{L^p(\mathbb{R}^N)}. \end{aligned}$$

Resuming, if (56) and (57) hold, then

$$\lambda^{-\frac{N}{q}} \|u\|_{L^q(\mathbb{R}^N)} \le C\lambda^{1-\frac{N}{p}} \|\nabla u\|_{L^p(\mathbb{R}^N)}, \quad \textit{for all } u \in W^{1,p}(\mathbb{R}^N) \textit{ and for all } \lambda > 0.$$

Choose now $u \in W^{1,p}(\mathbb{R}^N) \subseteq L^q(\mathbb{R}^N)$ such that $\|u\|_{L^q(\mathbb{R}^N)} \ne 0$ (for this it would be sufficient to take $u \in \mathcal{D}(\mathbb{R}^N)$ with $u \ne 0$). We deduce

$$\lambda^{\frac{N}{p} - \frac{N}{q} - 1} \le C \frac{\|\nabla u\|_{L^p(\mathbb{R}^N)}}{\|u\|_{L^q(\mathbb{R}^N)}}, \qquad \textit{for all } \lambda > 0. \tag{58}$$

If $\frac{N}{p} - \frac{N}{q} - 1 > 0$, then, letting $\lambda \to +\infty$ in (58), we obtain a contradiction. Similarly, if $\frac{N}{p} - \frac{N}{q} - 1 < 0$, we obtain a contradiction letting $\lambda \to 0^+$. As a consequence, necessarily,

$$\frac{N}{p} - \frac{N}{q} - 1 = 0, \qquad i.\ e. \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{N}.$$

Lemma 23. *Let $N \ge 2$ and let $f_1, \dots, f_N \in L^{N-1}(\mathbb{R}^{N-1})$. Denote*

$$\tilde{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N) \in \mathbb{R}^{N-1}$$

and define

$$f(x) = f(x_1, \dots, x_N) = f_1(\tilde{x}_1)f_2(\tilde{x}_2)\dots f_N(\tilde{x}_N).$$

Then

$$f \in L^1(\mathbb{R}^N) \qquad \textit{and} \qquad \|f\|_{L^1(\mathbb{R}^N)} \leq \prod_{i=1}^{N} \|f_i\|_{L^{N-1}(\mathbb{R}^{N-1})}.$$

Proof. The case of $N = 2$ is the simplest. In fact, by hypothesis, we have $f_1,\, f_2 \in L^1(\mathbb{R})$ and $f(x_1,\, x_2) = f_1(x_2)f_2(x_1)$. Consequently $f \in L^1(\mathbb{R}^2)$ and

$$\begin{aligned} \|f\|_{L^1(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |f_1(x_2)f(x_1)|\, dx_1 dx_2 \\ &= \int_{\mathbb{R}} |f_1(x_2)|\, dx_2 \int_{\mathbb{R}} |f_2(x_1)|\, dx_1 = \|f_1\|_{L^1(\mathbb{R})}\|f_2\|_{L^1(\mathbb{R})}. \end{aligned}$$

Consider $N = 3$. Then $f_1,\, f_2,\, f_3 \in L^2(\mathbb{R}^2)$ and

$$f(x_1, x_2, x_3) = f_1(x_2, x_3)f_2(x_1, x_3)f_3(x_1, x_2).$$

The function

$$(x_2, x_3) \mapsto f_1^2(x_2, x_3)$$

is in $L^1(\mathbb{R}^2)$, so that, for almost every $x_3 \in \mathbb{R}$ and $x_1 \in \mathbb{R}$ respectively, the functions

$$x_2 \mapsto f_1^2(x_2, x_3) \qquad \text{and} \qquad x_2 \mapsto f_3^2(x_1, x_2)$$

are in $L^1(\mathbb{R})$. Hence for almost every $x_3 \in \mathbb{R}$ and $x_1 \in \mathbb{R}$ respectively, the functions

$$x_2 \mapsto f_1(x_2, x_3) \qquad \text{and} \qquad x_2 \mapsto f_3(x_1, x_2)$$

are in in $L^2(\mathbb{R})$. Consequently, for almost every $(x_1, x_3) \in \mathbb{R}^2$, the function

$$x_2 \mapsto f_1(x_2, x_3)f_3(x_1, x_2)$$

is in $L^1(\mathbb{R})$ and, from Cauchy-Schwarz,

$$\int_{\mathbb{R}} |f_1(x_2, x_3)f_3(x_1, x_2)|\, dx_2 \leq (\int_{\mathbb{R}} |f_1(x_2, x_3)|^2\, dx_2)^{\frac{1}{2}} (\int_{\mathbb{R}} |f_3(x_1, x_2)|^2\, dx_2)^{\frac{1}{2}}. \tag{59}$$

Now the functions

$$x_3 \mapsto \int_{\mathbb{R}} |f_1(x_2, x_3)|^2\, dx_2 \quad \text{and} \quad x_1 \mapsto \int_{\mathbb{R}} |f_3(x_1, x_2)|^2\, dx_2$$

are both $L^1(\mathbb{R})$ and then

$$(x_1, x_3) \mapsto (\int_{\mathbb{R}} |f_1(x_2, x_3)|^2\, dx_2)^{\frac{1}{2}} (\int_{\mathbb{R}} |f_3(x_1, x_2)|^2\, dx_2)^{\frac{1}{2}}$$

is in $L^2(\mathbb{R}^2)$. Hence, from (59), we have that the function

$$(x_1, x_3) \mapsto \int_{\mathbb{R}} |f_1(x_2, x_3)f_3(x_1, x_2)|\, dx_2$$

is in $L^2(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2} |\int_{\mathbb{R}} |f_1(x_2,x_3)f_3(x_1,x_2)|\,dx_2|^2\,dx_1dx_3$$

$$\leq \int_{\mathbb{R}^2} (\int_{\mathbb{R}} |f_1(x_2,x_3)|^2\,dx_2 \int_{\mathbb{R}} |f_3(x_1,x_2)|^2\,dx_2)\,dx_1dx_3$$

$$= \int_{\mathbb{R}} (\int_{\mathbb{R}} |f_1(x_2,x_3)|^2\,dx_2)\,dx_3 \cdot \int_{\mathbb{R}} (\int_{\mathbb{R}} |f_3(x_1,x_2)|^2\,dx_2)\,dx_1$$

$$= \|f_1\|^2_{L^2(\mathbb{R}^2))}\,\|f_1\|^2_{L^2(\mathbb{R}^2))}.$$

In conclusion the function

$$(x_1,x_3) \mapsto \int_{\mathbb{R}} |f_1(x_2,x_3)f_3(x_1,x_2)|\,dx_2$$

is in $L^2(\mathbb{R}^2)$ with norm $\leq \|f_1\|_{L^2(\mathbb{R}^2)}\,\|f_3\|_{L^2(\mathbb{R}^2)}$. We obtain that the function

$$x_1,x_3 \mapsto f_2(x_1,x_3) \cdot \int_{\mathbb{R}} |f_1(x_2,x_3)f_3(x_1,x_2)|\,dx_2$$

is in $L^1(\mathbb{R}^2)$ and, again from Cauchy-Schwarz, we deduce

$$\int_{\mathbb{R}^3} |f_2(x_1,x_3)f_1(x_2,x_3)f_3(x_1,x_2)|\,dx_1dx_2dx_3$$

$$= \int_{\mathbb{R}^2} |f_2(x_1,x_3)|(\int_{\mathbb{R}} |f_1(x_2,x_3)f_3(x_1,x_2)|\,dx_2)\,dx_1dx_3$$

$$\leq \|f_2\|_{L^2(\mathbb{R}^2)}\|f_1\|_{L^2(\mathbb{R}^2)}\,\|f_3\|_{L^2(\mathbb{R}^2)}.$$

The case $N \geq 4$ is obtained with an intricate procedure of recursion on N. The details can be found in [3, Ch. IX, Lemma IX.4]. □

Proof of Theorem 70. Let $u \in C^1_0(\mathbb{R}^N)$. We have, for all $j = 1, 2, \ldots, N$,

$$|u(x)| = |\int_{-\infty}^{x_j} \partial_j u(x_1,\ldots,x_{j-1},t,x_{j+1},\ldots,x_N)\,dt| \leq \int_{-\infty}^{+\infty} |\partial_j u(x_1,\ldots,t,\ldots,x_N)|\,dt.$$

We set, for all $j = 1, 2, \ldots, N$,

$$g_j(\tilde{x}_j) = \int_{-\infty}^{+\infty} |\partial_j u(x_1,\ldots,t,\ldots,x_N)|\,dt$$

and consequently

$$|u(x)|^N \leq \prod_{j=1}^{N} g_j(\tilde{x}_j).$$

Defining

$$f_j(\tilde{x}_j) = (g_j(\tilde{x}_j))^{\frac{1}{N-1}},$$

we have $f_j \in L^{N-1}(\mathbb{R}^{N-1})$ with

$$\|f_j\|_{L^{N-1}(\mathbb{R}^{N-1})} = (\int_{\mathbb{R}^{N-1}} |g_j(\tilde{x}_j)|\, d\tilde{x}_j)^{\frac{1}{N-1}} = \|\partial_j u\|_{L^1(\mathbb{R}^N)}^{\frac{1}{N-1}} \leq \|\nabla u\|_{L^1(\mathbb{R}^N)}^{\frac{1}{N-1}}$$

and

$$|u(x)|^{\frac{N}{N-1}} \leq \prod_{j=1}^{N} f_j(\tilde{x}_j).$$

We apply Lemma 23 and we obtain

$$\|u\|_{L^{\frac{N}{N-1}}(\mathbb{R}^N)}^{\frac{N}{N-1}} = \||u|^{\frac{N}{N-1}}\|_{L^1(\mathbb{R}^N)} \leq \prod_{j=1}^{N} \|f_j\|_{L^{N-1}(\mathbb{R}^{N-1})} \leq \prod_{j=1}^{N} \|\nabla u\|_{L^1(\mathbb{R}^N)}^{\frac{1}{N-1}}.$$

In conclusion, we have proved that if $u \in C_0^1(\mathbb{R}^N)$ then

$$\|u\|_{L^{\frac{N}{N-1}}(\mathbb{R}^N)} \leq \|\nabla u\|_{L^1(\mathbb{R}^N)}. \tag{60}$$

Remark that (60) is actually (55) in the case $p = 1$. Suppose $p > 1$. Now, for $t \geq 1$, we apply (60) to the function $v(x) = |u(x)|^{t-1}u(x)$. Remarking that

$$\partial_j v(x) = t|u(x)|^{t-1}\partial_j u(x)$$

we obtain

$$\|u\|_{L^{\frac{tN}{N-1}}(\mathbb{R}^N)}^t = \|v\|_{L^{\frac{N}{N-1}}(\mathbb{R}^N)} \leq \|\nabla v\|_{L^1(\mathbb{R}^N)} = t\||u|^{t-1}\nabla u\|_{L^1(\mathbb{R}^N)}.$$

Since $u \in C_0^1(\mathbb{R}^n)$, we infer

$$|u|^{t-1} \in L^{p'}(\mathbb{R}^N) \qquad \text{and} \qquad \nabla u \in L^p(\mathbb{R}^N) \quad \text{with} \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Using Hölder inequality we have

$$\||u|^{t-1}\nabla u\|_{L^1(\mathbb{R}^N)} \leq \||u|^{t-1}\|_{L^{p'}(\mathbb{R}^N)} \|\nabla u\|_{L^p(\mathbb{R}^N)} = \|u\|_{L^{p'(t-1)}(\mathbb{R}^N)}^{t-1} \|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

Putting all together we finally obtain

$$\|u\|_{L^{\frac{tN}{N-1}}(\mathbb{R}^N)}^t \leq t\|u\|_{L^{p'(t-1)}(\mathbb{R}^N)}^{t-1} \|\nabla u\|_{L^p(\mathbb{R}^N)}. \tag{61}$$

The trick is to choose $t \geq 1$ is such a way that

$$\frac{tN}{N-1} = p'(t-1) \qquad \text{i. e.} \qquad t = \frac{p}{N-p}(N-1). \tag{62}$$

With this choice, (61) becomes

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq \underbrace{\frac{p(N-1)}{N-p}}_{=C_{N,p}} \|\nabla u\|_{L^p(\mathbb{R}^N)} \qquad \text{with} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}.$$

Suppose now $u \in W^{1,p}(\mathbb{R}^N)$. From Friedrichs theorem we know that there exists a sequence $(u_n)_n$ in $\mathcal{D}(\mathbb{R}^N)$ such that $u_n \xrightarrow{n} u$ in $W^{1,p}(\mathbb{R}^N)$. It is not restrictive to suppose that $u_n \xrightarrow{n} u$ a. e.. This sequence is a Cauchy sequence in $W^{1,p}(\mathbb{R}^N)$ so that, from (55), it is a Cauchy sequence in $L^{p^*}(\mathbb{R}^N)$ and it converges a. e. to u. Consequently $u \in L^{p^*}(\mathbb{R}^N)$ and we can pass to the limit in (55). The proof is complete.

□

Corollary 19. *Let Ω be an open set of class C^1 in $\mathbb{R}^N$. Suppose that $\partial\Omega$ is bounded (or Ω is an half-space). Let p and let p^* as in Theorem 70.*

Then

$$W^{1,p}(\Omega) \subseteq L^{p^*}(\Omega)$$

and there exists $C > 0$ such that, for all $u \in W^{1,p}(\Omega)$,

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

Proof. Use the extension theorem (Theorem 69) and Theorem 70. In particular

$$\|u\|_{L^{p^*}(\Omega)} \leq \|Pu\|_{L^{p^*}(\mathbb{R}^N)} \leq C\|Pu\|_{W^{1,p}(\mathbb{R}^N)} \leq C'\|u\|_{W^{1,p}(\Omega)}.$$

□

Corollary 20 (case $p = N$)**.** *For all $q \in [N, +\infty[$,*

$$W^{1,N}(\mathbb{R}^N) \subseteq L^q(\mathbb{R}^N)$$

and there exists $C_q > 0$ such that, for all $u \in W^{1,N}(\mathbb{R}^N)$,

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C_q\|u\|_{W^{1,N}(\mathbb{R}^N)}.$$

Proof. Let $u \in C_0^1(\mathbb{R}^n)$. We know that (remember that, in the proof of Theorem 70, up to the point (61), we did not use the fact that $p < N$; this condition was only used in determining the correct t in (62))

$$\|u\|^t_{L^{\frac{tN}{N-1}}(\mathbb{R}^N)} \leq t\|u\|^{t-1}_{L^{p'(t-1)}(\mathbb{R}^N)} \|\nabla u\|_{L^p(\mathbb{R}^N)},$$

and this is valid also in the case $p = N$, so that

$$\|u\|^t_{L^{\frac{tN}{N-1}}(\mathbb{R}^N)} \leq t\|u\|^{t-1}_{L^{(t-1)\frac{N}{N-1}}(\mathbb{R}^N)} \|\nabla u\|_{L^N(\mathbb{R}^N)},$$

for all $t \geq 1$. We deduce

$$\|u\|_{L^{\frac{tN}{N-1}}(\mathbb{R}^N)} \leq C_t\|u\|^{1-\frac{1}{t}}_{L^{(t-1)\frac{N}{N-1}}(\mathbb{R}^N)} \|\nabla u\|^{\frac{1}{t}}_{L^N(\mathbb{R}^N)},$$

and finally, from Young inequality,

$$\|u\|_{L^{\frac{tN}{N-1}}(\mathbb{R}^N)} \leq C_t(\|u\|_{L^{(t-1)\frac{N}{N-1}}(\mathbb{R}^N)} + \|\nabla u\|_{L^N(\mathbb{R}^N)}). \tag{63}$$

Choosing $t = N$ in (63) we have

$$\|u\|_{L^{\frac{N^2}{N-1}}(\mathbb{R}^N)} \leq C_N(\|u\|_{L^N(\mathbb{R}^N)} + \|\nabla u\|_{L^N(\mathbb{R}^N)})$$

and, by interpolation, we infer that

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C_{N,q}\|u\|_{W^{1,N}(\mathbb{R}^N)} \tag{64}$$

for all $q \in [N, \frac{N^2}{N-1}] = [N, N + \frac{N}{N-1}]$. Choosing $t = N + 1$ in (63) we have

$$\|u\|_{L^{\frac{N^2+N}{N-1}}(\mathbb{R}^N)} \leq C_{N+1}(\|u\|_{L^{\frac{N^2}{N-1}}(\mathbb{R}^N)} + \|\nabla u\|_{L^N(\mathbb{R}^N)})$$

and, from (64),

$$\|u\|_{L^{\frac{N^2+N}{N-1}}(\mathbb{R}^N)} \le \tilde{C}_{N+1}(\|u\|_{L^N(\mathbb{R}^N)} + \|\nabla u\|_{L^N(\mathbb{R}^N)}).$$

Finally

$$\|u\|_{L^q(\mathbb{R}^N)} \le C_{N,q}\|u\|_{W^{1,N}(\mathbb{R}^N)}$$

for all $q \in [N + \frac{N}{N-1}, N + \frac{2N}{N-1}]$. Iterating this procedure we obtain that

$$\|u\|_{L^q(\mathbb{R}^N)} \le C_{N,q}\|u\|_{W^{1,N}(\mathbb{R}^N)}$$

for all $u \in C_0^1(\mathbb{R}^N)$ and for all $q \in [N, +\infty[$. An approximation procedure like in Theorem 70 gives the conclusion. □

Corollary 21. *Let Ω be an open set of class C^1 in $\mathbb{R}^N$. Suppose that $\partial\Omega$ is bounded (or Ω is an half-space). For all $q \in [N, +\infty[$,*

$$W^{1,N}(\Omega) \subseteq L^q(\Omega)$$

and there exists $C_q > 0$ such that, for all $u \in W^{1,N}(\Omega)$,

$$\|u\|_{L^q(\Omega)} \le C_q\|u\|_{W^{1,N}(\Omega)}.$$

Proof. Use the extension theorem (69) and Corollary 20. □

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21.1 Sobolev spaces in N space dimensions - 3

The content of this paragraph can be found in [3, Ch. IX].

21.1.1 Morrey theorem

Theorem 71 (Morrey). *Let $p \in]N, +\infty]$.*

Then

$$W^{1,p}(\mathbb{R}^N) \subseteq L^\infty(\mathbb{R}^N)$$

and there exists $C > 0$ such that, for all $u \in W^{1,p}(\mathbb{R}^N)$,

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le C\|u\|_{W^{1,p}(\mathbb{R}^N)}. \tag{65}$$

Moreover there exists $C' > 0$ such that, for all $u \in W^{1,p}(\mathbb{R}^N)$,

$$|u(x) - u(y)| \le C' \,\|\nabla u\|_{L^p(\mathbb{R}^N)}\, |x - y|^\alpha \qquad \text{for almost every } \; x, y \in \mathbb{R}^N, \tag{66}$$

where $\alpha = 1 - \frac{N}{p}$.

Proof. In the case $p = +\infty$, (65) is immediate and (66) has been proved in Remark 49. Let $p \in]N, +\infty[$. Let $u \in C_0^1(\mathbb{R}^N)$. Let Q be a cube in $\mathbb{R}^N$, containing 0 and such that the sides, of length $r > 0$, are parallel to the coordinate axes. Let $x \in Q$. We have

$$u(x) - u(0) = \int_0^1 v'(t)\, dt, \qquad \text{where} \;\; v(t) = u(tx).$$

Consequently

$$|u(x)-u(0)| \le \int_0^1 |v'(t)|\,dt \le \int_0^1 \sum_{j=1}^N |x_j|\,|\partial_j u(tx)|\,dt \le r \sum_{j=1}^N \int_0^1 |\partial_j u(tx)|\,dt. \tag{67}$$

Defining

$$\bar{u} = \frac{1}{|Q|}\int_Q u(x)\,dx = \frac{1}{r^N}\int_Q u(x)\,dx,$$

we have

$$\begin{aligned} |\bar{u}-u(0)| &= |\frac{1}{r^N}\int_Q u(x)\,dx - u(0)| \\ &= |\frac{1}{r^N}\int_Q (u(x)-u(0))\,dx| \le \frac{1}{r^N}\int_Q |u(x)-u(0)|\,dx, \end{aligned}$$

so that, from (67),

$$\begin{aligned} |\bar{u}-u(0)| &\le \frac{1}{r^{N-1}}\sum_{j=1}^N \int_Q (\int_0^1 |\partial_j u(tx)|\,dt)\,dx \\ &\le \frac{1}{r^{N-1}}\int_0^1 (\sum_{j=1}^N \int_Q |\partial_j u(tx)|\,dx)\,dt \\ &\le \frac{1}{r^{N-1}}\int_0^1 t^{-N}\,(\sum_{j=1}^N \int_{tQ} |\partial_j u(y)|\,dy)\,dt. \end{aligned}$$

Considering the fact that, for all $t \in [0,1]$, $tQ \subseteq Q$, from Hölder inequality we have

$$\int_{tQ} |\partial_j u(y)|\,dy \le |(t\,r)^N|^{\frac{1}{p'}} (\int_{tQ} |\partial_j u(y)|^p\,dy)^{\frac{1}{p}} \le r^{\frac{N}{p'}}\,t^{\frac{N}{p'}} (\int_Q |\partial_j u(y)|^p\,dy)^{\frac{1}{p}}.$$

Putting together we obtain

$$\begin{aligned} |\bar{u}-u(0)| &\le \frac{1}{r^{N-1}}\|\nabla u\|_{L^p(Q)}\,r^{\frac{N}{p'}} \int_0^1 t^{-N}\,t^{\frac{N}{p'}}\,dt \\ &\le r^{1+\frac{N}{p'}-N}\|\nabla u\|_{L^p(Q)} \int_0^1 t^{\frac{N}{p'}-N}\,dt = \frac{r^{1-\frac{N}{p}}}{1-\frac{N}{p}}\|\nabla u\|_{L^p(Q)}. \end{aligned}$$

This last inequality remains valid, by translation, for all cube Q with length side r, so that we have, for all for all cube Q with sides of length r,

$$|\bar{u}-u(x)| \le \frac{r^{1-\frac{N}{p}}}{1-\frac{N}{p}}\|\nabla u\|_{L^p(Q)}, \qquad \text{for all } \; x \in Q. \tag{68}$$

Consequently, for all x, $y \in Q$,

$$|u(x)-u(y)| \le |\bar{u}-u(x)| + |\bar{u}-u(y)| \le 2\frac{r^{1-\frac{N}{p}}}{1-\frac{N}{p}}\|\nabla u\|_{L^p(Q)}.$$

Since for every couple of points x, $y \in \mathbb{R}^N$ we can construct a cube of length side $r = 2|x-y|$ (with the sides parallel to the coordinate axes) containing x and y, we have, for all x, $y \in \mathbb{R}^N$,

$$|u(x) - u(y)| \leq 2\,\frac{(2|x-y|)^{1-\frac{N}{p}}}{1-\frac{N}{p}}\|\nabla u\|_{L^p(Q)} \leq C'|x-y|^{1-\frac{N}{p}}\|\nabla u\|_{L^p(\mathbb{R}^N)}$$

and (66) is proved for $u \in C^1_0(\mathbb{R}^N)$. To obtain (66) for $u \in W^{1,p}(\mathbb{R}^N)$ we use Friedrichs theorem as done in the proof of Sobolev-Gagliardo-Nirenberg theorem.

It remains to prove (65) in the case $p \in]N, +\infty[$. Let $u \in C^1_0(\mathbb{R}^N)$, let Q be a cube of side of length 1 and let $x \in Q$. From (68) we have

$$|u(x)| \leq |\bar{u}| + |\bar{u} - u(x)| \leq |\bar{u}| + C\|\nabla u\|_{L^p(Q)} \leq C'\|u\|_{W^{1,p}(Q)} \leq C'\|u\|_{W^{1,p}(\mathbb{R}^N)},$$

where C and C' depend only on p and N. Hence there exists $C > 0$ such that

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^N)}, \qquad \text{for all } \ u \in C^1_0(\mathbb{R}^N).$$

This last inequality can be proved for all $u \in W^{1,p}(\mathbb{R}^N)$ with the usual application of Friedrichs theorem. □

Remark 55. *The condition (66) says that, if $p > N$, $u \in W^{1,p}(\mathbb{R}^N)$ has an Hölder-continuous representative, i. e. we will write*

$$W^{1,p}(\mathbb{R}^N) \subseteq C^{0,\alpha}(\mathbb{R}^N),$$

where $\alpha = 1 - \frac{N}{p}$.

Corollary 22. *Let $p \in]N, +\infty[$. Let $u \in W^{1,p}(\mathbb{R}^N)$.*
Then

$$\lim_{|x| \to +\infty} u(x) = 0.$$

Proof. Exercise. □

Corollary 23. *Let Ω be an open set of class C^1 in $\mathbb{R}^N$. Suppose that $\partial\Omega$ is bounded (or Ω is an half-space). Let $p \in]N, +\infty]$.*
Then

$$W^{1,p}(\Omega) \subseteq L^\infty(\Omega)$$

and there exists $C > 0$ such that, for all $u \in W^{1,p}(\Omega)$,

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

Proof. Exercise. □

21.1.2 Rellich theorem

Theorem 72 (Rellich)**.** *Let Ω be an open bounded set of class C^1 in $\mathbb{R}^N$.*

i) If $p \in [1, N[$, then $W^{1,p}(\Omega) \subseteq L^q(\Omega)$, for all $q \in [1, p^*[$, with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.

ii) If $p = N$, then $W^{1,p}(\Omega) \subseteq L^q(\Omega)$, for all $q \in [1, +\infty[$.

iii) If $p \in]N, +\infty]$, then $W^{1,p}(\Omega) \subseteq C(\overline{\Omega})$.

All the above embeddings are continuous and compact.

Proof (sketch). Sobolev-Gagliardo-Nirenberg theorem, Morrey theorem and corollaries, give the continuity of the embedding

$$W^{1,p}(\Omega) \to L^q(\Omega)$$

in the the cases $p < N$ and $q = p^*$, $p = N$ and $q \in [p, +\infty[$ and $p > N$, $q = +\infty$ respectively. Since Ω is bounded, remarking that, for all $1 \leq r < q$

$$L^q(\Omega) \subseteq L^r(\Omega) \qquad \text{and} \qquad \|u\|_{L^r(\Omega)} \leq |\Omega|^{\frac{1}{r}-\frac{1}{q}} \|u\|_{L^q(\Omega)} \qquad \text{for all } \; u \in L^q(\Omega),$$

we have the continuity in all the cases quoted in points *i)*, *ii)* and *iii)*. Consequently only the compactness is of interest (remark also that, in any case, every compact operator is continuous).

The point *iii)* is the same as in the case of $N = 1$ and it is a consequence of (65), (66) and Ascoli-Arzelà theorem.

The point *ii)* is a consequence of the point *i)*, since, for Ω open bounded, a bounded set in $W^{1,p}(\Omega)$ is bounded also in $W^{1,r}(\Omega)$, for all $r \in [1, p[$.

The point *i)* will be proved using Riesz-Fréchet-Kolmogorov theorem (see [3, Cor. IV.26]). We recall that a set B is relatively compact in $L^q(\Omega)$ if the following two conditions hold.

a) For all $\varepsilon > 0$ and for all relatively compact set ω in Ω, there exists $\delta < \operatorname{dist}(\omega, \partial\Omega)$ such that,

$$\|\tau_h u - u\|_{L^q(\omega)} \leq \varepsilon, \quad \text{for all } h \in \mathbb{R}, \text{ with } |h| < \delta, \text{ and for all } u \in B.$$

b) For all $\varepsilon > 0$ there exists a relatively compact set ω in Ω such that

$$\|u\|_{L^q(\Omega \setminus \omega)} < \varepsilon, \quad \text{and for all } u \in B,$$

Let $q \in [1, p^*[$ and $\alpha \in]0, 1]$ such that

$$\frac{1}{q} = \frac{\alpha}{1} + \frac{1-\alpha}{p^*}.$$

Let ω be an open relatively compact set in Ω and $|h| < \operatorname{dist}(\omega, \partial\Omega)$. From interpolation inequality we have

$$\|\tau_h u - u\|_{L^q(\omega)} \leq \|\tau_h u - u\|_{L^1(\omega)}^{\alpha} \, \|\tau_h u - u\|_{L^{p^*}(\omega)}^{1-\alpha}.$$

From point *iii)* of Theorem 65,

$$\|\tau_h u - u\|_{L^1(\omega)} \leq |h| \, \|\nabla u\|_{L^1(\Omega)},$$

so that

$$\|\tau_h u - u\|_{L^q(\omega)} \leq (|h| \, \|\nabla u\|_{L^1(\Omega)})^{\alpha} \, (2\|u\|_{L^{p^*}(\omega)})^{1-\alpha} \leq C \, |h|^{\alpha}$$

and finally $\|\tau_h u - u\|_{L^q(\omega)} < \varepsilon$ for $|h|$ sufficiently small, for all u in a bounded set of $W^{1,p}(\Omega)$. Similarly, it is possible to choose ω open relatively compact set in Ω such that

$$\underbrace{\|u\|_{L^q(\Omega\setminus\omega)} \leq \|u\|_{L^{p^*}(\Omega\setminus\omega)} |\Omega \setminus \omega|^{1-\frac{q}{p^*}}}_{\text{Hölder}} \leq C |\Omega \setminus \omega|^{1-\frac{q}{p^*}} < \varepsilon.$$

□

21.1.3 Sobolev embeddings for $W^{m,p}$ spaces

Example 18. *Let $u \in W^{2,p}(\mathbb{R}^N)$, with $p \in [1, N[$. We know that*

$$u \in W^{1,p}(\mathbb{R}^N) \qquad \textit{and} \qquad \nabla u \in W^{1,p}(\mathbb{R}^N),$$

so that, by Sobolev embedding,

$$u \in L^{p^*}(\mathbb{R}^N) \qquad \textit{and} \qquad \nabla u \in L^{p^*}(\mathbb{R}^N), \qquad \textit{with} \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}.$$

In conclusion, if $p \in [1, N[$,

$$W^{2,p}(\mathbb{R}^N) \subseteq W^{1,p^*}(\mathbb{R}^N), \qquad \textit{with} \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}.$$

Example 19. *Let $u \in W^{2,p}(\mathbb{R}^N)$, with $p \in]\frac{N}{2}, N[$. We know that*

$$u \in W^{1,p^*}(\mathbb{R}^N), \qquad \textit{with} \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}.$$

We have

$$\frac{N}{2} < p < N \quad \textit{i. e.} \quad \frac{1}{N} < \frac{1}{p} < \frac{2}{N} \quad \textit{so that} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N} < \frac{1}{N} \quad \textit{i. e.} \quad p^* > N.$$

In conclusion, if $p \in]\frac{N}{2}, N[$,

$$W^{2,p}(\mathbb{R}^N) \subseteq L^{\infty}(\mathbb{R}^N).$$

Example 20. *Let $u \in W^{2,\frac{N}{2}}(\mathbb{R}^N)$. We know that*

$$u \in W^{1,p^*}(\mathbb{R}^N), \qquad \textit{with} \qquad \frac{1}{p^*} = \frac{2}{N} - \frac{1}{N} = \frac{1}{N},$$

so that $u \in W^{1,N}(\mathbb{R}^N)$. In conclusion

$$W^{2,\frac{N}{2}}(\mathbb{R}^N) \subseteq L^q(\mathbb{R}^N) \qquad \textit{for all} \;\; q \in [\frac{N}{2}, +\infty[.$$

Example 21. *Let $u \in W^{2,p}(\mathbb{R}^N)$, with $p \in [1, \frac{N}{2}[$. We know that*

$$u \in W^{1,p^*}(\mathbb{R}^N), \qquad \textit{with} \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}.$$

We have

$$p < \frac{N}{2} \quad \textit{i. e.} \quad \frac{2}{N} < \frac{1}{p} \quad \textit{so that} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N} > \frac{1}{N} \quad \textit{i. e.} \quad p^* < N.$$

In conclusion, if $p \in [1, \frac{N}{2}[$,

$$W^{2,p}(\mathbb{R}^N) \subseteq L^{p^{**}}(\mathbb{R}^N), \qquad \textit{with} \qquad \frac{1}{p^{**}} = \frac{1}{p*} - \frac{1}{N} = \frac{1}{p} - \frac{2}{N}.$$

Theorem 73. *Let $m \in \mathbb{N} \setminus \{0, 1\}$. Let $p \in [1, +\infty]$.*

i) If $p < \frac{N}{m}$, then $W^{m,p}(\mathbb{R}^N) \subseteq L^{p^{**}}(\mathbb{R}^N)$, with $\frac{1}{p^{**}} = \frac{1}{p} - \frac{m}{N}$.

ii) If $p = \frac{N}{m}$, then $W^{m,p}(\mathbb{R}^N) \subseteq L^q(\mathbb{R}^N)$, for all $q \in [\frac{N}{m}, +\infty[$.

iii) If $p > \frac{N}{m}$, then $W^{m,p}(\mathbb{R}^N) \subseteq L^\infty(\mathbb{R}^N)$.

All the above embeddings are continuous. Moreover if $m - \frac{N}{p} > 0$ (i.e. in the case iii)) and $m - \frac{N}{p}$ is not an integer, denoting by

$$k = \text{integer part of } \; m - \frac{N}{p}, \qquad \theta = \text{fractional part of } \; m - \frac{N}{p},$$

we have

$$W^{m,p}(\mathbb{R}^N) \subseteq C^k(\mathbb{R}^N),$$

and there exists $C > 0$ such that

$$\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\mathbb{R}^N)} \leq C\|u\|_{W^{m,p}(\mathbb{R}^N)}$$

and

$$|D^\alpha u(x) - D^\alpha u(y)| \leq C\|u\|_{W^{m,p}(\mathbb{R}^N)} \, |x - y|^\theta$$

for a. e. $x, y \in \mathbb{R}^N$ and for all α with $|\alpha| = k$ (we will say that $u \in C^{k,\theta}(\mathbb{R}^N)$).

Proof. Exercise. □

Corollary 24. *Let Ω be an open set of class C^1 in $\mathbb{R}^N$. Suppose that $\partial\Omega$ is bounded (or Ω is an half-space). Let $m \in \mathbb{N} \setminus \{0, 1\}$. Let $p \in [1, +\infty[$.*

i) If $p < \frac{N}{m}$, then $W^{m,p}(\Omega) \subseteq L^{p^{**}}(\Omega)$, with $\frac{1}{p^{**}} = \frac{1}{p} - \frac{m}{N}$.

ii) If $p = \frac{N}{m}$, then $W^{m,p}(\Omega) \subseteq L^q(\Omega)$, for all $q \in [\frac{N}{m}, +\infty[$.

iii) If $p > \frac{N}{m}$, then $W^{m,p}(\Omega) \subseteq L^\infty(\Omega)$.

All the above embeddings are continuous. Moreover, in the case iii), i. e. if $m - \frac{N}{p} > 0$ and $m - \frac{N}{p}$ is not an integer, denoting by

$$k = \text{integer part of } \; m - \frac{N}{p}, \qquad \theta = \text{fractional part of } \; m - \frac{N}{p},$$

we have

$$W^{m,p}(\Omega) \subseteq C^k(\overline{\Omega}),$$

and there exists $C > 0$ such that

$$\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\mathbb{R}^N)} \leq C\|u\|_{W^{m,p}(\Omega)}$$

and

$$|D^\alpha u(x) - D^\alpha u(y)| \leq C\|u\|_{W^{m,p}(\Omega)} \, |x - y|^\theta$$

for a. e. $x, y \in \mathbb{R}^N$ and for all α with $|\alpha| = k$. i. e. $u \in C^{k,\theta}(\overline{\Omega})$.

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22.1 Sobolev spaces in N space dimensions - 4

The content of this paragraph can be found in [3, Ch. IX].

22.1.1 The space $W_0^{1,p}(\Omega)$

Definition 46. *Let $p \in [1, +\infty[$. Let Ω be an open bounded set in $\mathbb{R}^N$. We denote by $W_0^{1,p}(\Omega)$ the closure of $C_0^1(\Omega)$ (or equivalently $C_0^\infty(\Omega)$) in $W^{1,p}(\Omega)$. We set*

$$W_0^{1,2}(\Omega) = H_0^1(\Omega).$$

Remark 56. *Let $p \in [1, +\infty[$. We have defined $W_0^{1,p}(I)$ and $W_0^{1,p}(\Omega)$ for $I \subseteq \mathbb{R}$ and $\Omega \subseteq \mathbb{R}^N$ open bounded set, in particular I is an open bounded interval. Suppose to define $W_0^{1,p}(I)$ and $W_0^{1,p}(\Omega)$ for I and Ω open sets different from $\mathbb{R}$ and $\mathbb{R}^N$ respectively. In such a case we know that*

$$W_0^{1,p}(I) \subsetneqq W^{1,p}(I).$$

In fact, for all $u \in W_0^{1,p}(I)$, u is continuous on $\overline{I}$ and $u_{|\partial I} = 0$. On the contrary, for Ω open set in $\mathbb{R}^N$, $N \geq 2$, it is not the case, in general. E. g. consider $\Omega = \mathbb{R}^2 \setminus \{0\}$,

$$W_0^{1,p}(\Omega) = W^{1,p}(\Omega)$$

for $p \in [1, 2]$. The proof is let as an exercise.

It is possible to prove the following result (details can be found in [3, Ch. IX]).

Theorem 74. *Let $p \in [1, +\infty[$. Let Ω be an open bounded set in $\mathbb{R}^N$. Let Ω be of class C^1. Let*

$$u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

Then

$$u \in W_0^{1,p}(\Omega) \qquad \text{if and only if} \qquad u_{|\partial\Omega} = 0.$$

Remark 57. *We remark that, given $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$,*

$$\text{if} \qquad u_{|\partial\Omega} = 0 \qquad \text{then} \qquad u \in W_0^{1,p}(\Omega),$$

without any hypothesis in the regularity of $\partial\Omega$. In fact consider a function $G \in C^1(\mathbb{R})$ such that,

$$G(s) = \begin{cases} s & \text{if } |s| \geq 2, \\ 0 & \text{if } |s| \leq 1, \end{cases}$$

and, for all $s \in \mathbb{R}$, $|G(s)| \leq |s|$. take

$$u_n(x) = \frac{1}{n} G(n\, u(x)).$$

From Theorem 67 we deduce that $u_n \in W^{1,p}(\Omega)$. Moreover $u_n \in C(\bar{\Omega})$ and $u_n(x) = 0$ if $|u(x)| < \frac{1}{n}$, hence $\operatorname{Supp} u_n$ is a compact set contained in Ω. It is possible, using the usual technique of approximation by convolution with a family of mollifiers, to show that $u_n \in W_0^{1,p}(\Omega)$. It remains to prove that $u_n \xrightarrow{n} u$ in $W^{1,p}(\Omega)$.

Here a characterization of $W_0^{1,p}(\Omega)$, for $p > 1$.

Theorem 75 (Characterization of $W_0^{1,p}(\Omega)$, for $p > 1$)**.** *Let $p \in]1, +\infty]$. Let Ω be an open bounded set in $\mathbb{R}^N$. Let Ω be of class C^1. Suppose $u \in L^p(\Omega)$.*

Then the following conditions are equivalent.

i) $u \in W_0^{1,p}(\Omega)$.

ii) There exists $C > 0$ such that, for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and for all $j = 1, \dots, N$,

$$|\int_\Omega u\,\partial_j\varphi| \le C\|\varphi\|_{L^{p'}}, \qquad \textit{where } \frac{1}{p} + \frac{1}{p'} = 1.$$

iii) The function

$$\bar{u}(x) = \begin{cases} u(x) & \textit{if } x \in \Omega, \\ 0 & \textit{if } x \notin \Omega, \end{cases}$$

is in $W^{1,p}(\mathbb{R}^N)$.

Proof. Let *i)*. Let $(u_n)_n$ be a sequence in $\mathcal{D}(\Omega)$ such that $u_n \xrightarrow{n} u$ in $W^{1,p}(\Omega)$. Then, for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and for all $j = 1, \dots, N$,

$$|\int_\Omega u_n\,\partial_j\varphi| = |\int_\Omega \partial_j u_n\,\varphi| \le \|\nabla u_n\|_{L^p}\|\varphi\|_{L^{p'}}.$$

Passing to the limit in n, we obtain *ii)*.

Let *ii)*. We have

$$|\int_{\mathbb{R}^N} \bar{u}\,\partial_j\varphi| = |\int_\Omega u\,\partial_j\varphi| \le C\|\varphi\|_{L^{p'}}, \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^N).$$

Applying Theorem 65 we obtain that $\bar{u} \in W^{1,p}(\mathbb{R}^N)$, i. e. *iii)*.

Let finally *iii)*. Using a local change of variables and the partition of unity the point *i)* is obtained proving the following statement: let $u \in L^p(\{x \in B_{R,r} \,\big|\, x_N < 0\})$ and suppose that the function

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x \in B_{R,r} \text{ and } x_N < 0, \\ 0 & \text{if } x \in B_{R,r} \text{ and } x_N \ge 0, \end{cases}$$

is in $W^{1,p}(B_{R,r})$. Then

$$\psi u \in W_0^{1,p}(\{x \in B_{R,r} \,|\, x_N < 0\}) \qquad \text{for all } \psi \in \mathcal{D}(B_{R,r}).$$

It would be sufficient to take a family of mollifier $(\rho_n)_n$ such that

$$\operatorname{Supp}\rho_n \subseteq \{x \in \mathbb{R}^N \,|\, \frac{1}{2n} < x_n < \frac{1}{n}\}$$

and to consider $v_n = \rho_n * (\psi\bar{u})$ for obtaining, for n sufficiently big, $v_n \in C_0^\infty(\{x \in B_{R,r} \,|\, x_N < 0\})$ and $v_n \xrightarrow{n} \psi u$ in $W^{1,p}(\{x \in B_{R,r} \,|\, x_N < 0\})$. □

Remark 58. *In the previous proof the implications i) ⇒ ii) ⇒ iii) are true without any assumption on the regularity of $\partial\Omega$.*

We end this subparagraph showing the Poincaré inequality.

Theorem 76 (Poincaré inequality)**.** *Let $p \in [1, +\infty[$. Let Ω be an open bounded set in $\mathbb{R}^N$.*

Then there exists $C > 0$ (depending on N, p and Ω) such that

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \qquad \textit{for all} \;\; u \in W_0^{1,p}(\Omega),$$

i. e. $\|\nabla u\|_{L^p(\Omega)}$ is an equivalent norm in $W_0^{1,p}(\Omega)$.

Proof. Let $u \in W_0^{1,p}(\Omega)$. As said in Remark 58,

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } \; x \in \Omega, \\ 0 & \text{if } \; x \notin \Omega, \end{cases}$$

is in $W^{1,p}(\mathbb{R}^N)$ without any condition on the boundary of Ω.

Consequently, if $p \in [1, N[$, from Sobolev-Gagliardo-Nirenberg theorem,

$$\|\bar{u}\|_{L^{p^*}(\mathbb{R}^N)} \leq C_{N,p}\|\nabla \bar{u}\|_{L^p(\mathbb{R}^N)}.$$

But, on one side, being Ω bounded,

$$\|u\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{p}-\frac{1}{p^*}}\|u\|_{L^{p^*}(\Omega)} = |\Omega|^{\frac{1}{p}-\frac{1}{p^*}}\|\bar{u}\|_{L^{p^*}(\mathbb{R}^N)}.$$

On the other side

$$\|\nabla \bar{u}\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\Omega)}.$$

Hence

$$\|u\|_{L^p(\Omega)} \leq \underbrace{C_{N,p}\,|\Omega|^{\frac{1}{p}-\frac{1}{p^*}}}_{=C(N,p,|\Omega|)} \|\nabla u\|_{L^p(\Omega)}.$$

If, on the contrary, $p \in [N, +\infty[$, let $q = \frac{Np}{N+p}$. Remark that, since we can suppose $N \geq 2$, $q \in [1, N[$. We have $q < p$, so that $\bar{u} \in L^q(\mathbb{R}^N)$ and $\nabla \bar{u} \in L^q(\mathbb{R}^N)$. Moreover

$$\frac{1}{q^*} = \frac{1}{q} - \frac{1}{N} = \frac{1}{p}.$$

Consequently, again from Sobolev-Gagliardo-Nirenberg theorem,

$$\|\bar{u}\|_{L^p(\mathbb{R}^N)} = \|\bar{u}\|_{L^{q^*}(\mathbb{R}^N)} \leq C_{N,q}\|\nabla \bar{u}\|_{L^q(\mathbb{R}^N)}.$$

But

$$\|\nabla \bar{u}\|_{L^q(\mathbb{R}^N)} = \|\nabla u\|_{L^q(\Omega)} \leq |\Omega|^{\frac{1}{q}-\frac{1}{p}}\|\nabla u\|_{L^p(\Omega)},$$

and the conclusion follows also in this case. □

22.1.2 Examples of boundary value problems

Example 22 (Homogeneous Dirichlet problem)**.** *Let Ω be a bounded open set in $\mathbb{R}^N$, of class C^1. Let $f \in C(\overline{\Omega})$.*

Find $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{cases} -\Delta u + u = f & \textit{in } \; \Omega, \\ u = 0 & \textit{in } \; \partial\Omega. \end{cases} \tag{69}$$

As in the case of $N = 1$, the strategy for solving it will be the following.

a) Introduce a modified (weak) problem. The correct setting of this modified problem will be crucial.

b) Solve the weak problem, using some suitable functional analysis results.

c) Check that the solution of the weak problem, with the conditions of the classical problem, is enough regular to be the solution of the classical problem.

We remark that the more difficult point will be the point c).

a) Let $\tilde{f} \in L^2(\Omega)$.

Find $w \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla w \cdot \nabla v + \int_\Omega w\, v = \int_\Omega \tilde{f}\, v, \qquad \text{for all} \;\; v \in H_0^1(\Omega). \tag{70}$$

Problem (70) is the weak homogeneous Dirichlet problem.

Remark 59. *If u is a solution of the classical problem, then u is a solution of the weak one, if $\tilde{f} = f$. In fact, suppose that $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ is a solution to the classical problem, then, multiplying the first line of (69) by v and integrating on Ω,*

$$\int_\Omega (-\Delta u + u)\, v = \int_\Omega f\, v, \qquad \textit{for all} \;\; v \in H_0^1(\Omega).$$

and, integrating by parts,

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega u\, v = \int_{]0,1[} f\, v, \qquad \textit{for all} \;\; v \in H_0^1(\Omega).$$

Finally, since $u = 0$ on $\partial\Omega$, we have that $u \in H_0^1(\Omega)$.

b) Let's solve problem (70). We use Lax-Milgram theorem (see [3, Cor. V.8]). We choose as Hilbert space H the space $H_0^1(\Omega)$, as bilinear form a, the form

$$a(w, v) = \int_\Omega \nabla w \cdot \nabla v + \int_\Omega w\, v$$

and as ϕ, element of H', the functional

$$\phi : H_0^1(\Omega) \to \mathbb{R}, \qquad \phi(v) = \int_\Omega \tilde{f}\, v.$$

The existence and uniqueness of the solution $w \in H_0^1(\Omega)$ follows.

c) To show that weak solutions to (70) are in fact regular (continuous, with continuous first and second derivatives) is a complicate matter, which presuppose also a certain regularity of the border of Ω. Details can be found in [3, Ch. IX, Par. 6]. Here we suppose, a priori, that w, solution of the weak problem, is in $C^1(\overline{\Omega}) \cap C^2(\Omega)$. From Theorem 74 we have $w_{|\partial\Omega} = 0$ and moreover, since w is in $C^2(\Omega)$,

$$\int_\Omega \nabla w \cdot \nabla \varphi + \int_\Omega w\, \varphi = \int_\Omega (-\Delta w + w)\varphi = \int_\Omega f \varphi, \qquad \text{for all} \;\; \varphi \in \mathcal{D}(\Omega).$$

As a consequence $-\Delta w + w - f = 0$ almost eveywhere in Ω, but, since $-\Delta w + w - f$ is a continuous function in Ω, $-\Delta w + w = f$ everywhere in Ω.

Example 23 (Non-homogeneous Dirichlet problem)**.** *Let Ω be a bounded open set in $\mathbb{R}^N$, of class C^1. Let $f \in C(\overline{\Omega})$. Let $g \in C^1(\partial\Omega)$.*

Find $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{cases} -\Delta u + u = f & \textit{in } \Omega, \\ u = g & \textit{in } \partial\Omega. \end{cases} \tag{71}$$

It is possible to prove that there exists a function $\tilde{g} \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $g = \tilde{g}$ on $\partial\Omega$. Introducing the set

$$K = \{v \in H^1(\Omega) \,|\, v - \tilde{g} \in H_0^1(\Omega)\},$$

from Theorem 74, K does not depend on $\tilde{g}$ but only on g. The weak non-homogeneous Dirichlet problem associated to (71) is the following: find $w \in K$ such that

$$\int_\Omega \nabla w \cdot \nabla v + \int_\Omega w\, v = \int_\Omega f\, v, \qquad \text{for all } \ v \in H^1(\Omega). \tag{72}$$

By Stampacchia's theorem (see [3, Th. V.6]), (72) has a unique solution. It is possible to prove that, under particular regularity conditions on $\partial\Omega$, the solution of (72) has a sufficient regularity and it is the solution to (71).

Example 24 (Homogeneous Neumann problem)**.** *Let Ω be a bounded open set in $\mathbb{R}^N$, of class C^1. Let $f \in C(\overline{\Omega})$.*

Find $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{cases} -\Delta u + u = f & \textit{in } \Omega, \\ \partial_n u = 0 & \textit{in } \partial\Omega, \end{cases} \tag{73}$$

where $\partial_n u$ denote the external normal derivative of u with respect to $\partial\Omega$, i. e. $\partial_n u = \nabla u \cdot n$, where n is the external normal vector to $\partial\Omega$ of length 1*.*

The weak homogeneous Neumann problem is the following: find $w \in H^1(\Omega)$ such that

$$\int_\Omega \nabla w \cdot \nabla v + \int_\Omega w\, v = \int_\Omega f\, v, \qquad \text{for all } \ v \in H^1(\Omega). \tag{74}$$

Lax-Milgram theorem guarantees that (74) has a unique solution. Again, under particular regularity conditions on $\partial\Omega$, it is possible to prove that the solution of (74) has a sufficient regularity and it is the solution to (73).

22.1.3 Maximum principle for the Dirichlet problem

Theorem 77. *Let Ω be a bounded open set in $\mathbb{R}^N$. Let $f \in L^2(\Omega)$. Let $u \in H^1(\Omega) \cap C(\overline{\Omega})$ be such that*

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega w\, v = \int_\Omega f\, v, \qquad \textit{for all } \ v \in H_0^1(\Omega). \tag{75}$$

Then, for all $x \in \Omega$,

$$\min\{\min_{\partial\Omega} u, \inf_\Omega f\} \le u(x) \le \max\{\max_{\partial\Omega} u, \sup_\Omega f\}$$

where here above inf *and* sup *denote* inf ess *and* sup ess *respectively.*

Proof. Let $G \in C^1(\mathbb{R})$ such that

$$G(s) = \begin{cases} 0 & \text{if } \ s \leq 0, \\ \text{strictly increasing} & \text{if } \ s > 0 \end{cases}$$

and $G'(s) \leq M$, for all $s \in \mathbb{R}$.

Let

$$K = \max\{ \max_{\partial\Omega} u, \sup_{\Omega} \}$$

and suppose that $K < +\infty$. We show that $u(x) \leq K$ for all $x \in \Omega$. Consider

$$v(x) = G(u(x) - K).$$

$v \in H^1(\Omega)$ (see Theorem Theorem 67) and, for all $x \in \partial\Omega$,

$$v(x) = G(u(x) - K) = 0.$$

Consequently $v \in H_0^1(\Omega)$. We use v inside (75), taking into account that

$$\nabla v(x) = G'(u(x) - K)\nabla u(x).$$

We have

$$\int_\Omega G'(u-K)\nabla u \cdot \nabla u + \int_\Omega u\, G(u-K) = \int_\Omega f\, G(u-K)$$

i. e.

$$\underbrace{\int_\Omega |\nabla u|^2\, G'(u-K)}_{\geq 0} + \int_\Omega (u-K)\, G(u-K) = \underbrace{\int_\Omega (f-K)\, G(u-K)}_{\leq 0}.$$

We obtain

$$\int_\Omega (u-K)\, G(u-K) \leq 0.$$

Remarking finally that the function $s \mapsto s\, G(s)$ is nonnegative, we have that

$$(u(x) - K)\, G(u(x) - K) = 0 \qquad \text{for all } \ x \in \Omega,$$

and hence $u(x) - K \leq 0$ for all $x \in \Omega$. The computation to show that $u(x) \geq \min\{ \min_{\partial\Omega} u, \inf_\Omega f \}$ is similar. $\square$

Appendix

Young's inequality

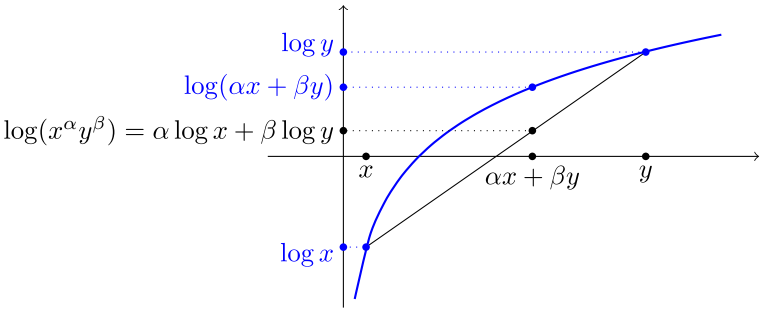
Let $\alpha,\ \beta \in [0,\ 1]$, with $\alpha + \beta = 1$ and let $x,\ y \in\,]0, +\infty[$.

Then

$$x^\alpha y^\beta \le \alpha x + \beta y.$$

In fact it is sufficient to remark that, since the logarithmic function is concave,

$$\log(x^\alpha y^\beta) = \alpha \log x + \beta \log y \le \log(\alpha x + \beta y).$$



Equivalently, for $p,\ q \in [0,\ 1]$, with $p + q = 1$ and for $a,\ b \in\,]0, +\infty[$,

$$ab \le p\, a^{\frac{1}{p}} + q\, b^{\frac{1}{q}}.$$

Lax-Milgram Theorem

Theorem 78. ([3, Cor. V.8]). *Let $a(u, v)$ be a bilinear, continuous and coercive form, defined on the Hilbert space H.*

Then, for all $\phi \in H'$, there exists a unique $u \in H$ such that

$$a(u, v) = \phi(v), \qquad \text{for all} \quad v \in H. \tag{76}$$

Moreover, if a is symmetric, then u, solution of (76), is characterized by

$$u \in H \qquad \text{and} \qquad \frac{1}{2}a(u, u) - \phi(u) = \min_{v \in H}\{\frac{1}{2}a(v, v) - \phi(v)\}.$$

References

[1] Adams, Robert A. "Sobolev Spaces". Academic Press, New York-San Francisco-London, 1975, xviii+268 pp.

[2] Bony, Jean-Michel. "Cours d'analyse: théorie des distributions et analyse de Fourier". Les Éditions de l'École Polytechnique, Palaiseau 2001. *Free download at the address*
http://www.cmap.polytechnique.fr/ massot/MAP431_web/MAP431_Histoire_Massot/Livres_Bony_Lutzen/Jean-Michel%20Bony-Cours%20d%27analyse.%20The%CC%81orie%20des%20distributions %20et%20analyse%20de%20Fourier-Ellipses(2001).pdf

[3] Brezis, Haïm. "Analyse fonctionnelle. (French) [Functional analysis] Théorie et applications. [Theory and applications]". Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree] Masson, Paris, 1983, xiv+234 pp.

[4] Cartan, Henri. "Théorie élémentaire des functions analytiques d'une ou plusieurs variables complexes". (French) Avec le concours de Reiji Takahashi. Enseignement des Sciences. Hermann, Paris 1961, 232 pp.

[5] Dirac, Paul. "The Principles of Quantum Mechanics". Oxford, at the Clarendon Press, 1930. x+257 pp.

[6] Giusti, Enrico. "Analisi matematica 2. Terza edizione interamente riveduta e ampliata". Bollati Boringhieri, Torino, 2003, xi+307 pp.

[7] Hewitt, Edwin; Stromberg, Karl. "Real and abstract analysis. A modern treatment of the theory of functions of a real variable". Third printing. Graduate Texts in Mathematics, No. 25. Springer-Verlag, New York-Heidelberg, 1975, x+476 pp.

[8] Hörmander, Lars. "Linear partial differential operators". Die Grundlehren der mathematischen Wissenschaften, Bd. 116 Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963, vii+287 pp.

[9] Katsuura, Hidefumi. *Continuous nowhere-differentiable functions–an application of contraction mappings.* Amer. Math. Monthly 98, no. 5 (1991), 411–416.

[10] Kolmogorov, A. N.; Fomin, S. V. "Introductory real analysis". Revised English edition. Translated from the Russian and edited by Richard A. Silverman Prentice-Hall, Inc., Englewood Cliffs, N.J., 1970, xii+403 pp. *Italian edition: "Elementi di teoria delle funzioni e di analisi funzionale". Editori Riuniti Univ. Press, 2012.*

[11] McCarthy, John. *An everywhere continuous nowhere differentiable function.* Amer. Math. Monthly 60 (1953), 709. *Free download at the address* http://jmc.stanford.edu/articles/weierstrass/weierstrass.pdf

[12] Meyers, Norman G.; Serrin, James. *H=W.* Proc. Nat. Acad. Sci. U.S.A. 51 (1964), 1055–1056.

[13] Rudin, Walter. "Principles of mathematical analysis". Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976, x+342 pp. *Free download at the address* https://notendur.hi.is/vae11/%C3%9Eekking/principles_of_mathematical_analysis_walter_rudin.pdf

[14] Rudin, Walter. "Real and complex analysis". Third edition. McGraw-Hill Book Co., New York, 1987, xiv+416 pp.

[15] Schwartz, Laurent. "Théorie des distributions". (French) Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée. Hermann, Paris 1966 xiii+420 pp.

[16] Schwartz, Laurent. "Un mathématicien aux prises avec le siècle", Editeur Odile Jacob, Paris, 1997, 528 pp.

[17] Stein, Elias. "Singular integrals and differentiability properties of functions". Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J., 1970, xiv+290 pp.

[18] Treves, François. "Topological vector spaces, distributions and kernels". Academic Press, New York-London, 1967, xvi+624 pp.