Mathematics of Time Value

The general expression for computing the present value of future cash flows is as follows:

$$PV = \sum_t \frac{C_t}{(1+r_t)^t} \qquad (1)$$

This expression allows for variations in cash flows over time and for variations in required rates of return over time. We could generalize further by allowing for the possibility that different cash flows in the same period have different degrees of risk and, therefore, possibly different required rates of return.

To illustrate, consider a project that requires investing $1000 today and another $1000 in one year. Assume that if you decide to undertake the project, the investment in year 1 is certain to be needed. You have determined that the appropriate rate for discounting certain future cash flows is 6 percent. On the positive side, the project is expected to produce cash inflows from operations for three years. The expected cash flows are $500 in year 1, $1000 in year 2, and $2000 in year 3. Given the uncertainties of these cash flows, you have concluded that the appropriate discount rates are 12 percent for year 1, 11 percent per year for year 2, and 10 percent per year for year 3. In addition, the investment can be sold in year 3 for an expected salvage value of $500. Because of the high uncertainty about what the salvage value actually will be, the appropriate discount rate for the salvage value cash flow is 20 percent per year.

Suppose we pose the question of whether you should invest $1000 today to acquire the future cash flows (both positive and negative) of the project. The answer depends on the present value of future cash flows. Using Eq. (8.1A), the present value of the project, is as follows:

$$PV = \frac{-1000}{1.06} + \frac{500}{1.12} + \frac{1000}{1.11^2} + \frac{2000}{1.10^3} + \frac{500}{1.20^3}$$

$$PV = -943.40 + 446.43 + 811.62 + 1502.63 + 289.35$$

$$PV = 2106.63$$

Since $2106.63 is greater than the $1000 initial investment that would be required for the project, it should be accepted.

Alternatively, the question could have been posed in terms of net present value (NPV), which is simply the excess of the present value computed above over the $1000 initial investment. Since the net present value of $1106.63 is positive, the project should be accepted. The correct decision does not depend on which of the two decision rules is used, but in doing the calculations, it is important to keep track of whether you are computing a present value or a net present value.

Shortcuts to Computing Present Value

The discrete calculation method of computing present value (as described in Eq. (1) and illustrated above) is always available. But for projects with cash flows spread over many periods, that approach can be cumbersome. There are several shortcuts for calculating present value under these conditions. The shortcuts often are useful for valuing new ventures. We do not show the derivations of these shortcuts but they are covered in most principles texts.

Perpetuities

A perpetuity is an investment that is expected to return the same cash flow each period, forever. If it is appropriate to discount each cash flow at the same periodic rate (i.e., the required rate of return is constant over time), then the value of a perpetuity that is expected to generate cash flows beginning at period one can be determined using the following expression:

$$PV = \frac{C}{r} \tag{2}$$

where C is the expected cash flow beginning in period one and r is the required rate of return that is appropriate for each periodic cash flow.

To illustrate, suppose you invest $100 in a savings account that is expected to return 7 percent per year. The expected annual cash flow from such an investment is $7, and if the expected rate of return is equal to the required rate of return, then the present value of the account is $100. If, however, the required rate of return is only 6 percent, then the expected annual cash flow of $7 results in a present value that is higher than the investment. Using Eq. (2), the present value is $116.67. This is the value of the right to receive an expected annual cash flow of $7 forever if the required rate of return is 6 percent. If you think of the savings account as an investment with a cost of $100, then the net present value of the account, given the 6 percent required rate of return, is $16.67.

Growing Perpetuities

Suppose the perpetuity, instead of returning a constant dollar amount per year, is expected to return an amount, beginning at period one, that will grow at a constant rate per period. It turns out that if the expected growth rate is constant, such an investment can be valued by adjusting the discount rate to reflect expected growth, as shown in the following expression:

$$PV = \frac{C_1}{r-g} \tag{3}$$

where $C1$ is the cash flow that is expected to be received in the first period and g is the expected rate of growth of cash flows.

Some intuition about Eq. (3) can be gained if you think of the growth rate of a future cash flow that is due to inflation and g as the expected annual rate of inflation. In that case, r is the nominal required rate of return (the one you could infer from observable market information) and r - g is the real required rate of return (the rate in excess of what would be needed to compensate for inflation). You can value the investment either by discounting the nominal (growing) cash flow at the nominal cost of capital, r, or by discounting real cash flows of C per year at the real cost of capital $(r - g)$. The answer is the same either way, but the second approach is easier.

It often is possible to value the shares of stock of an established company using Eq. (3). Suppose that next year's dividend on a stock is expected to be $5, that the required rate of return is 12 percent, and that the expected rate of dividend growth is 4 percent. The resulting share value is $62.50 (the $5 dividend divided by 0.08). The value of the stock depends on dividends rather than earnings since dividends are the cash flows the investor actually receives. Nonetheless, share valuation often is focused on earnings or operating cash flow since those are measures of the firm's ability to pay dividends in the future. Brealey and Myers (2003) provide a discussion of the conditions under which valuing the earnings stream is equivalent to valuing the stream of cash flows the investor will receive.

Equation (3) is closely related to a common approach to valuing new ventures and public offerings of common stock. Under that approach, a terminal value is calculated by projecting out new venture earnings or cash flows to a point when the operations of the venture are expected to have stabilized, and then capitalizing the earnings or cash flow numbers using a multiplier. The multiplier is usually determined based on the price/earnings or price/cash flow ratios of comparable public companies. The terminal value is then used as if it were a single cash flow, and it is discounted back to present value at the required rate of return. The logic behind using price/earnings or price/cash flow ratios to determine terminal value is that, if earnings or cash flows are retained by the business, they are expected to earn their cost of capital. It does not matter whether the cash flows are paid out immediately or higher cash flows are paid out in later periods. To connect this valuation method to Eq. (3), recognize that the price/earnings or price/cash flow ratio can be an estimate of $(1/(r - g))$ in the equation.

Annuities

Suppose instead of a level perpetuity, an investment is expected to return a constant cash flow for a certain period of time. A mortgage, for example, normally has a constant monthly payment for a certain number of months, after which no further payments are made. Equation (4) gives the expression for valuing a level annuity.

$$PV = \frac{C}{r}\left(1 - \left(\frac{1}{1+r}\right)^{t}\right) \tag{4}$$

where the first cash flow occurs in period one and the last occurs in period t.

To illustrate, consider investment in a mortgage that is expected to pay $1000 per month for 360 months and where the required rate of return is 1 percent per month. Using Eq. (4) we can determine that the present value of the mortgage is $97,218.

Growing Annuities

Finally, there is a simple method of valuing growing annuities. Long-term leases with rent escalators and expected future salaries to the time of retirement are examples of problems that can be modeled as growing annuities. The expression for the value of a growing annuity combines the reasoning that underlies Eqs. (3) and (4):

$$PV = \frac{C_1}{r-g}\left(1 - \left(\frac{1+g}{1+r}\right)^{t}\right) \tag{5}$$

Equation (5) gives the value of a growing annuity that begins in period one and ends in period t.

Suppose you begin working next year, with an expected starting salary of $50,000 and that you expect to work for 30 years. You expect that your earnings will grow at an annual rate of 6 percent on average, and you have decided to value the earnings using a required rate of return of 14 percent. Using Eq. (5), the present value of expected future earnings is $554,545.

Timing of Cash Flows

All of the shortcuts illustrated so far assume that the cash flows begin at time one and yield a value as of time zero. If you are evaluating an investment with different timing, you still can use the shortcuts, but you need to combine them with other calculations to find the correct values. For example, if you are valuing an investment in a project that is expected to yield a growing perpetuity where the first cash flow is expected to be in year 5, Eq. (3) will give you the value as of year 4. You then need to discount that value for four more years at the required rate of return to find the value as of year zero.

Questions and Problems

1. A venture requires an investment of $5 million today and is expected to return $25 million in five years. The required rate of return is 16 percent. What is the NPV of the opportunity?

2. As an alternative to question 1, suppose it is possible to invest $2 million today and that doing so would result in a 40 percent chance of needing to invest another $5 million in year 3. If the second-round investment is made, the expected cash flow is $50 million. If not, the initial investment is worthless. The required return on the second-round investment is 14 percent. The required return on the first-round investment is 20 percent.
   a. Find the NPV at year 3 of the second-round investment.

b. Using this result as the net value of the option to invest at year 3, find the NPV of the opportunity to invest at time zero. Remember that there is only a 40 percent chance that the second-round investment will be made.

3. If you invest $5 million today, you can acquire an investment that is expected to pay $1 million per year forever, with the first payment in one year. If you invest only $3.5 million today, you can acquire the same investment, but the first cash flow will not be received until three years from today. If the required return on investment is 15 percent, which is better?

4. Your cost of capital is 15 percent; would you rather invest in an asset that pays $1 million in year 1 and grows at a rate of 10 percent per year forever, or one that pays $3 million in year 1 and declines at a rate of 10 percent per year forever?

5. What required rate of return would make you indifferent between the two opportunities described in question 4?

6. An entrepreneur has been offered the same amount of debt financing from two sources. One lender requires debt service payments (principal and interest) of $1 million per year for five years. The other requires debt service payments beginning at $800,000 in the first year, and growing at a rate of 8 percent per year through the fifth year.
   a. Suppose the entrepreneur believes the true cost of capital for borrowing (given default and other risk) is 12 percent. Which offer is better?
   b. Suppose both lenders are offering to lend $4 million. See if you can find or estimate the interest rate each is using.
   c. Assuming the entrepreneur wishes to borrow $4 million and has a cost of capital of 12 percent, what should be the annual payment of the level payment loan? What should be the first-year payment of the loan with the growing schedule of repayments?