Appendix E

Exercises

E.1 Pre-requisites

1. *Solve the following ODE problems:*

(a) *Solve using the method of variations of constants:*

$$\begin{aligned} y'' &= -y' + t \\ y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

(b) *Solve using an integrating factor:*

$$y' + 2\,t\,y = t$$

2. *Solve the following difference equations:*

(a) *Solve and write down the first* 5 *terms of the solution for the case* $r = 0.3$, $\alpha = 1000$

(starting with y_0):

$$y_{n+1} = r\, y_n + \alpha$$

$$y_0 = \alpha$$

(b) *Consider the ODE*

$$y' = a\, y; \quad a \in \mathbb{R}^-$$

Write down and solve the difference equation resulting from the numerical solution of the ODE with the second-order Adams-Bashfort method using a time-step $h = -2/a$

3. *Numerical quadrature:*

(a) *Derive the trapezoidal quadrature rule (report all mathematical steps).*

(b) *Consider a Gauss-quadrature formula on the interval $[0, 1]$, using the values of the inte-*

grating function at the assigned quadrature points 0 *and* 1 *and at a free quadrature point* $0 < a < 1$*:*

$$\int_0^1 f(x)\,dx \approx w_0\,f(0) + w_a\,f(a) + w_1\,f(1)$$

Write down the equations that allow to identify w_0, w_a, w_1 *and* a *so as to obtain a quadrature formula with the maximum possible degree of precision.*

4. *Initial-value problems and shooting method:*

   (a) *Devise a procedure to identify the stability region of the 3rd-order Adams-Moulton method.*

(b) Set up a procedure to solve the following BVP by the shooting method:

$$y^{iv} + f(x)\,y'' + g(x)\,y' + h(x)\,y = r(x), \quad a < x < b$$

$$y(a) = \alpha_1$$

$$y'(a) = \alpha_2$$

$$y(b) = \beta_1$$

$$y'(b) = \beta_2$$

(c) *The following method was suggested as a means of solving ODEs:*

$$\begin{aligned} y^* &= y_n + \frac{1}{3}\, h\, f(x_n, y_n) \\ y_{n+1} &= y_n + \frac{1}{2}\, h\, \left[3\, f(x_n + \frac{h}{3}, y^*) - f(x_n, y_n) \right] \end{aligned}$$

i. *Classify this method in as many ways as you can, for example, accuracy (oder), explicit/implicit, multipoint, ...*

ii. *Determine its order of accuracy (more specifically, the LTE).*

iii. *Discuss the stability of this method using* $y' = \alpha\, y$ *for real* α *only.*

iv. *Would you recommend use of this method?*

5.

5. Wile E. Coyote (WEC) is chasing Road Runner (RR). WEC runs so that he is always traveling directly toward RR. If RR runs at a speed of 15 m/s and stays in a straight line (this problem is not very realistic) and WEC runs at 20 m/s, how long does it take WEC to catch RR if they are initially 150 m apart in the direction perpendicular to the rabbit's path?

Derive the differential equation for the WEC's path and solve it numerically. The equations are singular at the instant of catch (so is what happens to RR), so you will need to be careful in this region. It is recommended that you compute the distance between the two at each time step and extrapolate it to the estimate its value at the next time step. Continue the calculation until the extrapolated value is negative; this indicates that capture will occur within the following time step. Then use extrapolation to find the time of capture.

Note: An analytical solution to this problem exists. It can be found in the book by Davis (1962).

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